

# Spherical Cubes and Rounding in High Dimensions

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## Abstract

What is the least surface area of a shape that tiles  $\mathbb{R}^d$  under translations by  $\mathbb{Z}^d$ ? Any such shape must have volume 1 and hence surface area at least that of the volume-1 ball, namely  $\Omega(\sqrt{d})$ . Our main result is a construction with surface area  $O(\sqrt{d})$ , matching the lower bound up to a constant factor of  $2\sqrt{2\pi/e} \approx 3$ . The best previous known was only slightly better than the cube, having surface area on the order of  $d$ .

We generalize this to give a construction that tiles  $\mathbb{R}^d$  by translations of any full rank discrete lattice  $\Lambda$  with surface area  $2\pi\|V^{-1}\|$ , where  $V$  is the matrix of basis vectors of  $\Lambda$ , and  $\|\cdot\|$  denotes the Frobenius norm. We show that our bounds are optimal within constant factors for rectangular lattices. Our proof is via a random tessellation process, following recent ideas of Raz citeRaz08 in the discrete setting.

Our construction gives an almost optimal noise-resistant rounding scheme to round points in  $\mathbb{R}^d$  to rectangular lattice points.