Optimal Online Discrepancy Minimization

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Warmup: edge orientation
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\[
\begin{align*}
\begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 1 & -1 & -1 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & 0 & 0
\end{pmatrix}
\end{align*}
\]
Beck-Fiala Theorem (1981)

For any vectors $v_1, \ldots, v_T \in [-1, 1]^n$ with at most $d$ nonzeros each,

$$\|x_1 v_1 + \cdots + x_T v_T\|_\infty < 2d$$

for some choice of signs $x_1, \ldots, x_T \in \{\pm 1\}.$
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Can we improve the above bound to $O(\sqrt{d})$?
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The Komlós Conjecture

For $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \leq 1$, do there exist $x_1, \ldots, x_T \in \{\pm 1\}$ with

$$\|x_1 v_1 + \cdots + x_T v_T\|_\infty \leq O(1)$$
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Best known bound $O(\sqrt{\log \min(n, T)})$ (Banaszczyk ’98, BDG ’16)
Introduction to online discrepancy

- Player given vectors $v_1, \ldots, v_T \in \mathbb{R}^n$ one at a time
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- Example: unit vectors, $\ell_2$ discrepancy
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\text{Adaptive adversary can always pick } v_t \text{ so that } \| \sum_{i=1}^t x_i v_i \|_2 \geq \sqrt{T}
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- *Adaptive* adversary can always pick $v_t$ so that $\| \sum_{i=1}^t x_i v_i \|_2 \geq \sqrt{T}$
- Player can also ensure $\leq \sqrt{T}$
Example II: Spencer’s hyperbolic cosine algorithm

- Player given vectors $v_1, \ldots, v_T \in [-1, 1]^n$ one at a time
- Find $x_1, \ldots, x_T \in \{\pm 1\}$ so that $\|x_1 v_1 + \cdots + x_t v_t\|_\infty$ small for all $t \in [T]$
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- Move to the position $p_t := p_{t-1} + x_t v_t$ that minimizes the potential

$$\sum_{i=1}^{n} (e^{\alpha p_t(i)} + e^{-\alpha p_t(i)})$$
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- $\alpha := \sqrt{\frac{2 \log(2n)}{T}}$ ensures $\|x_1 v_1 + \cdots + x_t v_t\|_\infty \leq \sqrt{2T \log(2n)}$
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- $\alpha := \sqrt{\frac{2 \log(2n)}{T}}$ ensures $\|x_1 v_1 + \cdots + x_t v_t\|_\infty \leq \sqrt{2T \log(2n)}$
- Matching lower bound $\Omega(\sqrt{n \log(2n)})$ for $T = n$
Oblivious adversary

- Suppose adversary picks unit vectors $v_1, \ldots, v_T$ in advance
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- Player still receives one at a time and must pick signs online
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- Player still receives one at a time and must pick signs online
- If player deterministic, same as adaptive
- What if player can use randomization?
Special case: edge orientation [Kalai ’01]

- Suppose edge vectors of the form $(0, 0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$
Special case: edge orientation \cite{Kalai01}

– Suppose edge vectors of the form $(0, 0, \ldots, 0, 1, 0, \ldots, 0, -1, 0, \ldots, 0)$
– For each coordinate $i \in [n]$, draw an infinite random $s_i \in \{0, 1\}^\mathbb{N}$
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  \[ k := \text{first position where } s_i(k) \neq s_j(k), \text{ i.e. } \{s_i(k), s_j(k)\} = \{0, 1\} \]
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Theorem [Kalai ’01]
The above algorithm achieves $\|\sum_{i=1}^{t} x_i v_i\|_\infty \leq O(\log T)$ after $T$ rounds.
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The above algorithm achieves \(\| \sum_{i=1}^{t} x_i v_i \|_\infty \leq O(\log T)\) after \(T\) rounds.

- Imbalance at a vertex upper bounded by the longest prefix ever used
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Set $x_t := 1$ with probability $\frac{1}{2} - \frac{\langle p_t, v_t \rangle}{c}$, else $x_t := -1$; $c = O(\log(nT))$
Self-balancing random walk [ALS ’20]

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**Theorem [ALS ’20]**

All prefix sums $p_t = \sum_{i=1}^{t} x_i v_i$ are $O(\sqrt{c})$-subgaussian.
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- $X$ is $C$-subgaussian:

\[\iff \mathbb{E}[\exp(\langle X, u \rangle^2 / C^2)] \leq 2 \text{ for all } u \in S^{n-1}\]
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**Corollary [ALS ’20]**

All prefix sums $\| \sum_{i=1}^{t} x_i v_i \|_\infty \leq O(\log(nT))$ with high probability.
Gaussian fixed point random walk [LSS ’21]

- Fix a parameter $\sigma \geq 1$
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Construct Markov chain on $\mathbb{R}$ with $0, \pm 1$ steps and stationary $\mathcal{N}(0, \sigma^2)$
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- Upon receiving vector $\nu_t$: set $p_t := p_{t-1} + M_\sigma(\langle p_{t-1}, \nu_t \rangle) \cdot \nu_t$
- Invariant: $p_t \sim N(0, \sigma^2 I_n)$ at all times

Theorem \cite{LSS21}

For $\sigma := p \log T$, all prefix sums are $2\sigma$-subgaussian and all steps are $\pm 1$.

Technical: construct $M_\sigma$ so that $\Pr[M_\sigma(x) = 0] \leq e^{-\sigma^2}$ for all $x \in \mathbb{R}$. 
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- Output $p_T - p_0$.  

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Our contribution

Theorem [Kulkarni, R., Rothvoss ’23]

For $\|v_t\|_2 \leq 1$, there is an online algorithm against an oblivious adversary which keeps all prefix sums 10-subgaussian. In particular,

$$\|\sum_{i=1}^{t} x_i v_i\|_\infty \leq O(\sqrt{\log T}) \text{ for all } t \in [T] \text{ with high probability.}$$
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**Theorem [Kulkarni, R., Rothvoss ’23]**

For any $n \geq 2$, there is a strategy for an oblivious adversary that yields a sequence of unit vectors $v_1, ..., v_T \in \mathbb{R}^n$ so that for any online algorithm, with probability at least $1 - 2^{-\text{poly}(T)}$,

$$\max_{t \in [T]} \left\| \sum_{i=1}^{t} x_i v_i \right\|_\infty \gtrsim \sqrt{\log T}.$$
Lower bound for an oblivious adversary

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- Proof sketch: split time horizon into blocks of size $k := \Theta(\log T)$
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- Within each block, guess all $k$ signs chosen by the player
- Simulate the strategy of adaptive adversary to get $\Omega(\sqrt{k})$ w.p. $2^{-k}$
- One of the blocks will succeed with probability $1 - (1 - 2^{-k})^{T/k}$.
ε-nets

P ⊆ \mathbb{R}^n \text{ so that, for all } \|v\|_2 \leq 1, \text{ there is } p \in P \text{ with } \|p - v\|_2 \leq \varepsilon.
$\varepsilon$-nets

- $P \subseteq \mathbb{R}^n$ so that, for all $\|v\|_2 \leq 1$, there is $p \in P$ with $\|p - v\|_2 \leq \varepsilon$.
- There exists an $\varepsilon$-net with $|P| \leq (3/\varepsilon)^n$. 

![Image of a sphere with points distributed evenly, likely representing an $\varepsilon$-net]

Overview of the algorithm

Theorem [Kulkarni, R., Rothvoss '23]

Let $T = (V, E)$ be a rooted tree with vectors $\|v_e\|_2 \leq 1$ on edges. Then there is a distribution $D$ over $\{-1, 1\}^E$ so that for $x \sim D$, $P_e \in P$ $x_e v_e$ is 10-subgaussian for every $i \in V$. 

[Diagram: A rooted tree with a series of vertices labeled as "children labeled with $\epsilon$-net" and a label "depth $T$".]
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![Diagram of a rooted tree with annotations for children labeled with $\epsilon$-net and $v'_1 \approx v_1$.]
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![Diagram of a rooted tree with annotations: start, children labeled with $\varepsilon$-net, $v'_1 \approx v_1$, $v'_2 \approx v_2$, depth $T$.]
Overview of the algorithm

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Let \( T = (V, E) \) be a rooted tree with vectors \( \|v_e\|_2 \leq 1 \) on edges. Then there is a distribution \( \mathcal{D} \) over \( \{-1, 1\}^E \) so that for \( x \sim \mathcal{D} \),

\[
\sum_{e \in P_i} x_e v_e \text{ is 10-subgaussian for every } i \in V.
\]
Banaszczyk prefix balancing

**Theorem [Banaszczyk ’12]**

For any $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \leq 1$ and any convex body $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq 1 - \frac{1}{2T}$, there are signs $x_1, \ldots, x_T \in \{\pm 1\}$ so that

$$\sum_{i=1}^{t} x_i v_i \in 5K \quad \forall t = 1, \ldots, T.$$
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For any convex body \( K \subseteq \mathbb{R}^n \) with \( \gamma_n(K) \geq \frac{1}{2} \) and \( u \in \mathbb{R}^n \) with \( \|u\|_2 \leq \frac{1}{5} \), there is a convex body \( (K * u) \subseteq (K + u) \cup (K - u) \) with \( \gamma_n(K * u) \geq \gamma_n(K) \).
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- Define \( K_T := K \) and \( K_{t-1} := (K_t * v_t) \cap K \).
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- Define \( K_T := K \) and \( K_{t-1} := (K_t * v_t) \cap K \).
- Show by induction \( \gamma(K_t) \geq 1 - \frac{T-t+1}{2T} \), then iteratively find \( x_1, \ldots, x_T \)
For any $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_i\|_2 \leq 1$ and any convex body $K \subseteq \mathbb{R}^n$ with $\gamma_n(K) \geq 1 - \frac{1}{2T}$, there are signs $x_1, \ldots, x_T \in \{\pm 1\}$ so that

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Banaszczyk prefix balancing for trees

**Theorem [Banaszczyk ’12]**

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**Theorem [Kulkarni, R., Rothvoss ’23]**

Let $T = (V, E)$ be a rooted tree with vectors $\|v_e\|_2 \leq 1$ on edges. Let $K \subseteq \mathbb{R}^n$ be a convex body with $\gamma_n(K) \geq 1 - \frac{1}{2|E|}$. Then there are signs $x \in \{-1, 1\}^E$ so that for every root-vertex path $P_i$,

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- Analogous proof with $K_i := \left( \bigcap_{j \in \text{children}_i} (K_j * v_{\{i,j\}}) \right) \cap K.$
Theorem [Kulkarni, R., Rothvoss ’23]

Let $T = (V, E)$ be a rooted tree with vectors $\|v_e\|_2 \leq 1$ on edges. Then there is a distribution $\mathcal{D}$ over $\{-1, 1\}^E$ so that for $x \sim \mathcal{D}$,

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is 10-subgaussian for every $i \in V$. 

Idea: clone each edge $N$ times, find a coloring, sample random clone
Cloning: coloring $\Rightarrow$ distribution

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- Idea: clone each edge $N$ times, find a coloring, sample random clone
- Define a convex body $K$ and show $\gamma_{Nn}(K) \geq 1 - \frac{1}{N^{1+\delta}} \geq 1 - \frac{1}{2N|E|}$
Body of subgaussian distributions

Take any $C > 2$ and define

$$K := \left\{ (y^{(1)}, \ldots, y^{(N)}) \in \mathbb{R}^{Nn} \mid Y \sim \{y^{(1)}, \ldots, y^{(N)}\} \text{ is } C \text{-subgaussian} \right\}.$$
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- By a net argument, suffices to consider a single unit vector $w \in S^{n-1}$:

$$K_w := \left\{ (y^{(1)}, \ldots, y^{(N)}) \in \mathbb{R}^{Nn} \mid E_{\ell \sim [N]} \left[ \exp \left( \frac{1}{C^2} \left\langle w, y^{(\ell)} \right\rangle^2 \right) \right] \leq 2 \right\}$$
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- $X_{\ell} := \exp(\frac{1}{C^2}g_{\ell}^2)$ satisfy $\mathbb{E}[X_{\ell}^p] < \infty$ for $p < C^2/2$ (want $p > 2$)
Lemma

Let $p \geq 2$ and $X_1, \ldots, X_N$ be centered, indep. r.v.'s with $\mathbb{E}[|X_i|^p] = O_p(1)$. Then

$$\Pr[X_1 + \cdots + X_N > N] \leq \frac{O_p(1)}{N^{p/2}}.$$
Concentration for heavy-tailed random variables

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$$\leq \sum_{i=1}^{N} \mathbb{E}[X_i^2] + \sum_{i \neq j} \mathbb{E}[X_i] \mathbb{E}[X_j]$$
$$= \frac{O(1)}{N^2} + 0 = O(1/N).$$
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- In general, follows by Markov + Rosenthal’s inequality:
Concentration for heavy-tailed random variables

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▶ In general, follows by Markov + Rosenthal’s inequality:

Rosenthal ’70

Let \( p \geq 2 \) and \( X_1, \ldots, X_N \) centered, indep. r.v.’s with \( \mathbb{E}[|X_\ell|^p] < \infty \). Then

\[
\mathbb{E}[|X_1 + \cdots + X_N|^p]^{1/p} \leq 2^p \cdot \max \left\{ \left( \sum_{i=1}^{N} \mathbb{E}[|X_i|^p] \right)^{1/p}, \left( \sum_{i=1}^{N} \mathbb{E}[X_i^2] \right)^{1/2} \right\}.
\]
Online algorithm

- Given $T$, build depth $T$ tree where children are labeled with $\epsilon$-net
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$$\mathcal{D}_t^* = \Pi_{\{\pm 1\}^E} (\mathcal{D}_{t+1}^*).$$
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Open problems

Polynomial time algorithm

Given oblivious $v_1, \ldots, v_T \in \mathbb{R}^n$ with $\|v_t\|_2 \leq 1$, does there exist a polynomial time online algorithm against an oblivious adversary which keeps all signed prefix sums $O(1)$-subgaussian?
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Oblivious edge orientation
Given oblivious edge vectors \(v_1, \ldots, v_T \in \mathbb{R}^n\), can we find online signs \(x_1, \ldots, x_T \in \{\pm 1\}\) so that \(\|\sum_{i=1}^{T} x_i v_i\|_\infty \leq O(\sqrt[3]{\log T})\) w.h.p.?
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- Main theorem: \( O(\sqrt{\log T}) \), also \( \Omega(\sqrt[3]{\log \min(n, T)}) \) [AANRSW’98]
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Given oblivious $v_1, \ldots, v_n \in [-1, 1]^n$, can we find online signs $x_1, \ldots, x_n \in \{\pm 1\}$ so that $\|\sum_{i=1}^n x_i v_i\|_\infty \leq O(\sqrt{n})$ w.h.p.?
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Open problems

Polynomial time algorithm
Given oblivious $v_1, \ldots, v_T \in \mathbb{R}^n$ with $||v_t||_2 \leq 1$, does there exist a polynomial time online algorithm against an oblivious adversary which keeps all signed prefix sums $O(1)$-subgaussian?

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Given oblivious edge vectors $v_1, \ldots, v_T \in \mathbb{R}^n$, can we find online signs $x_1, \ldots, x_T \in \{\pm 1\}$ so that $\| \sum_{i=1}^T x_i v_i \|_1 \leq O\left(3^{\sqrt{\log T}}\right)$ w.h.p.?

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Thanks for your attention!