UPPER TAIL LARGE DEVIATIONS IN FIRST PASSAGE PERCOLATION

YUZHOU GU

1. Introduction

Consider the grid graph $\mathbb{Z}^2$ where there is an edge between any two vertices with Euclidean distance 1. Let ν be a probability measure supported on the interval $[0, b]$ with continuous density. Let each edge have length iid chosen from ν. This defines a random metric $\mathbf{PT}(\cdot, \cdot)$ ("passage time") on $\mathbb{Z}^2$. Fix a unit vector $\vec{v} \in \mathbb{R}^2$. A standard fact (see [ADH17]) says that the limit $\lim_{n\to\infty} \frac{1}{n} \mathbf{PT}(\vec{0}, n\vec{v})$ exists. Let $\mu = \mu(\nu, \vec{v})$ denote this limit. Kesten [Kes86] studied the large deviation properties of $\mathbf{PT}(0, n\vec{v})$. He proved that

(1) for any $\zeta \in (0, \mu)$, the limit

$$\lim_{n\to\infty} -\frac{\log \mathbb{P}(\mathbf{PT}(\vec{0}, n\vec{v}) \le (\mu - \zeta)n)}{n}$$

exists and is $\in (0, \infty)$;

(2) for any $\zeta \in (0, b - \mu)$, we have

$$\begin{aligned} 0 &< \liminf_{n\to\infty} -\frac{\log \mathbb{P}(\mathbf{PT}(\vec{0}, n\vec{v}) \ge (\mu + \zeta)n)}{n^2} \\ &\le \limsup_{n\to\infty} -\frac{\log \mathbb{P}(\mathbf{PT}(\vec{0}, n\vec{v}) \ge (\mu + \zeta)n)}{n^2} < \infty. \end{aligned}$$

Therefore the lower tail has speed n while the upper tail has speed n^2. The intuition is that to lower the passage time by $\Theta(n)$, we only need to lower the length of $\Theta(n)$ edges, while to increase the passage time by $\Theta(n)$, we need to increase the length of $\Theta(n^2)$ edges.

It was left open whether a rate function exists for the upper tail large deviation. Recently, Basu-Ganguly-Sly [BGS17] answered this question in the affirmative.

Theorem 1 (Basu-Ganguly-Sly [BGS17])**.** *The limit*

$$\lim_{n\to\infty} -\frac{\log \mathbb{P}(\mathbf{PT}(\vec{0}, n\vec{v}) \ge (\mu + \zeta)n)}{n^2}$$

exists and is $\in (0, \infty)$.

Remark 2. (1) The condition imposed on ν is not the weakest possible for Theorem 1 to hold.
(2) Theorem 1 holds also for the first passage percolation in $\mathbb{Z}^d$, with the speed (denominator) replaced by n^d.

In this expository paper we study the proof of Theorem 1 in [BGS17]. We emphasize high-level ideas and often omit details of proof.

2. Overview of the proof

For simplicity, let $\mathscr{U}_\zeta(n)$ denote the upper tail large deviation event $\mathbf{PT}(\vec{0}, n\vec{u}) \geq (\mu+\zeta)n$. The proof of Theorem 1 is in two main parts.

Proposition 3. *For each $\epsilon' \in (0,\zeta)$ and $\epsilon > 0$, there exists N_0 and H_0 such that for all $n > N_0$ and $m > nH_0$ we have*

$$\frac{1}{m^2}\log \mathbb{P}(\mathscr{U}_{\zeta-\epsilon'}(m)) \geq \frac{1}{n^2}\log \mathbb{P}(\mathscr{U}_\zeta(n)) - \epsilon.$$

Proposition 4. *For each $\epsilon > 0$, there exists $\epsilon' > 0$ such that for all n large enough we have*

$$\frac{1}{n^2}\log \mathbb{P}(\mathscr{U}_{\zeta-\epsilon'}(n)) \leq \frac{1}{n^2}\log \mathbb{P}(\mathscr{U}_\zeta(n)) + \epsilon.$$

Proof that Proposition 3 + 4 implies Theorem 1. It is not hard to see that the two propositions imply for all $\epsilon > 0$, there exists N_0 such that for all $n > N_0$, there exists $M_0 = M_0(n)$ such that for all $m > M_0$, we have

$$\frac{1}{m^2}\log \mathbb{P}(\mathscr{U}_\zeta(m)) \geq \frac{1}{n^2}\log \mathbb{P}(\mathscr{U}_\zeta(n)) - \epsilon.$$

Using this fact we can prove that

$$\liminf_{n\to\infty} \frac{1}{n^2}\log \mathbb{P}(\mathscr{U}_\zeta(n)) = \limsup_{n\to\infty} \frac{1}{n^2}\log \mathbb{P}(\mathscr{U}_\zeta(n)).$$

□

Let $\mathbf{Box}(\mathscr{C}n)$ denote the set $[-\mathscr{C}n, \mathscr{C}n]^2$. Before we prove Proposition 3 and 4, we need the following result.

Lemma 5. *There exists $\alpha > 0$ such that for any $\mathscr{C} > 0$, for n large enough, with probability $1 - o(1)$, for any two points $\vec{x}, \vec{y} \in \mathbf{Box}(\mathscr{C}n)$ with $|\vec{x} - \vec{y}| > \sqrt{n}$, we have $\mathbf{PT}(\vec{x}, \vec{y}) \geq \alpha|\vec{x} - \vec{y}|$.*

The proof of lemma is by a union bound. Then by triangle inequality, for some large enough constant $\mathscr{C}$, with probability $1 - o(1)$, the shortest path from $\vec{0}$ to $n\vec{v}$ lies inside $\mathbf{Box}(\mathscr{C}n)$. Note that by an application of FKG inequality, Lemma 5 is also true conditioned on $\mathscr{U}_\zeta(n)$.

Now let $\mathcal{E}$ denote the event that for any two points $\vec{x}, \vec{y} \in \mathbf{Box}(\mathscr{C}n)$ with $|\vec{x} - \vec{y}| > \sqrt{n}$, we have $\mathbf{PT}(\vec{x}, \vec{y}) \geq \alpha|\vec{x} - \vec{y}|$. Let $\mathscr{U}^*_\zeta = \mathscr{U}_\zeta \cap \mathcal{E}$. We have $\mathbb{P}(\mathscr{U}^*_\zeta) = (1 - o(1))\mathbb{P}(\mathscr{U}_\zeta)$. Therefore we only need to prove Proposition 3 and 4 with $\mathscr{U}_\zeta$ replaced by $\mathscr{U}^*_\zeta$. This enables us to work within a finite size box instead of working with infinitely many edges.

3. Proof of Proposition 4

The proof of Proposition 4 is easier, so we describe it first. Starting with an environment $\Pi \in \mathscr{U}^*_{\zeta-\epsilon}$, we increase the length of all edges slightly to get an environment $\Pi' \in \mathscr{U}^*_\zeta$. To implement this proof, we need to handle two types of "bad" edges.

1. If an edge e already has length x_e very close to b, then we cannot increase its length by an amount larger than $b - x_e$.
2. If an edge e has length x_e in a low density region of ν, then increasing its weight by a small amount results in a low probability event. So instead we need to increase its weight by a large amount, to some value close to b.

Let $\mathbf{H}_1$ be the set of edges of type (1), and $\mathbf{H}_2$ be the set of edges of type (2). Then with probability $1-o(1)$, we have $|\mathbf{H}_1|, |\mathbf{H}_2| \leq \epsilon_4 n^2$, where ϵ_4 depends on the choice of parameters in the definition of bad edges. So there exist two sets A_1, A_2 of size $O(\epsilon_4)n^2$ such that

$$\mathbb{P}(\{\mathbf{H}_1 \subseteq A_1\} \cap \{\mathbf{H}_2 \subseteq A_2\} | \mathscr{U}^*_{\zeta-\epsilon}) = \exp(-O(\epsilon_5)n^2).$$

Conditioned on this event, we

(1) preserve length of edges in A_1;
(2) increase length of edges in A_2 to a value close to b;
(3) increase length of all other edges by ϵ_7.

By choosing parameters carefully, the modified event is in $\mathscr{U}^*_\zeta$ and happens with probability $\exp(-O(\epsilon_6)n^2)$ conditioned on $\{\mathbf{H}_1 \subseteq A_1\} \cap \{\mathbf{H}_2 \subseteq A_2\} \cup \mathscr{U}^*_{\zeta-\epsilon}$; furthermore, we can let $\epsilon_4 \to 0$, $\epsilon_5 \to 0$, $\epsilon_6 \to 0$. This finishes the proof.

4. Proof of Proposition 3

This is the major part of the proof of Theorem 1.

4.1. Proof overview.

Fix n and m. We pick $(\frac{m}{n})^2$ similar events $\Pi_1, \dots, \Pi_{(\frac{m}{n})^2} \in \mathscr{U}^*_\zeta(n)$. Then we do cut-and-paste to get a dilated event Π in $\mathscr{U}^*_{\zeta-\epsilon'}(m)$.

Fix some integer j. We split $\mathbf{Box}(\mathscr{C}n)$ into $2^j \times 2^j$ tiles, each with size $\frac{\mathscr{C}n}{2^j} \times \frac{\mathscr{C}n}{2^j}$, and label these tiles using $[2^j] \times [2^j]$. Let $\mathbf{Tile}_{\mathscr{C}n}(j, v)$ denote the tile of $\mathbf{Box}(\mathscr{C}n)$ with label $v \in [2^j] \times [2^j]$. Each $\mathbf{Tile}_{\mathscr{C}m}(j, v)$ can be divided into $\frac{m}{n} \times \frac{m}{n}$ subtiles, each with size $\frac{\mathscr{C}n}{2^j} \times \frac{\mathscr{C}n}{2^j}$. We call these subtiles $\mathbf{Tile}_{\mathscr{C}m}(j, v, w)$ for $w \in [\frac{m}{n}] \times [\frac{m}{n}]$. Roughly speaking, we construct the event Π by letting $\mathbf{Tile}_{\mathscr{C}m}(j, v, w)$ be $\mathbf{Tile}_{\mathscr{C}n}(j, v)$ in Π_w.

We will define the meaning of "similar" so that for fixed $v \in [2^j] \times [2^j]$, these tiles have similar large-scale metric properties, and so that the $\mathbf{PT}(\vec{0}, m\vec{v})$ in Π is at least $(1-o(1))\,\mathbf{PT}(\vec{0}, n\vec{v})$ in Π_w for any $w \in [(\frac{m}{n})^2]$. Then $\Pi \in \mathscr{U}^*_{\zeta-\epsilon'}(m)$.

4.2. Base event.

We define **Base-Event**, the set from which the smaller events $\Pi_1, \dots, \Pi_{(\frac{m}{n})^2}$ are picked. It has two parts.

The first part is stability. Roughly, stability means that fixing a starting point and a direction, the metric in that direction is almost linear.

Definition 6. A tile is (δ, l, k)-stable if for every point $\vec{z}$ in tile, for every unit vector $\vec{u} \in \mathbb{R}^2$, for all $1 \leq k' \leq k$, we have

$$\frac{\sum_{1\leq i\leq k'} \mathbf{PT}(\vec{z}+(i-1)l\vec{u}, \vec{z}+il\vec{u})}{k'\,\mathbf{PT}(\vec{z}, \vec{z}+l\vec{u})} \in [\frac{1}{1+\delta}, 1+\delta].$$

The following lemma says that there exists a choice of parameters so that with constant probability, almost all tiles are stable.

Lemma 7. *Given small enough $\delta, \epsilon_1 > 0$, positive integer $m_1 \leq -\frac{1}{4}\log_2 \epsilon_1$, and positive integer J_1, there exists positive integer J_2 such that for all n large enough, conditioned on $\mathscr{U}^*_\zeta(n)$, there exists $j \in [J_1, J_2]$ such that with with probability at least $\frac{1}{J_2}$ the fraction of $v \in [2^j] \times [2^j]$ such that $\mathbf{Tile}_{\mathscr{C}n}(j, v)$ is not (δ, l, k)-stable is at most ϵ_1, where $l = \frac{n}{2^{j+m_1}}$ and $k = 2^{2m_1}$.*

Therefore there exists a set $A \subseteq [2^j] \times [2^j]$ of size $O(\epsilon_1 2^{2j})$ such that with probability $\exp(-o(n^2))$, all unstable tiles are in A. In **Base-Event**, we require that in every Π_w, all unstable tiles are in A.

The second part of **Base-Event** is large scale distances. Fix a discretization parameter η. Let $\mathbf{Grid}_{\mathscr{C}n}(j)$ be the points in $\mathbf{Box}(N) \cap \frac{n}{2^j}\mathbb{Z}^{\not{k}}$. Define $\mathbf{Proj}$: $\mathbf{Grid}_{\mathscr{C}n}(j+\frac{m_1}{2}) \times \mathbf{Grid}(\mathscr{C}n)(j+\frac{m_1}{2}) \to \mathbb{R}$ as the function

$$\mathbf{Proj}(\vec{x},\vec{y}) = \eta|\vec{x}-\vec{y}|\lfloor \frac{\mathbf{PT}(\vec{x},\vec{y})}{\eta|\vec{x}-\vec{y}|} \rfloor.$$

We can count that the number of possible choices of $\mathbf{Proj}$ is $\exp(o(n^2))$. So there exists a function P such that $\mathbf{Proj} = P$ with probability $\exp(-o(n^2))$. In **Base-Event**, we require that $\mathbf{Proj} = P$ in every Π_w.

Summing up, we have

$$\textbf{Base-Event} = \mathscr{U}^*_\zeta(n) \cap \{\text{unstable tiles} \subseteq A\} \cap \{\mathbf{Proj} = P\}.$$

The following lemma says that we do not lose much measure if we replace $\mathscr{U}^*_\zeta$ with **Base-Event**.

Lemma 8. *Given $\epsilon_4 > 0$, there exists a choice of parameters such that*

$$\frac{\log \mathbb{P}(\textbf{Base-Event})}{n^2} \geq \frac{\log \mathbb{P}(\mathscr{U}^*_\zeta(n))}{n^2} - \epsilon_4.$$

4.3. **Favorable event.** Now we can describe in detail the construction of the dilated event Π on $\mathbf{Box}(\mathscr{C}m)$. In fact, for technical reasons, we slightly increase the size of the box and work with $\mathbf{Box}(\mathscr{C}(1+2\epsilon_6)m)$. Starting from the construction described in Section 4.1, we add the following region.

(1) Between any two adjacent $\mathbf{Tile}_{\mathscr{C}m}(j,v)$'s, we insert a row/column of width $\epsilon_6\mathscr{C}m$.
(2) For fixed $v \in [2^j] \times [2^j]$, between any two adjacent $\mathbf{Tile}_{\mathscr{C}m}(j,v,w)$'s, we insert a row/column of width $\epsilon_6\mathscr{C}\frac{m}{2^j}$.

After inserting these columns/rows to $\mathbf{Box}(\mathscr{C}m)$, we get a $\mathbf{Box}(\mathscr{C}(1+2\epsilon_6)m)$. The inserted region is called corridor.

The event **Fav** is described as following.

(1) Each edge in corridor has length $\in [b-\epsilon_7, b]$.
(2) For $v \notin A$, $\mathbf{Tile}_{\mathscr{C}m}(j,v,w)$ is $\mathbf{Tile}_{\mathscr{C}n}(j,v)$ in Π_w.
(3) For $v \in A$, each edge in $\mathbf{Tile}_{\mathscr{C}m}(j,v)$ has length $\in [b-\epsilon_7, b]$.

Now we abuse notation and replace $(1+2\epsilon_6)m$ with m. Thus **Fav** is an event defined on $\mathbf{Box}(\mathscr{C}m)$.

Lemma 9. *Given ϵ_8 and ϵ_9, there exists a choice of parameters such that*

$$\frac{\log \mathbb{P}(\mathbf{Fav})}{m^2} \geq \frac{\log \mathbb{P}(\mathscr{U}^*_\zeta(n))}{n^2} - \epsilon_8$$

and **Fav** $\subseteq \mathscr{U}^*_{\zeta-\epsilon_9}(m)$.

The hard part of the proof is that $\mathbf{PT}(\vec{0}, m\vec{v}) \geq (\zeta - \epsilon_9)m$. Given a path from $\vec{0}$ to $m\vec{v}$, we first modify it to satisfy some useful conditions.

Lemma 10. *Conditioned on* **Fav**, *given any path α from $\vec{0}$ to $m\vec{v}$, we can construct a path β satisfying the following conditions.*

(1) *If β touches $\mathbf{Tile}_{\mathscr{C}m}(j,v)$, then it is large in $\mathbf{Tile}_{\mathscr{C}m}(j,v)$. Here a path is large in a tile with side length L means that if the path enters tile from point $\vec{x}$ and exits from point $\vec{y}$, then there exists a point $\vec{z}$ in tile such that*

$$\min\{|\vec{x}-\vec{z}|, |\vec{y}-\vec{z}|\} \geq \epsilon_6^2 L.$$

(2) *If β touches $\mathbf{Tile}_{\mathscr{C}m}(j,v,w)$, then it is large in $\mathbf{Tile}_{\mathscr{C}m}(j,v,w)$.*

(3) *β is regular, in the sense that whenever it exits $\mathbf{Tile}_{\mathscr{C}m}(j,v)$, it enters an adjacent tile using a completely vertical or completely horizontal path.*

(4) $|\alpha| \geq (1 - O(\epsilon_7 + \epsilon_6))|\beta|$.

The proof of lemma is by performing modifications step by step. The existences of corridors helps reduce short zig-zags between adjacent tiles.

Now that we have path β satisfying all these conditions. It can be decomposed into $\beta_1\chi_1\beta_2\cdots\chi_{s-1}\beta_s$ such that each β_i is an excursion in some $\mathbf{Tile}_{\mathscr{C}m}(j,v)$, and each χ_i is a (completely vertical or completely horizontal) path in corridor. Let $\vec{x}_i$ be the start point of β_i. Let $\vec{x}_i^S$ be the closest point in $\mathbf{Grid}_{\mathscr{C}n}(j+\frac{m_1}{2})$ to $\frac{n}{m}\vec{x}_i$.[1] Let $\vec{x}_i^{S\prime}$ be the point adjacent to $\vec{x}_i^S$ in the tile containing $\vec{x}_{i+1}^S$. Now let β_i^S be the shortest path from $\vec{x}_i^S$ to $\vec{x}_i^{S\prime}$ and β^S be the path concatenated from β_i^S for all i.

Lemma 11. *Given any $\epsilon_{11} > 0$, there exists a choice of parameters such that*

$$|\beta_i| \geq (1-\epsilon_{11})\frac{m}{n}|\beta_i^S|.$$

LHS is computed under **Fav** *and RHS is computed under any environment in* **Base-Event**.

The proof of Lemma 11 uses stability of tiles and the function **Proj** which governs the large-scale metric. This finishes the proof of Lemma 9 and therefore Proposition 3 holds.

References

[ADH17] A. Auffinger, M. Damron, and J. Hanson. *50 years of first-passage percolation*, volume 68. American Mathematical Soc., 2017.

[BGS17] R. Basu, S. Ganguly, and A. Sly. Upper tail large deviations in first passage percolation. *arXiv preprint arXiv:1712.01255*, 2017.

[Kes86] H. Kesten. Aspects of first passage percolation. In *École d'été de probabilités de Saint Flour XIV-1984*, pages 125–264. Springer, 1986.

[1] Here we are abusing notation by pretending that corridors do not exist. The correct thing to do is to first remove the corridors, and then do proper scaling of coordinates.