Spherical Cubes and Rounding in High Dimensions

Guy Kindler
Anup Rao
Ryan O'Donnell
Avi Wigderson

January 21, 2010

Abstract

What is the least surface area of a shape that tiles $\mathbb{R}^d$ under translations by $\mathbb{Z}^d$? Any such shape must have volume 1 and hence surface area at least that of the volume-1 ball, namely $\Omega(\sqrt{d})$. Our main result is a construction with surface area $O(\sqrt{d})$, matching the lower bound up to a constant factor of $2\sqrt{2\pi/e} \approx 3$. The best previous known was only slightly better than the cube, having surface area on the order of d.

We generalize this to give a construction that tiles $\mathbb{R}^d$ by translations of any full rank discrete lattice Λ with surface area $2\pi\|V^{-1}\|$, where V is the matrix of basis vectors of Λ, and $\|\cdot\|$ denotes the Frobenius norm. We show that our bounds are optimal within constant factors for rectangular lattices. Our proof is via a random tessellation process, following recent ideas of Raz citeRaz08 in the discrete setting.

Our construction gives an almost optimal noise-resistant rounding scheme to round points in $\mathbb{R}^d$ to rectangular lattice points. .