THE OBSTACLE PROBLEM FOR A FRACTIONAL MONGE–AMPÈRE EQUATION

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ABSTRACT. We study the obstacle problem for a degenerate fractional Monge–Ampère equation. We show the existence of a unique, globally Lipschitz and semiconcave classical solution, at which the fractional Monge–Ampère equation becomes locally uniformly elliptic. This uniform ellipticity is used to deduce local regularity of the solution and the free boundary.

1. INTRODUCTION

Let \( s \in (1/2, 1) \). Consider the fractional Monge–Ampère operator
\[
D_s u(x) = \inf \{ -(\Delta)^s [u \circ A](A^{-1}x) : A \in \mathcal{M} \}
\]
where \( u : \mathbb{R}^n \to \mathbb{R} \), \( n \geq 1 \), \( (\Delta)^s \) is the fractional Laplacian on \( \mathbb{R}^n \), and \( \mathcal{M} \) is the class of positive definite symmetric matrices \( A \) of size \( n \times n \) such that \( \det A = 1 \). This operator was first introduced by L. A. Caffarelli and F. Charro in [1] as a fractional analogue to the classical Monge–Ampère operator. In fact, if \( u \) is a convex \( C^2 \) function, then it can be checked that
\[
(\det D^2 u(x))^{1/n} = \inf \{ \Delta [u \circ A](A^{-1}x) : A \in \mathcal{M} \}.
\]
If, in addition, \( u \) is asymptotically linear at infinity, then
\[
\lim_{s \to 1} D_s u(x) = (\det D^2 u(x))^{1/n},
\]
see [1, Appendix A]. Like its local counterpart (1.2), the fractional operator (1.1) is degenerate elliptic. Indeed, matrices of the form \( A = \text{diag}(\varepsilon, 1/\varepsilon), \varepsilon > 0 \), in dimension 2, are in \( \mathcal{M} \), and they degenerate as \( \varepsilon \searrow 0 \). Thus, the existence and regularity theory for nonlocal elliptic equations previously developed in [4, 5, 6], see also [8, 10, 12], does not directly apply to equations involving (1.1). Nevertheless, Caffarelli and Charro established in [1, Theorem 3.1] that the operator \( D_s \) becomes uniformly elliptic as soon as \( D_s u \) is bounded below away from zero and \( u \) is globally Lipschitz and semiconcave. They considered the problem
\[
\begin{align*}
D_s \bar{u}(x) &= \bar{u}(x) - \phi(x) \quad \text{in } \mathbb{R}^n \\
\lim_{|x| \to \infty} (\bar{u} - \phi)(x) &= 0
\end{align*}
\]
where \( \phi \) is a function in \( C^{2,\sigma}(\mathbb{R}^n) \), \( \sigma > 0 \), that is strictly convex in compact sets and asymptotically linear at infinity (see section 2 for the precise definition of \( \phi \)). It is shown in [1] that there exists a unique, globally Lipschitz and semiconcave classical solution \( \bar{u} \) to (1.3). In addition, \( \bar{u} \) has the crucial property that \( \bar{u} > \phi \) in \( \mathbb{R}^n \). This and (1.3) imply that \( D_s u \) is locally uniformly bounded away from zero, making (1.1) a locally uniformly elliptic operator. As a consequence, known regularity theory [4, 5, 6] gives that \( \nabla \bar{u} \) is locally Hölder continuous.


Key words and phrases. Free boundary problems, fractional Monge–Ampère equation, degenerate elliptic nonlinear nonlocal equations, regularity estimates.

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In this paper, we investigate the following obstacle problem:

\[
\begin{cases}
\mathcal{D}_s u \geq u - \phi & \text{in } \mathbb{R}^n \\
u \leq \psi & \text{in } \mathbb{R}^n \\
\mathcal{D}_s u = u - \phi & \text{in } \{u < \psi\} \\
\lim_{|x| \to \infty} (u - \phi)(x) = 0.
\end{cases}
\]

(1.4)

We assume that the obstacle \( \psi \in C^{2,1}(\mathbb{R}^n) \) is such that \( \psi > \phi \) in \( \mathbb{R}^n \) and \( \psi \leq \bar{u} \) in some compact set \( K \).

Here and in the remainder of this work, \( \bar{u} \) denotes the solution to (1.3).

Obstacle problems for nonlocal operators appear in optimal control, mathematical finance, biology, and nonlinear elasticity. The regularity of solutions and free boundaries for this type of nonlinear problem for the fractional Laplacian was studied by Silvestre in [13], and by Caffarelli–Salsa–Silvestre in [3], and for more general homogeneous, translation invariant, purely nonlocal uniformly elliptic operators by Caffarelli–Ros-Oton–Serra in [2]. Our problem (1.4) does not fit into any of these previous settings. First, the last equation in (1.4) implies that \( u \) is not bounded. Second, and more importantly, \( \mathcal{D}_s \) given by (1.1) is degenerate elliptic. As a matter of fact, we are not aware of any literature dealing with regularity estimates for degenerate elliptic, purely nonlocal obstacle problems as in this paper.

On the other hand, obstacle problems for the classical Monge–Ampère equation (1.2) were considered by Lee [9], where the obstacle is above \( u \), and Savin [11], in which the obstacle lies below \( u \). Then, (1.4) can be seen as a fractional nonlocal counterpart of [9].

Our first result establishes the existence and global regularity of a unique classical solution to (1.4). For the necessary notation, see section 2.

**Theorem 1.1.** There exists a unique classical solution \( u \) to the obstacle problem (1.4). Moreover, \( u \) is globally Lipschitz continuous and semiconcave with constants no larger than

\[
M_1 = \max \left\{ \left[ \phi \right]_{\text{Lip}(\mathbb{R}^n)}, \left[ \psi \right]_{\text{Lip}(\mathbb{R}^n)} \right\} \quad \text{and} \quad M_2 = \max \left\{ \text{SC}(\phi), \text{SC}(\psi) \right\},
\]

respectively, and the contact set \( \{u = \psi\} \subset K \) is compact. Furthermore,

\[
u > \phi \quad \text{in } \mathbb{R}^n.
\]

(1.6)

The degenerate ellipticity of the fractional Monge–Ampère operator (1.1) prevents us from applying standard techniques used to prove existence and uniqueness for nonlocal uniformly elliptic obstacle problems [2, 13]. Therefore, to construct the solution \( u \) to (1.4), we need to devise a new strategy. This is one of the main contributions of this paper. To prove Theorem 1.1, we consider a family of obstacle problems of the form (1.4), but where \( \mathcal{D}_s \) is replaced by a truncated operator \( \mathcal{D}_s^\varepsilon \), which is defined in the same way as \( \mathcal{D}_s \), but by constraining the matrices appearing in (1.1) to have eigenvalues bounded from below by \( \varepsilon > 0 \) (see section 2 for the precise definition of \( \mathcal{D}_s^\varepsilon \)).

We build the solutions \( u_\varepsilon \) to such uniformly elliptic nonlocal problems as the largest subsolution sitting below \( \psi \) (see Theorem 3.8). A key feature of the family of solutions \( \{u_\varepsilon\}_{\varepsilon > 0} \) is that it is uniformly globally Lipschitz continuous and semiconcave with constants no larger than \( M_1 \) and \( M_2 \) (see (1.5)), respectively. At this point, we would like to use that viscosity solutions are stable under local uniform convergence. Yet again, the degenerate ellipticity of \( \mathcal{D}_s \) proves obstructive. The crucial, delicate step that will allow us to overcome this difficulty is to show that \( u = \inf_{\varepsilon > 0} u_\varepsilon \) remains strictly above \( \phi \) (see Lemma 3.9), that is, that (1.6) holds. This property finally allows us to conclude the proof of Theorem 1.1. See section 3 for details.

Next, we prove local Hölder estimates on \( \nabla u \) outside of the contact set \( \{u = \psi\} \) and across the free boundary \( \partial \{u < \psi\} \). Recall that, in Theorem 1.1, we established the important separation property (1.6). This and the global regularity of \( u \) permit us to apply [1, Theorem 3.1] to get that
where $\mathcal{D}^{\lambda}$ is a uniformly elliptic truncated version of (1.1), with ellipticity constants depending on the gap between $u$ and $\phi$ in $\mathcal{B}$. Then, using the regularity estimates for uniformly elliptic nonlocal equations from [2] and [12], we can prove our second main result.

**Theorem 1.2.** Let $u$ be the solution to the obstacle problem (1.4).

1. Let $\mathcal{O}$ be an open set and let $\mathcal{O}_\delta$, $\delta > 0$, be a $\delta$-neighborhood of $\mathcal{O}$ such that $\mathcal{O}_\delta \subset \{u < \psi\}$. There exists $\beta = \beta(n, s, \inf_{\mathcal{O}}(u - \phi), M_1, M_2) \in (0, 1)$ such that $u \in C^{1,2s+\beta-1}(\mathcal{O})$ and
   \[
   \|u\|_{C^{1,2s+\beta-1}(\mathcal{O})} \leq C(1 + \|u - \phi\|_{L^\infty(\mathbb{R}^n)}),
   \]
   where $C = C(\beta, \text{diam}(\mathcal{O})) > 0$.

2. Let $\mathcal{B}$ be a ball centered at the origin such that $\{u = \psi\} \subset \mathcal{B}$. There exists $\tau = \tau(n, s, \inf_{\mathcal{B}}(u - \phi), M_1, M_2) \in (0, 1)$ such that $u \in C^{1,\tau}(\mathcal{B})$ and
   \[
   \|u\|_{C^{1,\tau}(\mathcal{B})} \leq C(1 + \|\psi\|_{C^{1,\tau}(\mathcal{B})}),
   \]
   where $C = C(\tau, \text{diam}(\mathcal{B})) > 0$.

Theorem 1.2 demonstrates another important point of divergence between our obstacle problem and uniformly elliptic nonlocal obstacle problems. In [2], solutions are shown to be $C^{1,\tau}(\mathbb{R}^n)$. In contrast, since $\lim_{|x| \to \infty}(u - \phi)(x) = 0$, we cannot guarantee that $\mathcal{D}^{\lambda}$, when acting on $u$, will become globally uniformly elliptic. In particular, the H"older exponents $\beta$ and $\tau$ in Theorem 1.2 degenerate as $\mathcal{O}$ drifts to infinity and $\mathcal{B}$ increases in size, respectively.

To study the regularity of the free boundary $\partial\{u < \psi\}$ and the behavior of $u$ near free boundary points, we fix a ball $\mathcal{B}$ centered at the origin such that $\{u = \psi\} \subset \mathcal{B}$. Then, $u$ satisfies the obstacle problem (1.7). Observe that, unlike in [2], we do not know whether or not $\mathcal{D}^{\lambda}u = u - \phi$ in the part of the noncoincidence set $\{u < \psi\}$ that lies outside of $\mathcal{B}$. Hence, it is not clear that (after subtracting the obstacle from $u$ in (1.7)) similar arguments can be adapted to study the structure and regularity of the free boundary. In fact, it is well known that the behavior at infinity of solutions to nonlocal equations can have dramatic consequences on their local properties (see, for instance, [7]). Moreover, our Hölder estimates for $\nabla u$ degenerate at infinity. Nonetheless, the global regularity of $u$ proved in Theorem 1.1 gives us enough control at infinity to show that appropriate blow ups at regular points converge to the global profiles found in [2]. To state our next result, let

\[
\text{(1.8)} \quad v = \psi - u.
\]

Let $x_0 \in \partial\{u < \psi\} = \partial\{v > 0\}$ be a regular free boundary point (see Definition 5.1). By following [2], for $r > 0$, we define the rescalings

\[
v_r(x) = \frac{v(x_0 + rx)}{r^{1+s+\alpha}\theta(x_0, r)} \quad \text{for } x \in \mathbb{R}^n
\]

where

\[
\theta(x_0, r) = \sup_{\rho \geq r} \frac{||\nabla v(x_0 + \cdot)||_{L^\infty(B_{\rho})}}{\rho^{s+\alpha}}.
\]

**Theorem 1.3.** There exist a sequence $r_k \downarrow 0$, $1/4 \leq K_0 \leq 1$, and $e_0 \in S^{n-1}$ such that

\[
v_{r_k}(x) \to K_0(e_0 \cdot x)^{1+s} \quad \text{in } C^{1}_\text{loc}(\mathbb{R}^n), \quad \text{as } k \to \infty.
\]
It is known that blow up profiles ultimately drive the regularity of the free boundary. Since, in our case, we have local uniform ellipticity of $D_s$, global Lipschitz continuity of $u$, and convergence of blow ups to the same global solutions of Caffarelli–Ros-Oton–Serra [2], we can rely on their methods to obtain our last main result.

**Theorem 1.4.** Let $u$ be the solution to (1.4). Let $B$ be a ball centered at the origin such that $\{u = \psi\} \subset B$. There exists $\bar{\alpha} = \bar{\alpha}(n, s, \inf_{B}(u - \phi), M_1, M_2) \in (0, 1)$ such that the following holds: for any $\gamma \in (0, \bar{\alpha})$ and $\alpha \in (0, \bar{\alpha})$ such that $1 + s + \alpha < 2$ and for any $x_0 \in \partial\{u < \psi\}$,

1. either
   $$\liminf_{r \downarrow 0} \frac{|\{u = \psi\} \cap B_r(x_0)|}{|B_r(x_0)|} > 0 \quad \text{and} \quad \psi(x) - u(x) = cd^{1+s}(x) + o(|x - x_0|^{1+s+\alpha})$$

2. or
   $$\liminf_{r \downarrow 0} \frac{|\{u = \psi\} \cap B_r(x_0)|}{|B_r(x_0)|} = 0 \quad \text{and} \quad \psi(x) - u(x) = o(|x - x_0|^{\min\{2s+\gamma, 1+s+\alpha\}})$$

3. or
   $$\liminf_{r \downarrow 0} \frac{|\{u = \psi\} \cap B_r(x_0)|}{|B_r(x_0)|} > 0 \quad \text{and} \quad \psi(x) - u(x) = o(|x - x_0|^{1+s+\alpha})$$

where $d(x) = \text{dist}(x, \partial\{u < \psi\})$ and $c > 0$. Moreover, the set of points $x_0$ satisfying (1) is an open subset of the free boundary of class $C^{1,\gamma}$.

The paper is organized as follows. In section 2, we establish some preliminary results that will be needed for the rest of the work. The proofs of Theorems 1.1, 1.2, 1.3, and 1.4 are presented in sections 3, 4, 5, and 6, respectively.

2. Preliminaries

In this section, we recall some facts about the fractional Monge–Ampère operator $D_s$, problem (1.3), and uniformly elliptic nonlocal operators.

2.1. Notation. Let $O$ be an open subset of $\mathbb{R}^n$ and $f : O \to \mathbb{R}$. We denote the Lipschitz constant of $f$ in $O$ by

$$[f]_{\operatorname{Lip}(O)} = \sup_{x, y \in O, x \neq y} \frac{|f(x) - f(y)|}{|x - y|}.$$

For the second order incremental quotient of $f$ at $x$ in the direction of $y$, we write

$$\delta(f, x, y) = f(x + y) + f(x - y) - 2f(x).$$

When $O = \mathbb{R}^n$, we say that $f$ is semiconcave if there exists a constant $C > 0$ such that $\delta(f, x, y) \leq C|y|^2$ for all $x, y \in \mathbb{R}^n$. In this case,

$$\operatorname{SC}(f) = \sup_{x, y \in \mathbb{R}^n} \frac{\delta(f, x, y)}{|y|^2}$$

is the semiconcavity constant of $f$. Alternatively, $f$ is semiconcave if and only if $f(x) - C|x|^2/2$ is concave.

Let $\operatorname{USC}(O)$ (resp. $\operatorname{LSC}(O)$) be the set of functions that are upper (resp. lower) semicontinuous in $O$. Define

$$f^*(x) = \limsup_{r \to 0} \{f(y) : y \in O \text{ and } |y - x| < r\} \quad \text{for every } x \in O.$$
We call $f^*$ the upper semicontinuous envelope of $f$ in $\mathcal{O}$; it is the smallest $g \in \text{USC}(\mathcal{O})$ satisfying $f \leq g$.

**Remark 2.1.** A simple, useful property of the upper semicontinuous envelope $f^*$ of $f$ in $\mathcal{O}$ is that for any $x_0 \in \mathcal{O}$, there exist points $y_k \in \mathcal{O}$ such that

$$y_k \to x_0 \quad \text{and} \quad f(y_k) \to f^*(x_0), \quad \text{as} \ k \to \infty.$$ 

### 2.2. The fractional Monge–Ampère operator

We begin this subsection by providing some novel insight on the definition of the fractional Monge–Ampère operator $D_s u$ in (1.1), which may be of independent interest. Next, we precisely describe $\phi$. Then, we discuss the definition of viscosity solutions and some further properties of $D_s u$ and the problem (1.3).

Recall that

$$\mathcal{M} = \left\{ \text{symmetric positive definite matrices } A \text{ of size } n \times n \text{ such that } \det A = 1 \right\}.$$ 

For any $A \in \mathcal{M}$, we define the constant coefficient second order elliptic operator

$$L_A w(x) = -\Delta [w \circ A](A^{-1} x) = -\operatorname{tr}(A^2 D^2 w)(x),$$

see (1.2). Then, $L_A$ is nothing but a linear transformation of the Laplacian $-\Delta$. For $s \in (0, 1)$, consider the fractional power operator

$$L_A^s = -(L_A)^s \quad \text{in } \mathbb{R}^n.$$ 

**Lemma 2.2.** Let $w : \mathbb{R}^n \to \mathbb{R}$ such that

$$\int_{\mathbb{R}^n} \frac{|w(x)|}{(1 + |x|)^{n+2s}} \, dx < \infty.$$ 

Let $\mathcal{O}$ be an open set. If $w \in C^{2s+\delta}(\mathcal{O})$ or $w \in C^{1,2s+\delta-1}(\mathcal{O})$ when $s \geq 1/2$, for some $\delta > 0$, then, for any $x \in \mathcal{O}$,

$$L_A^s w(x) = c_{n,s} P.V. \int_{\mathbb{R}^n} \frac{w(y) - w(x)}{|A^{-1}(y - x)|^{n+2s}} \, dy$$

$$= \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{w(x + y) + w(x - y) - 2w(x)}{|A^{-1} y|^{n+2s}} \, dy$$

$$= -(\Delta)^s [w \circ A](A^{-1} x),$$

where $c_{n,s} = \frac{4^s \Gamma(n/2+s)}{\pi^{n/2} \Gamma(-s)} > 0$. As a consequence of (2.1), we have

$$D_s w(x) = \inf \left\{ L_A^s w(x) : A \in \mathcal{M} \right\}.$$ 

**Proof.** The idea is to first prove (2.1) for $w$ in the Schwartz class $\mathcal{S}$, by applying the method of semigroups as in [14, Lemma 5.1]. Then, for $w$ as in the hypotheses, one can use an approximation device exactly as done in [13, Proposition 2.4]. We just sketch the steps here. For $A \in \mathcal{M}$ and $w \in \mathcal{S}$, the heat semigroup generated by $L_A$ acting on $w$ is given explicitly by

$$e^{tL_A} w(x) = \int_{\mathbb{R}^n} \frac{e^{-|A^{-1} x|^2/(4t)}}{(4\pi t)^{n/2}} w(x - y) \, dy,$$

for $x \in \mathbb{R}^n$ and $t > 0$. Then, since $e^{tL_A} \mathbf{1}(x) = 1$, by Fubini’s theorem (see [14, Lemma 5.1]) and the change of variables $r = |A^{-1} x|^2/(4t)$,

$$L_A^s w(x) = -(L_A)^s w(x)$$

$$= \frac{1}{|\Gamma(-s)|} \int_0^\infty \left( e^{tL_A} w(x) - w(x) \right) \frac{dt}{t^{1+s}}$$

$$= \frac{1}{|\Gamma(-s)|} \int_0^\infty \int_{\mathbb{R}^n} \frac{e^{-|A^{-1} x|^2/(4t)}}{(4\pi t)^{n/2}} (w(x - y) - w(x)) \, dy \frac{dt}{t^{1+s}}.$$
\[
= \text{P.V.} \int_{\mathbb{R}^n} (w(x-y) - w(x)) \left[ \int_0^\infty \frac{e^{-|A^{-1}x|^2/(4t)}}{\Gamma(-s)}(4\pi t)^{n/2} t^{1+s} \right] dy \\
= c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{w(x-y) - w(x)}{|A^{-1}y|^{n+2s}} dy,
\]
as desired. The second identity in (2.1) follows immediately from the first one, and the third one is deduced via a simple change of variables. \(\square\)

Now, we give the precise description of the function \(\phi\) appearing in (1.3) and (1.4). Let \(\Gamma\) be a cone and \(\eta : \mathbb{R}^n \to \mathbb{R}\) be such that
\[
|\eta(x)| \leq a|x|^{-\epsilon}, \quad |\nabla \eta(x)| \leq a|x|^{-(1+\epsilon)}, \quad \text{and} \quad |D^2 \eta(x)| \leq a|x|^{-(2+\epsilon)}
\]
for some constants \(a > 0\) and \(\epsilon \in (0, n)\). We let \(\phi \in C^{2,\sigma}(\mathbb{R}^n)\), for some \(\sigma > 0\), be such that
\[
\phi(0) = 0, \quad \nabla \phi(0) = 0, \quad \text{and} \quad \phi = \Gamma + \eta \quad \text{near infinity.}
\]

We will work with viscosity solutions as defined in [1, Definition 2.1].

**Definition 2.3.** Let \(\mathcal{O}\) be an open subset of \(\mathbb{R}^n\). A function \(w : \mathbb{R}^n \to \mathbb{R}\) such that \(w \in \text{USC}(\overline{\mathcal{O}})\) (resp. \(w \in \text{LSC}(\overline{\mathcal{O}})\)) is called a viscosity subsolution (resp. supersolution) to \(\mathcal{D}_sw = w - \phi\) in \(\mathcal{O}\), which we denote by
\[
\mathcal{D}_sw \geq w - \phi \quad (\text{resp. } \mathcal{D}_sw \leq w - \phi) \quad \text{in } \mathcal{O},
\]
if whenever
- \(x_0\) is a point in \(\mathcal{O}\);
- \(\mathcal{N} \subset \mathcal{O}\) is an open neighborhood of \(x_0\);
- \(P\) is a \(C^2\) function on \(\overline{\mathcal{N}}\);
- \(P(x_0) = w(x_0)\); and
- \(P(x) > w(x)\) (resp. \(P(x) < w(x)\)) for every \(x \in \mathcal{N} \setminus \{x_0\}\);

then
\[
\mathcal{D}_s \vartheta(x_0) \geq \vartheta(x_0) - \phi(x_0) \quad (\text{resp. } \mathcal{D}_s \vartheta(x_0) \leq \vartheta(x_0) - \phi(x_0))
\]
where \(\vartheta\) is defined as
\[
\vartheta(x) = \begin{cases} 
P(x) & \text{if } x \in \mathcal{N} \\
w(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}.
\end{cases}
\]

When all of the items listed above are satisfied for some triplet \((P, x_0, \mathcal{N})\), we say that \(P\) is a \(C^2\) function touching \(w\) from above (resp. below) at \(x_0\) in \(\mathcal{N}\). A viscosity solution \(w\) is both a viscosity subsolution and a viscosity supersolution. In particular, solutions are continuous by definition.

From now on, any reference to a subsolution, supersolution, or solution will be in the viscosity sense.

Note that a semiconcave function can always be touched from above by a quadratic polynomial at any point.

**Remark 2.4.** Let \(P\) be a \(C^2\) function touching \(w\) from above (resp. below) at \(x_0\) in \(\mathcal{N}\). If \(\mathcal{N}'\) is any open subset of \(\mathcal{N}\) containing \(x_0\), then \(P\) is a \(C^2\) function that touches \(w\) from above (resp. below) at \(x_0\) in \(\mathcal{N}'\). Define \(\vartheta\) as in (2.2) and let
\[
\vartheta'(x) = \begin{cases} 
P(x) & \text{if } x \in \mathcal{N}' \\
w(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}'.
\end{cases}
\]
Then, \(\vartheta \geq \vartheta'\) (resp. \(\vartheta \leq \vartheta'\)) in \(\mathbb{R}^n\) and \(\vartheta(x_0) = \vartheta'(x_0)\), so that \(\delta(\vartheta, x_0, y) \geq \delta(\vartheta', x_0, y)\) for every \(y \in \mathbb{R}^n\) (resp. \(\delta(\vartheta, x_0, y) \leq \delta(\vartheta', x_0, y)\)). It follows that \(L_A^s \vartheta(x_0) \geq L_A^s \vartheta'(x_0)\) (resp. \(L_A^s \vartheta(x_0) \leq L_A^s \vartheta'(x_0)\)) for all matrices \(A \in \mathcal{M}\), which implies that
\[
\mathcal{D}_s \vartheta(x_0) \geq \mathcal{D}_s \vartheta'(x_0)
\]
Then, \( D_s \vartheta(x_0) \leq D_s \vartheta'(x_0) \), see (1.1). Therefore, if \( D_s w \geq w - \phi \) (resp. \( D_s w \leq w - \phi \)) in \( \mathbb{R}^n \), then, in order to check the viscosity solution condition from Definition 2.3, we can always restrict ourselves to working in a smaller neighborhood \( \mathcal{N}' \subset \mathcal{N} \) containing \( x_0 \).

From the definition of \( D_s \), we see that

1. If \( \tau_h w(x) = w(x + h) \), for some \( h \in \mathbb{R}^n \), then \( D_s(\tau_h w) = \tau_h(D_s w) \).
2. For any constant \( c \in \mathbb{R} \), \( D_s(w + c) = D_s w \).
3. \( D_s \) is a concave operator in the sense that, for any \( w_1, w_2 \),
   \[
   D_s \left( \frac{w_1 + w_2}{2} \right) \geq \frac{1}{2} D_s w_1 + \frac{1}{2} D_s w_2.
   \]

Let \( w \) be a viscosity subsolution (resp. supersolution) as in Definition 2.3. In the next lemma, we state that if \( w \) can be touched from above (resp. below) by a \( C^2 \) function at a point \( x \), then \( D_s w(x) \) can be computed classically. This is an important, typical feature of nonlocal equations, see also [5, Lemma 3.3].

**Lemma 2.5** (see [1, Lemma 2.2]). Let \( w : \mathbb{R}^n \to \mathbb{R} \) be asymptotically linear at infinity. If

\[
D_s w \geq w - \phi \quad \text{(resp. } D_s w \leq w - \phi \text{)} \quad \text{in } \mathcal{O} \subset \mathbb{R}^n
\]

in the viscosity sense and \( w \) can be touched by a \( C^2 \) function from above (resp. below) at a point \( x \in \mathcal{O} \), then \( \delta(w, x, y)/|A^{-1} y|^{n+2s} \in L^1(\mathbb{R}^n) \) for every \( A \in \mathcal{M} \) and

\[
D_s w(x) \geq w(x) - \phi(x) \quad \text{(resp. } D_s w(x) \leq w(x) - \phi(x) \text{)}
\]

in the classical sense.

Finally, we recall the comparison principle proved by Caffarelli and Charro.

**Theorem 2.6** (see [1, Theorem 4.1]). Let \( w_1 \in \text{USC}(\mathbb{R}^n) \) and \( w_2 \in \text{LSC}(\mathbb{R}^n) \) such that

\[
\begin{align*}
D_s w_1 & \geq w_1 - \phi & \text{in } \mathbb{R}^n \\
\lim_{|x| \to \infty} (w_1 - \phi)(x) & = 0 & \text{and} & \lim_{|x| \to \infty} (w_2 - \phi)(x) & = 0.
\end{align*}
\]

Then,

\[
w_1 \leq w_2 \quad \text{in } \mathbb{R}^n.
\]

2.3. **The truncated fractional Monge–Ampère operator.** For \( \varepsilon > 0 \), consider the class

\[
\mathcal{M}_\varepsilon = \{ A \in \mathcal{M} : \langle A \xi, \xi \rangle \geq \varepsilon |\xi|^2 \text{ for all } \xi \in \mathbb{R}^n \}.
\]

Since the matrices in \( \mathcal{M} \) have determinant one, not only are the eigenvalues of \( A \in \mathcal{M}_\varepsilon \) bounded from below, but they are also bounded from above. In particular,

\[
\varepsilon |\xi|^2 \leq \langle A \xi, \xi \rangle \leq \varepsilon^{1-n} |\xi|^2 \quad \text{for every } \xi \in \mathbb{R}^n.
\]

We define

\[
D^\varepsilon_A u(x) = \inf_{A \in \mathcal{M}_\varepsilon} L^s_A u(x).
\]

The kernels of \( L^s_A \), for \( A \in \mathcal{M}_\varepsilon \), satisfy

\[
\frac{\varepsilon^{n+2s}}{|y|^{n+2s}} \leq \frac{1}{|A^{-1} y|^{n+2s}} \leq \frac{\varepsilon^{(1-n)(n+2s)}}{|y|^{n+2s}}.
\]

Therefore, the truncated operator \( D^\varepsilon_A \) is uniformly elliptic in the sense of Caffarelli–Silvestre (see Lemma 2.8 and [5, Definition 3.1, Lemma 3.2]). We define the notion of viscosity subsolution, supersolution, and solution for \( D^\varepsilon_A \) exactly as in Definition 2.3. Moreover, by [5, Lemma 3.3], Lemma 2.5 also holds for \( D^\varepsilon_A \) in place of \( D_s \). Obviously, \( D^0_A = D_s \).

Caffarelli and Charro proved in [1, Theorem 3.1] that the operator \( D_s \) becomes uniformly elliptic provided that \( D_s w \) is bounded below away from zero and \( w \) is globally Lipschitz and semiconcave.
The next statement, which follows by closely examining the proof of [1, Theorem 3.1], will be crucial to proving our results.

**Theorem 2.7.** Let $\eta_0$, $L$, and $C$ be positive constants, and fix an open set $\mathcal{O} \subset \mathbb{R}^n$. There exists $\lambda = \lambda(n,s,\eta_0, L, C) > 0$ such that for any Lipschitz and semiconcave function $w$ with constants $L$ and $C$, respectively, if $0 < \varepsilon < \lambda$ and

$$D_s^\varepsilon w \geq \eta_0 > 0 \quad \text{in } \mathcal{O}$$

in the viscosity sense, then

$$D_s^\varepsilon w(x) = D_s^\lambda w(x) \quad \text{for every } x \in \mathcal{O}$$

in the classical sense.

**Proof.** The proof follows by precisely tracking the constants in the proof of Theorem 3.1 in [1]. Following their notation, fix

$$\varepsilon = \left( \frac{\eta_0}{2C_1} \right)^{\frac{n-1}{2s}} \quad \text{and} \quad 0 < \theta < \left( \frac{\mu_0}{\mu_1} \right)^{\frac{n-1}{2s}}$$

(this value of $\varepsilon$ is not to be confused with $0 \leq \varepsilon < \lambda$ in our hypotheses). Then, $\varepsilon$ and $\theta$ depend only on $n, s, \eta_0, L$, and $C$. Choose $\lambda = \min\{\varepsilon, \theta, 1\}$. We notice that if $0 \leq \varepsilon < \lambda$, then we can apply [1, Lemma 3.9] to deduce Proposition 3.3 in [1] with $D_s^\varepsilon$ in place of $D_s$. Thus, the statement of [1, Proposition 3.5], being a simple consequence of Proposition 3.3, holds in our setting. We end our proof exactly as in the proof of Theorem 3.1 in [1, pp. 12-13], in which $\theta = 1/k$. \qed

We close this section with some continuity and stability results. Their proofs are minor modifications of well known facts proved in [13, Propositions 2.4 and 2.6] and [5, Lemma 4.5] by using that $w$ satisfies (2.4) instead of being just bounded.

**Lemma 2.8.** Let $\mathcal{O}$ be an open set, $1/2 < s < 1$, and $w \in L^1_{\text{loc}}(\mathbb{R}^n)$ be such that

$$(2.4) \quad \int_{\mathbb{R}^n} \frac{|w(x)|}{(1 + |x|)^{n+2s}} \, dx < \infty.$$

Suppose that $w \in C^{1,2s+\mu-1}(\mathcal{O})$ for some $\mu > 0$. Then, for any positive definite symmetric matrix $A$ of size $n \times n$, $L^s_A w \in C^\mu(\mathcal{O})$ and

$$[L^s_A w]_{C^\mu(\mathcal{O})} \leq C[\nabla w]_{C^{2s+\mu-1}(\mathcal{O})},$$

where $C > 0$ depends only on $n, s, \mu$ and the largest eigenvalue of $A$. In particular, if $\varepsilon > 0$, then

$$(2.5) \quad \text{the family } \{L^s_A w : A \in \mathcal{M}_\varepsilon \} \text{ is equicontinuous in } \mathcal{O}.$$

Consequently, by taking the infimum over $A \in \mathcal{M}_\varepsilon$ above,

$$D^s_A w \in C(\mathcal{O}).$$

We say that a sequence $w_k \in \text{LSC}(\mathbb{R}^n)$, $k \geq 1$, $\Gamma$-converges to $w$ in a set $\mathcal{O}$ if the following two conditions hold:

- For any sequence $x_k \to x$ in $\mathcal{O}$, $\liminf_{k \to \infty} w_k(x_k) \geq w(x)$.
- For any $x \in \mathcal{O}$, there is a sequence $x_k \to x$ in $\mathcal{O}$ so that $\limsup_{k \to \infty} w_k(x_k) = w(x)$.

**Lemma 2.9.** Let $w_k \in \text{LSC}(\mathbb{R}^n) \cap L^1_{\text{loc}}(\mathbb{R}^n)$ be a sequence of functions such that

$$\int_{\mathbb{R}^n} \frac{|w_k(x)|}{(1 + |x|)^{n+2s}} \, dx \leq C < \infty$$

for all $k \geq 1$. Let $I$ be either $L^s_A$, for $A \in \mathcal{M}$, or $D^s_A$, for $\varepsilon > 0$, for any $1/2 < s < 1$. Suppose that

- $I w_k \leq f_k$ in $\mathcal{O}$;
- $w_k \to w$ in the $\Gamma$ sense in $\mathcal{O}$;
Given any open neighborhood $N$, Thus, $\hat{w}$ touches $w$ for any $k$, and

Moreover, since $\delta(x, y) \geq 0$ for every $x, y \in \mathbb{R}^n$. Hence, $L^s_A \delta \geq 0$ in $\mathbb{R}^n$ for every $A \in \mathcal{M}$, which implies that $\mathcal{D}_s w \geq 0 = \phi - \phi$ in $\mathbb{R}^n$.

Now, define

$$u(x) = \left( \sup \{ w(x) : w \in \mathcal{F} \} \right)^* \text{ for } x \in \mathbb{R}^n.$$ 

By construction,

$$u \in \text{USC}(\mathbb{R}^n), \quad \phi \leq u \leq \psi, \quad \text{and} \quad \lim_{|x| \to \infty} (u - \phi)(x) = 0.$$

In particular,

$$\int_{\mathbb{R}^n} \frac{|u(x)|}{(1 + |x|)^{n+2s}} \, dx < \infty.$$ 

Moreover, since $u - \psi$ is upper semicontinuous in $\mathbb{R}^n$, we have that

the noncoincidence set $\{u < \psi\}$ is open.

First, we will show that $u$, as defined in (3.2), is in the class $\mathcal{F}$, see Lemma 3.4. Before doing so, we need a couple of preliminary results.

**Lemma 3.1.** Let $w_1, w_2 \in \mathcal{F}$. Then,

$$w(x) = \max\{w_1(x), w_2(x)\} \in \mathcal{F}.$$ 

**Proof.** Evidently, $w \in \text{USC}(\mathbb{R}^n)$, $w \leq \psi$, and $\lim_{|x| \to \infty} (w - \psi)(x) \leq 0$. Let $P$ be a $C^2$ function touching $w$ from above at $x_0$ in $\mathcal{N}$. Without loss of generality, $P(x_0) = w(x_0) = w_1(x_0)$, so $P$ also touches $w_1$ from above at $x_0$ in $\mathcal{N}$. Let

$$\vartheta(x) = \begin{cases} P(x) & \text{ if } x \in \mathcal{N} \\ w(x) & \text{ if } x \in \mathbb{R}^n \setminus \mathcal{N} \end{cases} \quad \text{ and } \quad \vartheta_1(x) = \begin{cases} P(x) & \text{ if } x \in \mathcal{N} \\ w_1(x) & \text{ if } x \in \mathbb{R}^n \setminus \mathcal{N}. \end{cases}$$

Observe that $\vartheta(x_0) = \vartheta_1(x_0)$ and $\vartheta \geq \vartheta_1$ in $\mathbb{R}^n$, from which it follows that $\delta(\vartheta, x_0, y) \geq \delta(\vartheta_1, x_0, y)$ for any $y \in \mathbb{R}^n$. Therefore, given any matrix $A \in \mathcal{M},$

$$L^s_A \vartheta(x_0) \geq L^s_A \vartheta_1(x_0) \geq \mathcal{D}_s \vartheta_1(x_0) \geq \vartheta_1(x_0) - \phi(x_0) = \vartheta(x_0) - \phi(x_0).$$

Thus, $\mathcal{D}_s w \geq \vartheta - \phi$ in $\mathbb{R}^n$. \hfill $\Box$

**Lemma 3.2.** Let $u$ be as in (3.2) and let $P$ be a $C^2$ function touching $u$ from above at $x_0$ in $\mathcal{N}$. Given any open neighborhood $\mathcal{N}' \subset \subset \mathcal{N}$ that contains $x_0$, there exist functions $u_k \in \mathcal{F}$, points $x_k \in \mathcal{N}'$, and constants $d_k > 0$, for $k \geq 1$, such that

$$u_k \leq u_{k+1}, \quad x_k \to x_0, \quad d_k \searrow 0, \quad u_k(x_k) \to u(x_0),$$

and

$$P_k(y) = P(y) + \frac{|y - x_k|^2}{k} - d_k \quad \text{ touches } u_k \text{ from above at } x_k \text{ in } \mathcal{N}'.$$
Proof. Fix \( x_0 \in \mathbb{R}^n \). The proof is divided into two steps.

- **Step 1.** There exist points \( y_k \) and functions \( u_k \in \mathcal{F} \) with \( u_k \leq u_{k+1} \) such that 
  \[ y_k \to x_0 \quad \text{and} \quad u_k(y_k) \to u(x_0). \]

Indeed, by Remark 2.1, there exists a sequence of points \( y_k \) such that 
  \[ y_k \to x_0 \quad \text{and} \quad w(y_k) \equiv \sup_{w \in \mathcal{F}} w(y_k) \to u(x_0). \]

Let \( k \geq 1 \). There is a sequence \( \{w_{k,j}\}^\infty_{j=1} \in \mathcal{F} \) such that 
  \[ w_{k,j}(y_k) \nearrow w(y_k) \quad \text{as} \quad j \to \infty. \]

In particular, there exists \( J(k) > 0 \) such that 
  \[ 0 \leq w(y_k) - w_{k,j}(y_k) < 1/k \quad \text{for every} \quad j \geq J(k). \]

Without loss of generality we can let \( J(k) < J(k+1) \), for every \( k \geq 1 \). Define 
  \[ u_k(y) = \max \{w_{1,j(1)}(y), \ldots, w_{k,j(k)}(y)\} \quad \text{for} \quad y \in \mathbb{R}^n. \]

Then, \( u_k \leq u_{k+1} \) and, by Lemma 3.1, \( u_k \in \mathcal{F} \) for every \( k \geq 1 \). Finally, observe that, by the definition of \( u_k \), (3.6), (3.5), and (3.7), as \( k \to \infty, \)
  \[ |u(x_0) - u_k(y_k)| \leq |u(x_0) - w(y_k)| + (w(y_k) - u_k(y_k)) \]
  \[ \leq |u(x_0) - w(y_k)| + (w(y_k) - w_{k,j}(y_k)) \]

- **Step 2.** Let \( \mathcal{N}' \subset \subset \mathcal{N} \) be any open neighborhood of \( x_0 \). Without loss of generality, we can assume that the sequence \( y_k \) from Step 1 satisfies \( y_k \in \mathcal{N}' \) for all \( k \geq 1 \). Define 
  \[ d_k = \inf_{\mathcal{N}'} (P - u_k). \]

Notice that \( d_k \geq 0 \) is well defined because \( P - u_k \) is lower semicontinuous in \( \mathbb{R}^n \). Moreover, \( d_k \geq d_{k+1} \) as \( u_k \leq u_{k+1} \leq u \leq P \) in \( \mathcal{N} \). Also, 
  \[ 0 \leq d_k \leq P(y_k) - u_k(y_k) \to P(x_0) - u(x_0) = 0. \]

Let \( x_k \in \mathcal{N}' \) be such that 
  \[ P(x_k) - u_k(x_k) = d_k. \]

The set of points \( \{x_k\}^\infty_{k=1} \) is bounded, so, after passing to a subsequence, we can assume that 
  \( \{x_k\}^\infty_{k=1} \) is convergent in \( \mathcal{N}' \).

Let us show that \( x_k \to x_0 \). Suppose, to the contrary, that there exists a subsequence \( \{x_{k_j}\}^\infty_{j=1} \) of 
  \( \{x_k\}^\infty_{k=1} \) such that \( x_{k_j} \to x' \in \mathcal{N}' \), as \( j \to \infty \), with \( x' \neq x_0 \). Then, as \( P > u \) in \( \mathcal{N}' \setminus \{x_0\} \) and \( P - u \) is lower semicontinuous, 
  \[ 0 < P(x') - u(x') \leq \liminf_{j \to \infty} (P - u)(x_{k_j}) \leq \lim_{j \to \infty} d_{k_j} = 0, \]

which is a contradiction. Hence, \( x_k \to x_0 \), as desired.

This and (3.8) imply that \( u_k(x_k) \to u(x_0) \). By construction, \( P(x_k) - d_k = u_k(x_k) \) and \( P - u \) is lower semicontinuous, 
  \[ 0 < P(x') - u(x') \leq \liminf_{j \to \infty} (P - u)(x_{k_j}) \leq \lim_{j \to \infty} d_{k_j} = 0, \]

which is a contradiction. Hence, \( x_k \to x_0 \), as desired.

This and (3.8) imply that \( u_k(x_k) \to u(x_0) \). By construction, \( P(x_k) - d_k = u_k(x_k) \) and \( P - d_k \geq u_k \) in \( \mathcal{N}' \). So, \( P_k(y) \) as defined in (3.4) is a \( C^2 \) function that touches \( u_k \) from above at \( x_k \) in \( \mathcal{N}' \). \( \Box \)

**Remark 3.3.** In Lemma 3.2, we can modify the definition of \( P_k(y) \) in (3.4). Indeed, as the proof above shows, any function of the form 
  \[ P(y) + \varphi(y) - d_k, \]

where \( \varphi \) is a \( C^2 \) function such that \( \varphi(x_k) = 0 \) and \( \varphi(y) > 0 \) for all \( y \in \mathcal{N}' \setminus \{x_k\} \), will touch \( u_k \) from above at \( x_k \) in \( \mathcal{N}' \).
Lemma 3.4. Let $u$ be as in (3.2). Then,

$$D_s u \geq u - \phi \quad \text{in } \mathbb{R}^n.$$ 

In particular,

$$u \leq \bar{u},$$

where $\bar{u}$ is the solution to (1.3).

Proof. Let $P$ be a $C^2$ function touching $u$ from above at $x_0$ in $\mathcal{N}$. By Lemma 3.2, there exist functions $u_k \in \mathcal{F}$, points $x_k \in B_r(x_0) \subset \mathcal{N}$ for some $r > 0$, and constants $d_k > 0$ such that $u_k \leq u_{k+1} \leq u$, $d_k \to 0$, $u_k(x_k) \to u(x_0)$, and $P_k(y) = P(y) + \frac{1}{k} |y - x_k|^2 - d_k$ touches $u_k$ from above at $x_k$ in $B_r(x_0)$ for $k \geq 1$. Define the test functions

$$\vartheta(x) = \begin{cases} P(x) & \text{if } x \in B_r(x_0) \\ u(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0) \end{cases} \quad \text{and} \quad \vartheta_k(x) = \begin{cases} P_k(x) & \text{if } x \in B_r(x_0) \\ u_k(x) & \text{if } x \in \mathbb{R}^n \setminus B_r(x_0). \end{cases}$$

We recall that, by Remark 2.4, it is enough to use $\vartheta$ as defined above as a test function for $u$. Let $A \in \mathcal{M}$ and let $\Lambda_A$ denote the maximum eigenvalue of $A$. Then,

$$c_{n,s}^{-1} L_A^s \vartheta_k(x_k) = \lim_{\rho \to 0} \int_{B_r(x_0) \setminus B_{\rho}(x_0)} \frac{\vartheta_k(y) - \vartheta_k(x_k)}{A^{-1}(y - x_k)^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{\vartheta_k(y) - \vartheta_k(x_k)}{A^{-1}(y - x_k)^{n+2s}} \, dy$$

$$\leq \lim_{\rho \to 0} \left[ \int_{B_r(x_0) \setminus B_{\rho}(x_0)} \frac{P(y) - P(x_k)}{A^{-1}(y - x_k)^{n+2s}} \, dy + \frac{1}{k} \int_{B_r(x_0) \setminus B_{\rho}(x_0)} \frac{|y - x_k|^2}{A^{-1}(y - x_k)^{n+2s}} \, dy \right]$$

$$+ \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{u(y) - P(x_k)}{A^{-1}(y - x_k)^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{d_k}{A^{-1}(y - x_k)^{n+2s}} \, dy$$

$$\leq c_{n,s}^{-1} L_A^s \vartheta(x_k) + \Lambda_A^{s+2s} \left[ \frac{1}{k} \int_{B_r(x_0)} \frac{1}{|y - x_k|^{n+2s-2}} \, dy + \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{d_k}{|y - x_k|^{n+2s}} \, dy \right]$$

$$\leq c_{n,s}^{-1} L_A^s \vartheta(x_k) + C(k^{-1} + d_k),$$

where $C = C(n, s, \Lambda_A, r) > 0$ is independent of $k$. Here, we have used that

$$\int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{1}{|y - x_k|^{n+2s-2}} \, dy \to \int_{\mathbb{R}^n \setminus B_r(x_0)} \frac{1}{|y - x_0|^{n+2s-2}} \, dy$$

and

$$\int_{B_r(x_0)} \frac{1}{|y - x_k|^{n+2s-2}} \, dy \to \int_{B_r(x_0)} \frac{1}{|y - x_0|^{n+2s-2}} \, dy,$$

as $k \to \infty$. Hence, as $u_k$ is a subsolution,

$$\vartheta_k(x_k) - \phi(x_k) \leq D_s \vartheta_k(x_k) \leq L_A^s \vartheta(x_k) + C(k^{-1} + d_k).$$

Notice that $\vartheta_k(x_k) = P_k(x_k) \to P(x_0) = \vartheta(0)$ as $k \to \infty$. Together with Lemma 2.8, this implies that

$$\vartheta(x_0) - \phi(x_0) \leq L_A^s \vartheta(x_0).$$

Since $A \in \mathcal{M}$ was arbitrary, we obtain $\vartheta(x_0) - \phi(x_0) \leq D_s \vartheta(x_0)$, which means that $u$ is a subsolution to $D_s u \geq u - \phi$.

We have already seen that $\lim_{|x| \to \infty} (u - \phi)(x) = 0$. Thus, the comparison principle (Theorem 2.6) implies that $u \leq \bar{u}$. 

With Lemma 3.4 in hand, we can prove that the contact set $\{u = \psi\}$ is compact and that $u$ is Lipschitz and semiconcave with constants no larger than those of $\phi$ and $\psi$. 

\[\square\]
Lemma 3.5. Let $u$ be as in (3.2). Then,
\[
\{ u = \psi \} \text{ is compact.}
\]

Proof. We know that $u \leq \psi$ and that the noncoincidence set $\{ u < \psi \}$ is open, see (3.3). Therefore, the contact set $\{ u = \psi \}$ is closed. On the other hand, by Lemma 3.4, $\bar{u} < \psi \subset \{ u < \psi \}$, which implies that $\{ u = \psi \} \subset K$. Hence, the contact set is compact. \(\square\)

Recall the definition of $M_1$ and $M_2$ from the statement of Theorem 1.1.

Lemma 3.6. Let $u$ be as in (3.2). Then, $u$ is Lipschitz continuous and semiconcave with
\[
[u]_{\text{Lip}(\mathbb{R}^n)} \leq M_1 \quad \text{and} \quad \text{SC}(u) \leq M_2.
\]

Proof. Given any $h \in \mathbb{R}^n$, let us first show that
\[
(3.9) \quad w(x) = u(x + h) - M_1|h| \in \mathcal{F}.
\]

Indeed, $w \in \text{USC}(\mathbb{R}^n)$,
\[
\lim_{|x| \to \infty} (w - \phi)(x) = \lim_{|x| \to \infty} \left[ (u(x + h) - u(x)) + (u(x) - \phi(x)) \right] - M_1|h| \leq 0,
\]
and since $-M_1|h| \leq \psi(x) - \psi(x + h)$ and $u \leq \psi$,
\[
w(x) = u(x + h) - M_1|h| \leq u(x + h) - \psi(x + h) + \psi(x) \leq \psi(x).
\]

Finally, as $\mathcal{D}_s$ is translation invariant, $\mathcal{D}_sc = 0$ for any constant $c$, and $\phi(x + h) - \phi(x) \leq M_1|h|$, we find that
\[
\mathcal{D}_s w(x) = \mathcal{D}_s(\tau_h u)(x) = (\mathcal{D}_s u)(x + h) \geq u(x + h) - \phi(x + h)
\]
\[
= (u(x + h) - M_1|h|) - \phi(x + h) + M_1|h|
\]
\[
\geq w(x) - \phi(x),
\]
in the viscosity sense. Thus, (3.9) is proved. Now, by the maximality of $u$ in $\mathcal{F}$, $w \leq u$, which means that
\[
u(x + h) - u(x) \leq M_1|h|.
\]

Since $x$ and $h$ above are arbitrary, we conclude that $[u]_{\text{Lip}(\mathbb{R}^n)} \leq M_1$.

Given any $h \in \mathbb{R}^n$, let us first see that
\[
(3.10) \quad w(x) = \frac{u(x + h) + u(x - h) - M_2|h|^2}{2} \in \mathcal{F}.
\]

Indeed, $w \in \text{USC}$, and since $\delta(\phi, x, h) \leq M_2|h|^2$,
\[
(w - \phi)(x) = \frac{u(x + h) + u(x - h)}{2} - \frac{M_2|h|^2}{2} - \phi(x)
\]
\[
= \frac{(u - \phi)(x + h) + (u - \phi)(x - h)}{2} + \frac{\delta(\phi, x, h) - M_2|h|^2}{2}
\]
\[
\leq \frac{(u - \phi)(x + h) + (u - \phi)(x - h)}{2} \to 0
\]
as $|x| \to \infty$. Also, $u \leq \psi$ and $\delta(\psi, x, h) \leq M_2|h|^2$, which implies that
\[
w(x) = \frac{u(x + h) + u(x - h)}{2} - \frac{M_2|h|^2}{2} \leq \frac{\psi(x + h) + \psi(x - h)}{2} - \frac{M_2|h|^2}{2} \leq \psi(x).
\]

Finally, using the inequality $M_2|h|^2 - \delta(\phi, x, h) \geq 0$,
\[
\mathcal{D}_s w(x) \geq \frac{1}{2} \mathcal{D}_s(\tau_h u)(x) + \frac{1}{2} \mathcal{D}_s(\tau_{-h} u)(x)
\]
\[
= \frac{1}{2} \mathcal{D}_s u(x + h) + \frac{1}{2} \mathcal{D}_s u(x - h)
\]
Moreover, for any \( u \) in \( \mathcal{F} \), we have that \( w \leq u \). Hence,

\[
\begin{align*}
\frac{u(x + h) + u(x - h) - 2u(x)}{2} & \leq w(x) + M_2|h|^2 - \delta(\phi, x, h) - \phi(x) \geq w(x) - \phi(x),
\end{align*}
\]

in the viscosity sense. Thus, (3.10) is proved. By the maximality of \( u \) in \( \mathcal{F} \), we have that \( w \leq u \). Hence,

\[
0 \leq \mathcal{D}_s u(x) \leq C \left( 1 + \|u - \phi\|_{L^\infty(\mathbb{R}^n)} \right)
\]

for every \( x \in \mathbb{R}^n \), for some constant \( C = C(n, s, a, \epsilon, M_2) > 0 \).

**Proof.** As \( u \) is semiconcave on \( \mathbb{R}^n \) (see Lemma 3.6), it can be touched from above by a \( C^2 \) function at every point \( x \in \mathbb{R}^n \). Thus, Lemmas 3.4 and 2.5 imply that \( \mathcal{D}_s u(x) \) can be computed classically and \( \mathcal{D}_s u(x) \geq u(x) - \phi(x) \geq 0 \) for every \( x \in \mathbb{R}^n \). Since, for any \( x \in \mathbb{R}^n \), we have \( \delta(u, x, y)/|y|^{n+2s} \in L^1(\mathbb{R}^n) \) and \( \delta(\phi, x, y) \geq 0 \), we can estimate

\[
\begin{align*}
\mathcal{D}_s u(x) & \leq -(-\Delta)^s u(x) \\
& = c_{n,s} \int_{B_1} \frac{\delta(u, x, y)}{|y|^{n+2s}} dy + c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \frac{\delta(u, x, y)}{|y|^{n+2s}} dy \\
& \leq c_{n,s} \int_{B_1} \mathcal{SC}(u)|y|^2 dy + c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \frac{\delta(u - \phi, x, y)}{|y|^{n+2s}} dy + c_{n,s} \int_{\mathbb{R}^n \setminus B_1} \frac{\delta(\phi, x, y)}{|y|^{n+2s}} dy \\
& \leq C(M_2 + \|u - \phi\|_{L^\infty(\mathbb{R}^n)} - (-\Delta)^s \phi(x)) \\
& \leq C(1 + \|u - \phi\|_{L^\infty(\mathbb{R}^n)}).
\end{align*}
\]

In the last inequality, we have used that

\[
(3.11) \quad 0 \leq -(-\Delta)^s \phi \leq M_0 \quad \text{in } \mathbb{R}^n,
\]

for some constant \( M_0 = M_0(n, s, a, \epsilon) > 0 \), see [1, eq. (6.8)].

Before we can conclude the proof of Theorem 1.1, we need to consider the obstacle problem (1.4) for the truncated fractional Monge–Ampère operator defined in subsection 2.3.

**Theorem 3.8.** For any \( \epsilon > 0 \), there exists a unique classical solution \( u_\epsilon \) to the obstacle problem

\[
\begin{align*}
\mathcal{D}_s u_\epsilon & \geq u_\epsilon - \phi \quad \text{in } \mathbb{R}^n \\
u_\epsilon & \leq \psi \quad \text{in } \mathbb{R}^n \\
\mathcal{D}_s u_\epsilon & = u_\epsilon - \phi \quad \text{in } \{u_\epsilon < \psi\} \\
\lim_{|x| \to \infty} (u_\epsilon - \phi)(x) & = 0.
\end{align*}
\]

Moreover, \( u_\epsilon \) is Lipschitz and semiconcave with constants no larger than \( M_1 \) and \( M_2 \), respectively.

**Proof.** Fix \( \epsilon > 0 \). Parallel to (3.1), we define the class

\[
\mathcal{F}_\epsilon = \left\{ w \in \text{USC}(\mathbb{R}^n) : \mathcal{D}_s w \geq w - \phi \text{ in } \mathbb{R}^n, w \leq \psi, \text{ and } \lim_{|x| \to \infty} (w - \phi)(x) \leq 0 \right\}.
\]
Then, \( \phi \in \mathcal{F}_\varepsilon \). By replacing \( u \) by \( u_\varepsilon \) and \( D_s \) by \( D_s^\varepsilon \) in the arguments of Lemmas 3.1–3.4 and Lemma 3.6, we deduce that

\[
u_\varepsilon(x) = \left( \sup \{ w(x) : w \in \mathcal{F}_\varepsilon \} \right)^* \quad \text{for } x \in \mathbb{R}^n
\]

is the largest function in \( \mathcal{F}_\varepsilon \), \( \phi \leq u_\varepsilon \leq \psi \), \( \lim_{|x| \to \infty} (u_\varepsilon - \phi)(x) = 0 \),

\[
(3.12) \quad [u_\varepsilon]_{\text{Lip}(\mathbb{R}^n)} \leq M_1 \quad \text{and} \quad \text{SC}(u_\varepsilon) \leq M_2,
\]

and the noncoincidence set \( \{ u_\varepsilon < \psi \} \) is open.

It remains to prove that

\[
D_s^\varepsilon u_\varepsilon = u_\varepsilon - \phi \quad \text{in } \{ u_\varepsilon < \psi \}.
\]

We argue by contradiction. Specifically, we will show that if \( D_s^\varepsilon u_\varepsilon = u_\varepsilon - \phi \) fails in the open set \( \{ u_\varepsilon < \psi \} \), then \( u_\varepsilon \) is not maximal in \( \mathcal{F}_\varepsilon \). To this end, let \( x_0 \in \{ u_\varepsilon < \psi \} \) and \( P \) be a \( C^2 \) function touching \( u_\varepsilon \) from below at \( x_0 \) in \( \mathcal{N} \) such that for

\[
\vartheta(x) = \begin{cases} P(x) & \text{for } x \in \mathcal{N} \\ u_\varepsilon(x) & \text{for } x \in \mathbb{R}^n \setminus \mathcal{N}, \end{cases}
\]

we have

\[
D_s^\varepsilon \vartheta(x_0) > \vartheta(x_0) - \phi(x_0).
\]

Recall that \( D_s^\varepsilon \vartheta \) is continuous in \( \mathcal{N} \) (see Lemma 2.8). Hence, given

\[
0 < \tau < D_s^\varepsilon \vartheta(x_0) - (\vartheta(x_0) - \phi(x_0)),
\]

there exists a ball \( B_r(x_0) \subset \subset \mathcal{N} \cap \{ u_\varepsilon < \psi \} \) such that

\[
(3.13) \quad D_s^\varepsilon \vartheta(z) \geq \vartheta(z) - \phi(z) + \tau \quad \text{for every } z \in B_r(x_0).
\]

Next, we lift \( P \) in \( \mathcal{N} \) by a small amount \( d > 0 \) (to be fixed) so that \( \{ u_\varepsilon < P + d \} \subset \subset B_r(x_0) \) and \( \{ P + d < \psi \} \subset \{ u_\varepsilon < P + d \} \). We then set

\[
u'_\varepsilon(x) = \begin{cases} P(x) + d & \text{if } x \in \{ u_\varepsilon < P + d \} \\ u_\varepsilon(x) & \text{otherwise}, \end{cases}
\]

and notice that \( u'_\varepsilon \) is continuous, \( u'_\varepsilon \geq u_\varepsilon \) in \( \mathbb{R}^n \), and \( \lim_{|x| \to \infty} (u'_\varepsilon - \phi)(x) = 0 \). If we can show that \( u'_\varepsilon \) is a subsolution to \( D_s^\varepsilon w = w - \phi \), then \( u_\varepsilon \) is not maximal in \( \mathcal{F}_\varepsilon \) because we constructed \( u'_\varepsilon \) in such a way that

\[
u'_\varepsilon(x_0) = P(x_0) + d > P(x_0) = u_\varepsilon(x_0).
\]

Therefore, we now prove that

\[
(3.14) \quad u'_\varepsilon \text{ is a subsolution to } D_s^\varepsilon w = w - \phi \text{ in } \mathbb{R}^n.
\]

Let \( P' \) be a \( C^2 \) function touching \( u'_\varepsilon \) from above at \( x' \) in \( \mathcal{N}' \). We have two cases to consider.

- **Case 1.** \( x' \in \{ u'_\varepsilon = u_\varepsilon \} \). Since \( \delta(u'_\varepsilon, x', y) \geq \delta(u_\varepsilon, x', y) \), by Lemma 2.5 (which is also valid for the uniformly elliptic case), we see that

\[
D_s^\varepsilon u'_\varepsilon(x') \geq D_s^\varepsilon u_\varepsilon(x') \geq u_\varepsilon(x') - \phi(x') = u'_\varepsilon(x') - \phi(x'),
\]

and (3.14) follows.

- **Case 2.** \( x' \in \{ u'_\varepsilon > u_\varepsilon \} \). Define

\[
\vartheta'(x) = \begin{cases} P'(x) & \text{if } x \in \mathcal{N}' \\ u'_\varepsilon(x) & \text{if } x \in \mathbb{R}^n \setminus \mathcal{N}'. \end{cases}
\]
Remark 2.4 allows us to assume that $N' \subset \{ u'_\varepsilon > u_\varepsilon \} = \{ u_\varepsilon < P + d \} \subset B_r(x_0)$. Observe that $P' - d \geq u'_\varepsilon - d = P$ in $N'$ and $P'(x') - d = P(x')$. Then,

$$c_{n,s}^{-1} L_A^s \vartheta'(x') = \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus \mathbb{B}_\rho(x')} \frac{P(y) - u'_\varepsilon(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus \mathbb{N}'} \frac{u'_\varepsilon(y) - u'_s(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy$$

$$\geq \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus \mathbb{B}_\rho(x')} \frac{P(y) - P(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy + I$$

(3.15)

$$\geq \lim_{\rho \to 0} \int_{\mathbb{R}^n \setminus \mathbb{B}_\rho(x')} \frac{P(y) - P(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy + I$$

We estimate the integral $I$ from below. Since $u'_\varepsilon \geq u_\varepsilon$ in $\mathbb{R}^n$, $u'_s(x') = P(x') = P(x') + d$, and $u'_\varepsilon \geq P + d$ in $N \setminus N'$, we find that

$$I = \int_{\mathbb{R}^n \setminus \mathbb{N}'} \frac{u'_\varepsilon(y) - u'_s(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy + \int_{\mathbb{N}'} \frac{u'_\varepsilon(y) - u'_s(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy$$

$$\geq \int_{\mathbb{R}^n \setminus \mathbb{N}'} \frac{u_\varepsilon(y) - P(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy - \int_{\mathbb{R}^n \setminus \mathbb{N}'} \frac{d}{|A^{-1}(y - x')|^{n+2s}} \, dy$$

(3.16)

$$+ \int_{\mathbb{N}'} \frac{P(y) - P(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy.$$

From (3.15), (3.16) and the definition of $\vartheta$, we get

$$c_{n,s}^{-1} L_A^s \vartheta'(x') \geq \lim_{\rho \to 0} \int_{\mathbb{N} \setminus \mathbb{B}_\rho(x')} \frac{P(y) - P(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus \mathbb{N}'} \frac{u_\varepsilon(y) - P(x')}{|A^{-1}(y - x')|^{n+2s}} \, dy$$

$$- \frac{d}{|A^{-1}(y - x')|^{n+2s}} \, dy$$

$$= c_{n,s}^{-1} L_A^s \vartheta(x') - C_d,$$

where $C = C(n, s, \varepsilon, N) > 0$ is independent of $A$. By taking the infimum over all $A \in \mathcal{M}_\varepsilon$ above and using (3.13), we deduce that

$$D^s \vartheta'(x') \geq D^s \vartheta(x') - C_d \geq \vartheta(x') - \vartheta(x') + \tau - C_d = \vartheta(x') - \vartheta(x') + \tau - (C + 1)d.$$

Thus, by choosing $d > 0$ sufficiently small, it follows that

$$D^s \vartheta'(x') \geq \vartheta(x') - \vartheta(x').$$

This completes the proof of (3.14) and the theorem. 

Let $u_\varepsilon$ and $f_\varepsilon$ be as in Theorem 3.8 and its proof. First, notice that if $\varepsilon_0 \geq \varepsilon$, then $D^s u_\varepsilon \geq D^s u_\varepsilon$. In particular, $u_\varepsilon \in f_{\varepsilon_0}$. Hence, by the maximality of $u_{\varepsilon_0}$ in $f_{\varepsilon_0}$, we have that $u_{\varepsilon_0} \geq u_\varepsilon$. In other words, the sequence of functions $u_\varepsilon$ is decreasing as $\varepsilon \searrow 0$. Let

$$u_0(x) = \inf_{\varepsilon > 0} u_\varepsilon(x) \quad \text{for } x \in \mathbb{R}^n.$$ 

(3.17)

Then, $u_0$ is well-defined because $\phi \leq u_\varepsilon \leq \psi$ for every $\varepsilon > 0$. Clearly,

$$\phi \leq u_0 \leq \psi \quad \text{and} \quad \lim_{|x| \to \infty} (u_0 - \phi)(x) = 0.$$

Moreover, (3.12) and Arzelà–Ascoli’s theorem imply that $u_0$ is the local uniform (decreasing) limit of $u_\varepsilon$ and that $u_0$ is Lipschitz continuous with $[u_0]_{\text{Lip}(\mathbb{R}^n)} \leq M_1$.

**Lemma 3.9.** Let $u_0$ be as in (3.17). Then

$$u_0 > \phi.$$
Proof. Let us argue by contradiction. Suppose that there is a point $x_0$ such that $u_0(x_0) = \phi(x_0)$. Then, as $\phi < \psi$ in $\mathbb{R}^n$, we have $x_0 \in \{ u_0 < \psi \}$. Since $\phi \in C^{2,\sigma}(\mathbb{R}^n)$ is strictly convex in compact sets and asymptotically close to a cone at infinity, we can find a function $\varphi \in C^{2,\sigma}(\mathbb{R}^n)$ that is also strictly convex in compact sets, asymptotically close to a cone at infinity, and touches both $u_0$ and $\phi$ from below in $B_r(x_0)$ at $x_0$ for some $r > 0$. As $u_0$ is the local uniform limit of $u_\varepsilon$, there exist points $x_\varepsilon \in B_r(x_0)$ such that $x_\varepsilon \to x_0$ and $u_\varepsilon$ can be touched from below at $x_\varepsilon$ in $B_r(x_0)$ by

$$
\varphi_\varepsilon(x) = \varphi(x) - \varepsilon \omega(|x - x_\varepsilon|) + d_\varepsilon \quad \text{for } x \in \mathbb{R}^n.
$$

Here, $d_\varepsilon \searrow_0 0$ and $\omega = \omega(t)$ is convex, strictly increasing in $(0, r)$, smooth in $(0, r)$, linear in $\mathbb{R} \setminus [0, r)$, $\omega(0^+) = \omega'(0^+) = 0$, and such that $\varphi_\varepsilon$ is strictly convex in compact sets. Because $\varphi_\varepsilon$ is convex and touches the supersolution $u_\varepsilon$ from below at $x_\varepsilon$,

$$
0 \leq D_s \varphi_\varepsilon(x_\varepsilon) \leq D_s^\varepsilon \varphi_\varepsilon(x_\varepsilon) \leq \varphi_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) = u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon).
$$

As the sets $\{ u_\varepsilon < \psi \}$ are increasing and $x_\varepsilon \to x_0$, we have $x_\varepsilon \in \{ u_\varepsilon < \psi \}$ for all $\varepsilon > 0$ sufficiently small. Moreover,

$$
u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon) \to 0 \quad \text{as } \varepsilon \searrow 0.
$$

This and (3.19) imply that

$$
0 \leq D_s \varphi_\varepsilon(x_\varepsilon) \to 0 \quad \text{as } \varepsilon \searrow 0.
$$

Using this last inequality, we will prove that there exists a direction $e_0 \in S^{n-1}$ such that

$$
(-\Delta)^s_{e_0} \varphi(x_0) = \frac{c_{n,s}}{2} \int_\mathbb{R} \frac{\delta(\varphi, x_0, te_0)}{|t|^{1+2s}} \, dt \leq 0.
$$

This clearly contradicts the convexity of the nonconstant function $\varphi$. In turn, $u_0 > \phi$, as desired.

To deduce (3.21), suppose, to the contrary, that

$$
(-\Delta)^s_{e_0} \varphi(x_\varepsilon) \geq \mu > 0 \quad \text{for all } e \in S^{n-1}.
$$

Since the family $\{ -(-\Delta)^s_{e_0} \varphi \}_{e \in S^{n-1}}$ is equicontinuous (see (2.5)),

$$
\inf_{e \in S^{n-1}} \{ -(-\Delta)^s_{e_0} \varphi(x_\varepsilon) \} \geq \frac{\mu}{2} > 0 \quad \text{for all } \varepsilon \text{ sufficiently small}.
$$

The function $\omega \equiv \omega(|\cdot|)$ in (3.18) is radially symmetric and convex. Hence, $(-\Delta)^s_{e_0} \varphi(0)$ is a negative constant independent of $e \in S^{n-1}$. Therefore, we can ensure that $\varepsilon(-\Delta)^s_{e_0} \varphi(0) \geq -\mu/4$ provided $\varepsilon$ is sufficiently small, independently of the direction $e \in S^{n-1}$. Collecting these last two facts and (3.18), we deduce that

$$
(-\Delta)^s_{e_0} \varphi_\varepsilon(x_\varepsilon) = (-\Delta)^s_{e_0} \varphi(x_\varepsilon) + \varepsilon(-\Delta)^s_{e_0} \varphi \geq \frac{\mu}{4} > 0,
$$

uniformly in $e \in S^{n-1}$ and for all $\varepsilon$ sufficiently small. It is show in the proof of [1, Proposition 3.5] that an estimate of the form (3.23) readily yields the existence of a positive constant $\tau = \tau(n, s, \mu, [\varphi]_{\text{Lip}(\mathbb{R}^n)}, \text{SC}(\varphi))$ such that $D_\varepsilon \varphi_\varepsilon(x_\varepsilon) \geq \tau$. This estimate is uniform in $\varepsilon$, a contradiction to (3.20). Thus, (3.22) cannot hold. In other words, there are directions $e_k \in S^{n-1}$ such that

$$
(-\Delta)^s_{e_k} \varphi(x_0) = \frac{c_{n,s}}{2} \int_\mathbb{R} \frac{\delta(\varphi, x_0, te_k)}{|t|^{1+2s}} \, dt \leq \frac{1}{k},
$$

for each $k \geq 1$. The compactness of $S^{n-1}$ allows us to assume, without loss of generality, that $e_k \to e_0$ for some $e_0 \in S^{n-1}$, as $k \to \infty$. The continuity of $\varphi$ gives

$$
\frac{\delta(\varphi, x_0, te_k)}{|t|^{1+2s}} \to \frac{\delta(\varphi, x_0, te_0)}{|t|^{1+2s}},
$$

as $k \to \infty$. Then if $\lim_{k \to \infty} \delta(\varphi, x_0, te_k) = 0$, we have

$$
\frac{\delta(\varphi, x_0, te_0)}{|t|^{1+2s}} = 0,
$$

and in particular $\delta(\varphi, x_0, te_0) = 0$. Therefore, $\lim_{k \to \infty} \delta(\varphi, x_0, te_k) = 0$. This contradiction shows that $\lim_{k \to \infty} \delta(\varphi, x_0, te_k)$ does not exist.
as $k \to \infty$. Since $\varphi$ is convex and has linear growth at infinity,

$$0 \leq \frac{\delta(\varphi, x_0, t e_k)}{|t|^{1+2s}} \leq \min\{2[\varphi]_{\text{Lip}(\mathbb{R}^n)}|t|, |\text{SC}(\varphi)|||t||^2\} \in L^1(\mathbb{R}),$$

uniformly in $k \geq 1$. Thus, we can apply the dominated convergence theorem to (3.24) to get

$$-(-\Delta)_{\partial\varphi}^s \varphi(x_0) = -\lim_{k \to \infty} (-\Delta)_{\partial\varphi}^s \varphi(x_0) \leq 0,$$

as desired. \hfill \Box

**Lemma 3.10.** Let $u_0$ be as in (3.17). Then,

$$\mathcal{D}_s u_0 \leq u_0 - \phi \quad \text{in} \quad \{u_0 < \psi\}.$$

**Proof.** Let $x_0 \in \{u_0 < \psi\}$ be a point at which $u_0$ can be touched from below by a $C^2$ function in a neighborhood $\mathcal{N} \subset \subset \{u_0 < \psi\}$. As $u_\varepsilon$ decreases locally uniformly to $u_0$, we can find a sequence of points $x_\varepsilon \to x_0$ and $C^2$ functions $P_\varepsilon$ that touch $u_\varepsilon$ from below at $x_\varepsilon$ in a common neighborhood $\mathcal{N}' \subset \subset \mathcal{N}$. By Lemma 3.9, we have $u_\varepsilon \geq u_0 > \phi$. Then, by Theorem 2.7, there exists $\lambda > 0$ such that

$$\mathcal{D}_s u_0(x_0) = \mathcal{D}_s^\lambda u_0(x_0) \quad \text{and} \quad \mathcal{D}_s^\varepsilon u_\varepsilon(x_\varepsilon) = \mathcal{D}_s^\lambda u_\varepsilon(x_\varepsilon)$$

for every $\varepsilon$ such that $0 < \varepsilon < \lambda$. Now, since $u_\varepsilon$ is a supersolution in $\{u_\varepsilon < \psi\}$ (see Theorem 3.8) that can be touched from below by a $C^2$ function at $x_\varepsilon$ in $\mathcal{N}'$, we can apply Lemma 2.5. Consequently,

$$\mathcal{D}_s^\lambda u_\varepsilon(x_\varepsilon) = \mathcal{D}_s^\varepsilon u_\varepsilon(x_\varepsilon) \leq u_\varepsilon(x_\varepsilon) - \phi(x_\varepsilon).$$

By Lemma 2.9,

$$\lim_{\varepsilon \to 0} \mathcal{D}_s^\lambda u_\varepsilon(x_\varepsilon) = \mathcal{D}_s^\lambda u_0(x_0).$$

Thus, by letting $\varepsilon \to 0$ in (3.25),

$$\mathcal{D}_s u_0(x_0) = \mathcal{D}_s^\lambda u_0(x_0) \leq u_0(x_0) - \phi(x_0).$$

\hfill \Box

With the following result we can conclude the proof of Theorem 1.1.

**Lemma 3.11.** Let $u$ and $u_0$ be as in (3.2) and (3.17), respectively. Then,

$$u = u_0.$$

**Proof.** Let us show that $u_0 \in \mathcal{F}$. Let $A \in \mathcal{M}$. If $\varepsilon$ is smaller than the minimum eigenvalue of $A$, then $A \in \mathcal{M}_\varepsilon$, and, as such, we have $L_A u_\varepsilon \geq \mathcal{D}_s^\varepsilon u_\varepsilon \geq u_\varepsilon - \phi$ in $\mathbb{R}^n$. Therefore, by Lemma 2.9, we find that $L_A u_0 \geq u_0 - \phi$ in $\mathbb{R}^n$. As $A \in \mathcal{M}$ was arbitrary, it follows that

$$\mathcal{D}_s u_0 \geq u_0 - \phi \quad \text{in} \quad \mathbb{R}^n.$$

Therefore, $u_0 \in \mathcal{F}$ and $u_0 \leq u$. For the opposite inequality, observe that

$$\mathcal{D}_s^\varepsilon u \geq \mathcal{D}_s u \geq u - \phi \quad \text{in} \quad \mathbb{R}^n.$$

Whence, $u \in \mathcal{F}_\varepsilon$ for all $\varepsilon > 0$, and by the maximality of $u_\varepsilon$ in $\mathcal{F}_\varepsilon$, $u \leq u_\varepsilon$ for all $\varepsilon > 0$. Thus, from the definition of $u_0$, we determine that $u \leq u_0$. \hfill \Box

**Proof of Theorem 1.1.** The proof follows from Lemmas 3.4–3.7 and Lemmas 3.9–3.11. \hfill \Box
4. Proof of Theorem 1.2

We first prove Theorem 1.2(1), that is, the local Hölder continuity of $\nabla u$ in the noncoincidence set.

Proof of Theorem 1.2(1). Let $O$ and $O_\delta$ be as in the statement. Since $u > \phi$ in $\mathbb{R}^n$, by Theorem 2.7, there exists $\lambda = \lambda(n, s, \inf_{O_\delta} (u - \phi), M_1, M_2) > 0$ such that

$$D_s u(x) = D^\lambda_s u(x) \quad \text{for all } x \in O_\delta.$$

For any $A \in \mathcal{M}_\lambda$, let

$$b_A(x) = L^s_A \phi(x) - (u - \phi)(x).$$

Since $u$ and $\phi$ are Lipschitz, $\phi \in C^{2,\sigma}(\mathbb{R}^n)$ and satisfies (3.11), and $\lim_{|x| \to \infty} (u - \phi)(x) = 0$, we deduce that

$$\sup_{A \in \mathcal{M}_\lambda} \left\{ \|b_A\|_{L^\infty(O_\delta)} + |b_A|_{\text{Lip}(O_\delta)} \right\} \leq C_0,$$

where $C_0 = C_0(n, s, \lambda, M_0, M_1, \text{SC}(\phi)) > 0$. Let $w = u - \phi$. We have

$$L^s_A w + b_A(x) = L^s_A u - (u - \phi) \quad \text{in } O_\delta.$$

By taking the infimum over all $A \in \mathcal{M}_\lambda$ above, we see that $w$ solves

$$\begin{cases}
\inf_{A \in \mathcal{M}_\lambda} \{ L^s_A w + b_A(x) \} = 0 \quad \text{in } O_\delta \\
w = u - \phi \in L^\infty(\mathbb{R}^n),
\end{cases}$$

with $b_A$ satisfying the uniform estimate (4.1). From Theorem 1.3(b) in [12], the conclusion follows.  

Next we prove Theorem 1.2(2), which establishes that $\nabla u$ is Hölder continuous across the free boundary. Recall that the contact set $\{u = \psi\}$ is compact, see Lemma 3.5. Let $B$ be as in the statement. Set $CB = \{Cx : x \in B\}$ with $C > 0$. By Theorem 2.7, there exists

$$\lambda = \lambda(n, s, \inf_{4B} (u - \phi), M_1, M_2) > 0$$

such that

$$D_s u = D^\lambda_s u \quad \text{in } 4B.$$

Define

$$c_A(x) = (u - \phi)(x) - L^s_A \psi(x).$$

Since $\sup_{A \in \mathcal{M}_\lambda} [L^s_A \psi]_{\text{Lip}(\mathbb{R}^n)} < \infty$ and $u, \phi \in \text{Lip}(\mathbb{R}^n)$, up to dividing by a constant depending on $\lambda$, we can assume that

$$\sup_{A \in \mathcal{M}_\lambda} [c_A]_{\text{Lip}(\mathbb{R}^n)} = 1.$$

We subtract the obstacle and let $v$ be as in (1.8). For any $A \in \mathcal{M}$, we have

$$L^s_A v + c_A(x) = -L^s_A u + (u - \phi),$$

so that

$$\sup_{A \in \mathcal{M}_\lambda} \{ L^s_A v + c_A(x) \} = -D^\lambda_s u + (u - \phi).$$
Therefore, from (4.3) and up to dividing $v$ by a normalizing constant depending on $\lambda$, we get
\begin{align*}
\begin{cases}
v \geq 0 & \text{in } \mathbb{R}^n \\
D^2 v(x) \geq -\text{Id} & \text{for a.e. } x \in \mathbb{R}^n \\
\sup_{A \in \mathcal{M}_\lambda} \{L^s_A v(x) + c_A(x)\} = 0 & \text{in } \{v > 0\} \cap 4\mathcal{B} \\
\sup_{A \in \mathcal{M}_\lambda} \{L^s_A v(x) + c_A(x)\} \leq 0 & \text{in } \mathbb{R}^n \\
|\nabla v(x)| \leq 1 & \text{for a.e. } x \in \mathbb{R}^n.
\end{cases}
\end{align*}
(4.6)

Finally, consider the extremal Pucci operators
$$M^+_\lambda w(x) = \sup_{A \in \mathcal{M}_\lambda} L^s_A w(x) \quad \text{and} \quad M^-_\lambda w(x) = \inf_{A \in \mathcal{M}_\lambda} L^s_A w(x).$$

To prove Theorem 1.2(2), we need the following rescaled version of a regularity result from [2].

**Proposition 4.1** (see [2, Proposition 2.4]). Let $\alpha \in (0, s)$, $1 + s + \alpha < 2$, $K > 0$, and $R \geq 1$. If $w$ satisfies
\begin{align*}
\begin{cases}
w \geq 0 & \text{in } \mathbb{R}^n \\
D^2 w(x) \geq -K \text{Id} & \text{for a.e. } x \in B_{2R} \\
M^+_\lambda (w - w(\cdot - h)) \geq -K|h| & \text{in } \{w > 0\} \cap B_R \\
|\nabla w(x)| \leq K(1 + |x|^{s+\alpha}) & \text{for a.e. } x \in \mathbb{R}^n,
\end{cases}
\end{align*}

then there exist $0 < \tau < 1$ and $C > 0$, depending on $\alpha$ and $\lambda$, such that
$$\|w\|_{L^\infty(B_{R/2})} + R\|\nabla w\|_{L^\infty(B_{R/2})} + R^{1+\tau} [\nabla w]_{C^{\tau}(B_{R/2})} \leq CKR^2.$$

**Proof of Theorem 1.2(2).** Fix $\mathcal{B}$ as in the statement. Let $\lambda$ be as in (4.2), and let $v$ be as in (1.8). Observe that, by (4.6) and (4.5),
$$M^+_\lambda (v - v(\cdot - h))(x) \geq (L^s_A v + c_A)(x) - (L^s_A v + c_A)(x - h) - (c_A(x) - c_A(x - h))$$
$$\geq (L^s_A v + c_A)(x) - |h|,$$
which gives
$$M^+_\lambda (v - v(\cdot - h)) \geq \sup_{A \in \mathcal{M}_\lambda} \{L^s_A v + c_A(x)\} - |h| = -|h| \quad \text{in } \{v > 0\} \cap 2\mathcal{B}.$$With this and (4.6), we can apply Proposition 4.1 and conclude that $v \in C^{1,\tau}(\mathcal{B})$, with the corresponding estimate. \hfill \Box

5. **Proof of Theorem 1.3**

In order to prove Theorem 1.3, we consider $v = \psi - u$ as in (1.8). We showed, in section 4, that $v$ satisfies the locally uniformly elliptic obstacle problem (4.6) with ellipticity constants $\lambda > 0$ (as defined in (4.2)) and $1/\lambda^{(n-1)(n+2s)}$. Before proceeding with the proof, we define regular free boundary points, the constant $\bar{\alpha} > 0$, and the rescalings we will use to determine the blow up sequence. Here, we follow [2].

**Definition 5.1.** Let $\nu : (0, \infty) \to (0, \infty)$ be a nonincreasing function with
$$\lim_{r \to 0^+} \nu(r) = \infty.$$
We say that a free boundary point $x_0 \in \partial \{v > 0\}$ is **regular with modulus $\nu$** if
$$\sup_{\rho \geq r} \frac{\sup_{B_\rho(x_0)} v}{\rho^{1+s+\alpha}} \geq \nu(r)$$
for some $\alpha \in (0, s)$ such that
$$1 + s + \alpha < 2.$$
**Definition 5.2.** Let $\bar{\alpha} = \bar{\alpha}(n, s, \lambda) > 0$ be the minimum of the following three constants:
- The $\alpha > 0$ of the interior $C^\alpha$ estimate given by [5, Theorem 11.1];
- The $\alpha > 0$ of the boundary $C^\alpha$ estimate for $u/d^s$ given by [10, Proposition 1.1];
- The $\alpha > 0$ of the interior $C^{2s + \alpha}$ estimate for convex equations given by [4, Theorem 1.1] and [12, Theorem 1.1].

Without loss of generality and for the rest of this section, we assume that $x_0 = 0$ is a regular free boundary point with modulus $\nu$. As in [2], in the case $1 + s + \alpha \geq 2s + \bar{\alpha}$, we further assume that

$$\liminf_{\rho \downarrow 0} \frac{|\{v = 0\} \cap B_\rho|}{|B_\rho|} > 0.$$

In particular, there exists $c_0 > 0$ such that

$$\frac{|\{v = 0\} \cap B_\rho|}{|B_\rho|} \geq c_0 > 0 \text{ for all } \rho \text{ sufficiently small.} \quad (5.1)$$

By following [2], for $r > 0$, we define the rescalings

$$v_r(x) = \frac{v(rx)}{r^{1+s+\bar{\alpha}}\theta(r)} \quad \text{for } x \in \mathbb{R}^n \quad (5.2)$$

where

$$\theta(r) = \sup_{\rho \geq r} \frac{||\nabla v||_{L^\infty(B_\rho)}}{\rho^{s+\bar{\alpha}}}.$$ 

Then, $\theta$ is nonincreasing and $\theta(r) \geq \nu(r)$ for all $r > 0$, see [2, Lemma 5.4].

**Proof of Theorem 1.3.** We first prove that for any $R > 0$, $\|v_r\|_{C^{1,\tau}(B_R)}$ is uniformly bounded for all $r > 0$ sufficiently small, where $\tau \in (0, 1)$ is as in Proposition 4.1. Observe that

$$v_r \geq 0 \quad \text{in } \mathbb{R}^n, \quad (5.3)$$

and, since $1 + s + \alpha < 2$ and $\theta(r) \geq \nu(r)$, for all $r < 1$,

$$D^2v_r(x) = \frac{r^2}{r^{1+s+\bar{\alpha}}\theta(r)} D^2v(rx) \geq -\frac{1}{\nu(r)} \text{Id} \quad \text{for a.e. } x \in \mathbb{R}^n.$$

Let $c_A(x)$ be as in (4.4) and define

$$c_{A,r}(x) = \frac{c_A(rx)}{r^{1-s+\alpha}\theta(r)}.$$

Then, for every $r < 1$, since $\alpha \in (0, s)$ and (4.5) holds,

$$[c_{A,r}]_{\text{Lip}(\mathbb{R}^n)} \leq \frac{r^{s-\alpha}}{\theta(r)} [c_A]_{\text{Lip}(\mathbb{R}^n)} \leq \frac{1}{\nu(r)}.$$

Hence, following the proof of Theorem 1.2(2) in section 4, we see that

$$M^+_r(v_r - v_r(\cdot - h))(x) \geq -\frac{|h|}{\nu(r)} \quad \text{for all } x \in \{v_r > 0\} \cap B_4R \quad (5.5)$$

and for all $r < 1$ sufficiently small. Finally,

$$\|\nabla v_r\|_{L^\infty(B_R)} = \frac{R^{s+\alpha}\|\nabla v\|_{L^\infty(B_{Rr})}}{(Rr)^{s+\alpha}\theta(r)} \leq R^{s+\alpha}, \quad (5.6)$$

which implies that

$$|\nabla v_r(x)| \leq 2(1 + |x|^{s+\alpha}) \quad \text{for all } x \in \mathbb{R}^n.$$

Thus, by Proposition 4.1, $\|v_r\|_{C^{1,\tau}(B_R)}$ is uniformly bounded for all $r > 0$ sufficiently small.
Next, for any $k \geq 1$, we choose
\begin{equation}
(5.7) \quad r_k \geq \frac{1}{k}
\end{equation}
such that
\begin{equation}
(5.8) \quad \frac{\|\nabla v\|_{L^\infty(B_{r_k})}}{r_k^{\alpha}} \geq \frac{1}{2} \theta(1/k) \geq \frac{1}{2} \theta(r_k).
\end{equation}
Since $\|\nabla v\|_{L^\infty(\mathbb{R}^n)} \leq 1$ and $\theta(1/k) \geq \nu(1/k) \to \infty$ as $k \to \infty$, we have that $r_k \to 0$ as $k \to \infty$. In addition, from (5.8) it follows that
\begin{equation}
(5.9) \quad \|\nabla v_{r_k}\|_{L^\infty(B_1)} \geq \frac{1}{2}.
\end{equation}
Moreover, in the case $1 + s + \alpha \geq 2s + \bar{\alpha}$, by (5.1),
\begin{equation}
(5.10) \quad \frac{|\{v_{r_k} = 0\} \cap B_{\rho}|}{|B_{\rho}|} = \frac{|\{v = 0\} \cap B_{r_{k,\rho}}|}{|B_{r_{k,\rho}}|} \geq c_0 > 0 \quad \text{for all } \rho \text{ sufficiently small.}
\end{equation}
By Arzelà–Ascoli’s theorem and a standard diagonal argument, there exists $v_0 \in C^1_{\text{loc}}(\mathbb{R}^n)$ such that
\[ v_{r_k} \to v_0 \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^n). \]
Additionally, by (5.3) and (5.4),
\[ \begin{cases} v_0 \geq 0 & \text{in } \mathbb{R}^n, \\ D^2 v_0(x) \geq 0 & \text{for a.e. } x \in \mathbb{R}^n, \end{cases} \]
and, by (5.6) and (5.9),
\[ \frac{1}{2} \leq \|\nabla v_0\|_{L^\infty(B_R)} \leq R^{s+\alpha} \quad \text{for all } R \geq 1. \]
Also, in the case $1 + s + \alpha \geq 2s + \bar{\alpha}$, from (5.10), we find that
\[ \liminf_{\rho \searrow 0} \frac{|\{v_0 = 0\} \cap B_\rho|}{|B_\rho|} \geq c_0 > 0. \]
Next, as $R > 0$ in (5.5) was arbitrary, by passing to the limit as $|h| \to 0$ and $r = r_k \to 0$, we get
\[ M_\lambda^+(\partial_{e} v_0) \geq 0 \quad \text{in } \{v_0 > 0\} \]
and for all $e \in S^{n-1}$. Furthermore, by arguing as we did to obtain (5.5), for any $R > 0$ and any nonnegative probability measure $\mu$ with compact support, we find that
\[ M_\lambda^+(v_r - \int v_r(\cdot - h) \, d\mu(h)) \geq -\frac{|h|}{\nu(r)} \quad \text{in } \{v_r > 0\} \cap B_R \]
provided $r$ is sufficiently small. As a consequence,
\[ M_\lambda^+(v_0 - \int v_0(\cdot - h) \, d\mu(h)) \geq 0 \quad \text{in } \{v_0 > 0\}. \]
Therefore, applying the classification results in [2, Theorems 7.1, 7.2] to $v_0$, we finally obtain
\[ v_0(x) = K_0(e_0 \cdot x)^{1+s}, \]
for some $1/4 \leq K_0 \leq 1$ and $e_0 \in S^{n-1}$, as desired. \qed
6. Proof of Theorem 1.4

As in section 5, we assume, without loss of generality, that \( x_0 = 0 \) is a regular free boundary point with modulus \( \nu \).

Proof of Theorem 1.4. Consider \( v_r \) as in (5.2), for \( r = r_k \) given by (5.7). From Theorem 1.3 and its proof, we have that given any \( \delta_0 > 0 \) and \( R_0 \geq 1 \), there exists \( r_0 = r_0(\delta_0, R_0, \alpha, \nu, \lambda) \in (0, 1) \) such that for all \( 0 < r_k < r_0 \),

\[
\begin{align*}
M_X^+(\partial_e v_{r_k}) & \geq -\delta_0 \quad \text{in} \quad \{v_{r_k} > 0\} \cap B_{R_0} \\
M_X^-(\partial_e v_{r_k}) & \leq \delta_0 \quad \text{in} \quad \{v_{r_k} > 0\} \cap B_{R_0},
\end{align*}
\]

for all \( e \in S^{n-1} \), and

\[
|v_{r_k}(x) - K_0(e_0 \cdot x)^1+|^s + |\nabla v_{r_k}(x) - (1 + s)K_0(e_0 \cdot x)^s e_0| < \delta_0 \quad \text{for every} \quad x \in B_{R_0}.
\]

In addition, for every \( R \geq 1 \), we have

\[
\|\nabla v_{r_k}\|_{L^\infty(B_{R})} \leq R^{s+\alpha},
\]

for all \( k \geq 1 \), see (5.6). With these estimates in hand, we can argue exactly as in sections 8.2 and 8.3 of [2] and deduce that Theorem 1.4 holds. \( \square \)

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References