ARITHMETIC INTERSECTIONS OF MODULAR GEODESICS

HENRI DARMON AND JAN VONK

Abstract. The arithmetic, $p$-arithmetic, and incoherent intersections between pairs of closed geodesics on a modular or Shimura curve are defined, and some of their expected algebraicity and factorisation properties are examined. These properties follow in a special case from the conjectures of [DV] on the RM values of rigid meromorphic cocycles, and are inspired from the recent work of James Rickards [Ri] and Xavier Guitart, Marc Masdeu and Xavier Xarles [GMX].

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Introduction

Let $\mathcal{H}$ denote the Poincaré upper half plane endowed with its usual action of $\text{SL}_2(\mathbb{Z})$ by Möbius transformations. The arithmetic quotient

$$X := \text{SL}_2(\mathbb{Z})\backslash \mathcal{H} = \text{Spec}(\mathbb{C}[j])$$

is equipped with a rich collection of cycles indexed by primitive integral binary quadratic forms

$$F(x, y) = ax^2 + bxy + cy^2 = a(x - \tau y)(x - \tau' y), \quad \tau \in \mathcal{H} \cup \mathbb{R} \cup \{\infty\}.$$ 

When $F$ has negative discriminant $D := b^2 - 4ac$, its root $\tau \in \mathcal{H}$ is a CM point whose associated $j$-value generates an abelian extension of the imaginary quadratic field $\mathbb{Q} (\sqrt{D})$. When $F$ has positive discriminant, it gives rise to the oriented geodesic on $\mathcal{H}$ going from $\tau$ to $\tau'$, where

$$\tau = \frac{-b + \sqrt{D}}{2a}, \quad \tau' = \frac{-b - \sqrt{D}}{2a}, \quad \sqrt{D} > 0.$$ 

This open geodesic maps to a closed modular geodesic on $X$, which is of real dimension one. Since it is not an algebraic cycle on $X$, its relevance to the generation of class fields of $\mathbb{Q} (\sqrt{D})$ is less immediately apparent.

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The goal of this note is to propose an *arithmetic intersection theory* for modular geodesics, attaching to a pair of such geodesics certain numerical invariants that are rich enough to (ostensibly) generate class fields of real quadratic fields. The predicted algebraicity of these quantities is a by-product of the approach of [DV] to explicit class field theory based on the RM values of rigid meromorphic cocycles, but avoids the latter notion and offers a somewhat complementary perspective. The authors hope that this perspective might appeal to certain readers and, by casting a different light on the conjectures of [DV], make them more amenable to other types of generalisations.

**Topological intersections.** Let $\gamma_1 = (\tau_1, \tau'_1)$ and $\gamma_2 = (\tau_2, \tau'_2)$ be two distinct geodesics on $H$ attached to a pair of indefinite binary quadratic forms $F_1$ and $F_2$. The fixing of an orientation on $H$ determines the signed topological intersection of $\gamma_1$ and $\gamma_2$, defined as

$$\gamma_1 \cdot \gamma_2 := \begin{cases} 0 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ do not intersect;} \\ 1 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ intersect positively;} \\ -1 & \text{if } \gamma_1 \text{ and } \gamma_2 \text{ intersect negatively,} \end{cases}$$

where the orientation conventions are illustrated below:

![Figure 1. The topological intersection number $\gamma_1 \cdot \gamma_2$](image)

If $\Gamma \subset \text{SL}_2(\mathbb{Z})$ is any congruence subgroup, then the stabiliser subgroups $\Gamma_1$ and $\Gamma_2$ of $\tau_1$ and $\tau_2$ are infinite cyclic modulo torsion. Let $\Sigma$ and $\Sigma_{12}$ be the double coset spaces

$$\Sigma := (\Gamma \times \Gamma)/(\Gamma_1 \times \Gamma_2), \quad \Sigma_{12} := \Gamma \Gamma_1 / \Gamma \Gamma_2,$$

which are in bijection via the map $(g_1, g_2) \mapsto g_1^{-1} g_2$. The sum

$$\gamma_1 \cdot \gamma_2 | \Gamma := \sum_{(g_1, g_2) \in \Sigma} (g_1 \gamma_1 \cdot g_2 \gamma_2) = \sum_{g \in \Sigma_{12}} (\gamma_1 \cdot g \gamma_2)$$

represents the topological intersection of the oriented closed geodesics arising from the image of $\gamma_1$ and $\gamma_2$ in the quotient Riemann surface $X_\Gamma := \Gamma \backslash H$. (In particular, this quantity vanishes when $\Gamma = \text{SL}_2(\mathbb{Z})$, since $\text{SL}_2(\mathbb{Z}) \backslash H$ has genus zero.) Although the index set $\Sigma_{12}$ is infinite, the sums on the right of (1) involve only finitely many non-zero terms. Basic facts about modular geodesics and their topological intersections, when $\Gamma$ is any discrete subgroup of $\text{SL}_2(\mathbb{R})$ acting on $H$ with compact or finite volume quotient, are recalled in Section 1.

**Arithmetic intersections.** The cross-ratio

$$(\gamma_1; \gamma_2) := (\tau_1, \tau'_1; \tau_2, \tau'_2) := \frac{(\tau_1 - \tau_2)(\tau'_1 - \tau'_2)}{(\tau_1 - \tau'_2)(\tau'_1 - \tau_2)} \in \mathbb{R}^\times$$
Theorem 1. The infinite product in (4) converges absolutely p-adically.

This theorem is proved in Section 3 as a special case of a more general result involving p-arithmetic subgroups of indefinite quaternion algebras over \( \mathbb{Q} \). The p-adic number \((\gamma_1 \ast \gamma_2)_p\) is called the p-arithmetic intersection number of the geodesics \( \gamma_1 \) and \( \gamma_2 \).

The main interest of the p-arithmetic intersection number lies in its relevance to explicit class field theory for real quadratic fields. Let \( H_1 \) and \( H_2 \) be the narrow ring class fields attached to the discriminants \( D_1 \) and \( D_2 \) respectively, and let \( H_{12} \) denote their compositum, viewed as a subfield of \( \overline{\mathbb{Q}}_p \), after fixing a p-adic embedding of this field.
Section 3 formulates a general conjecture on the algebraicity of $p$-arithmetic intersections. In the following special case, it follows from the conjectures of [DV]:

**Conjecture 2.** If $p = 2, 3, 5, 7$, or $13$, then $(\gamma_1 \ast \gamma_2)_t \in H_{12}$.

For example, let $\tau_1 = (1 + \sqrt{5})/2$ be the golden ratio, and let $\tau_2 = (15 + \sqrt{321})/8$ be a real quadratic irrationality of discriminant $321 = 3 \cdot 107$. The discriminants 5 and 321 have narrow class numbers 1 and 6 respectively, and the primitive integral binary quadratic form with root $\tau_2$ generates the narrow class group of discriminant 321. The smallest prime $p$ for which $(\frac{\tau_2}{p}) = (\frac{321}{p}) = -1$ is $p = 7$, and Conjecture 9 asserts the algebraicity of the 7-arithmetic intersection of $\gamma_1 = (\tau_1, \tau_1')$ and $\gamma_2 = (\tau_2, \tau_2')$. The quantity $(\gamma_1 \ast \gamma_2)_t$ was computed to 300 digits of 7-adic precision, and a rational recognition algorithm suggests that it is the square of a quantity satisfying the palindromic polynomial

$$g(t) = 7881253325449 \cdot t^{12} + a_{11} t^{11} + \ldots + a_1 t + 7881253325449,$$

whose other coefficients are listed below:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$a_n = a_{12-n}$</th>
<th>$n$</th>
<th>$a_n = a_{12-n}$</th>
<th>$n$</th>
<th>$a_n = a_{12-n}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>16711393316898</td>
<td>2</td>
<td>15589096918207</td>
<td>3</td>
<td>1349942214176</td>
</tr>
<tr>
<td>4</td>
<td>-8232873610095</td>
<td>5</td>
<td>-7408349176266</td>
<td>6</td>
<td>-4016169195897</td>
</tr>
</tbody>
</table>

The fact that the splitting field of $g(t)$ is contained in the compositum of $\mathbb{Q}(\sqrt{5})$ and the narrow Hilbert class field of $\mathbb{Q}(\sqrt{321})$ provides convincing evidence for Conjecture 2.

Experiments like this indicate that $p$-arithmetic intersection numbers are typically non-trivial. Theoretical insights into how often this happens can be obtained by seeking to understand the prime factorisations of $p$-arithmetic intersection numbers, a theme which is taken up in Section 5.

For instance, the constant coefficient in (5) factors as

$$a_0 = 7881253325449 = 7^4 \cdot 23^2 \cdot 47^2 \cdot 53^2.$$

It is no coincidence that the primes that occur in this factorisation are inert in both $\mathbb{Q}(\sqrt{5})$ and $\mathbb{Q}(\sqrt{321})$, and arise among the factors of the quantity $(5 \cdot 321 - t^2)/(4 \cdot 7)$ when it is an integer, which are listed in the table below:

<table>
<thead>
<tr>
<th>$t$</th>
<th>$(5 \cdot 321 - t^2)/(4 \cdot 7)$</th>
<th>$t$</th>
<th>$(5 \cdot 321 - t^2)/(4 \cdot 7)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3 \cdot 19</td>
<td>11</td>
<td>53</td>
</tr>
<tr>
<td>17</td>
<td>47</td>
<td>25</td>
<td>5 \cdot 7</td>
</tr>
<tr>
<td>31</td>
<td>23</td>
<td>39</td>
<td>3</td>
</tr>
</tbody>
</table>

This suggests that $p$-arithmetic intersection numbers ought to admit explicit factorisations just like those obtained by Gross and Zagier [GZ] for differences of singular moduli.

Let $m_p$ be a prime of $\mathbb{Q}(\sqrt{D_1 D_2})$ above $p$. Since $(\gamma_1 \ast \gamma_2)_t$ is defined $p$-adically, its $m_p$-adic valuation can be understood unconditionally by elementary manipulations, of the kind used to prove [DV, Thm. 3.26]. These arguments lead to the identity

$$\text{ord}_{m_p}((\gamma_1 \ast \gamma_2)_t) = \text{ord}_{m_p}((\gamma_1 \ast \gamma_2)_t).$$

The $q$-adic valuation of the $p$-arithmetic intersection when $q \neq p$ lies ostensibly deeper, and is related to an arithmetic intersection on a discrete subgroup of a suitable indefinite quaternion algebra. More precisely, let $B(pq)$ be the indefinite quaternion algebra ramified at $p$ and $q$, and let $R(pq)$ be a maximal order in $B(pq)$, which is unique up to conjugation. After fixing an identification of $B(pq) \otimes \mathbb{R}$ with $M_2(\mathbb{R})$, the group
Γ(pq) := R(pq)× is a discrete subgroup of SL2(ℝ). If the primes p and q are both non-split in \( \mathbb{Q}(\sqrt{D_1}) \) and \( \mathbb{Q}(\sqrt{D_2}) \), Section 3 explains how the geodesics \( γ_1 \) and \( γ_2 \), or their hyperbolic conjugacy classes, can be transferred to similar objects \( γ_1^♭, γ_2^♭ \) for \( Γ(pq) \), depending on the choice of primes \( m_p \) and \( m_q \) of \( \mathbb{Q}(\sqrt{D_1 D_2}) \) above \( p \) and \( q \) respectively. One can then consider the arithmetic intersection number

\[
(γ_1^♭ * γ_2^♭)_{Γ(pq)} \in \mathbb{Q}(\sqrt{D_1 D_2})^×.
\]

**Conjecture 3.** Assume as before that \( Γ_p = SL_2(\mathbb{Z}[1/p]) \) with \( p = 2, 3, 5, 7, \) or \( 13 \). If at least one of \( D_1 \) or \( D_2 \) is a non-zero square modulo \( p \) or \( q \), then

\[
\text{ord}_Ω ((γ_1 * γ_2)_{Γ_p}) = 0,
\]

for all primes \( Ω \) of \( H_{12} \) above \( m_q \). Otherwise, \( Ω \) can be chosen so that

\[
\text{ord}_Ω ((γ_1 * γ_2)_{Γ_p}) = \text{ord}_{m_q} ((γ_1^♭ * γ_2^♭)_{Γ(pq)}).
\]

An extension of Conjecture 3 to the setting of general primes \( p \) and general \( p \)-arithmetic groups arising from quaternion algebras over \( \mathbb{Q} \) is described in Section 4 where it is formulated as a relation between arithmetic intersection numbers and certain conjectural “incoherent intersection numbers” between compatible systems of geodesics on an “incoherent collection” of Shimura curves.

1. **Topological intersections**

The topological intersections described in the introduction extend to more general discrete subgroups of \( SL_2(ℝ) \), such as those arising from indefinite quaternion algebras over \( \mathbb{Q} \), and the article will place itself in this more general setting throughout. This is not merely done for the sake of extra generality. Arithmetic intersections on quaternion algebras arise naturally in the factorisation of \( p \)-arithmetic intersections on \( SL_2(\mathbb{Z}) \) predicted by Conjecture 3 of the Introduction, and formulating the theory for all quaternion algebras at once leads to a richer, more consistent picture.

Let \( B \) be an indefinite quaternion algebra over \( \mathbb{Q} \), viewed as a subring of \( M_2(ℝ) \) by fixing an identification \( B \otimes ℝ = M_2(ℝ) \). The set \( S \) of rational primes \( p \) for which \( B \otimes \mathbb{Q}_p \) is a division ring is of finite, even cardinality. The \( \mathbb{Q} \)-vector space \( B_0 \) of elements of \( B \) of trace zero, equipped with the trace form

\[
\langle b_1, b_2 \rangle = \frac{1}{2} \text{Trace}(b_1 b_2), \quad \text{for } b_1, b_2 \in B_0,
\]

is a non-degenerate quadratic space over \( \mathbb{Q} \) of real signature \( (2,1) \), on which the group \( B^× \) acts isometrically by conjugation. Let \( R \) be a maximal order in \( B \), which is unique up to conjugation by \( B^× \). The group \( Γ := R^×_1 \) of elements of reduced norm one preserves the lattice \( R_0 := R \cap B_0 \). It also acts discretely on \( \mathcal{H} \) by Möbius transformations via the chosen inclusion of \( B \) in \( M_2(ℝ) \). The quotient \( Γ \setminus \mathcal{H} \) has finite volume, and is even a compact Riemann surface when \( B \neq M_2(ℝ) \).

Let \( \hat{γ} = (τ, τ') \) be the open geodesic on \( \mathcal{H} \) joining the endpoints \( τ, τ' \in ℝ \), and suppose that it maps onto a closed geodesic \( γ \) on \( Γ \setminus \mathcal{H} \). Let \( L_τ \) and \( L_{τ'} \) be the lines in \( ℝ^2 \) spanned by the column vectors \( \begin{pmatrix} τ \\ 1 \end{pmatrix} \) and \( \begin{pmatrix} τ' \\ 1 \end{pmatrix} \). The ring

\[
\mathcal{O}_γ := \{ a \in R \subset M_2(ℝ) \text{ such that } a \text{ preserves } L_τ \text{ and } L_{τ'} \}
\]

is isomorphic to a real quadratic order and is equipped with a canonical map \( \mathcal{O}_γ \hookrightarrow ℝ \oplus ℝ \) sending \( a \in \mathcal{O}_γ \) to its eigenvalues on the two eigen-lines \( L_τ \) and \( L_{τ'} \). The discriminant of \( \mathcal{O}_γ \) is called the **discriminant** of the
geodesic $\gamma$. The primes $q \in S$ are non-split in $O_\gamma$, since $Q_q \times Q_q$ is not a subring of $B \otimes Q_q$ when $q \in S$. It will be frequently assumed, in order to lighten the exposition, that the $q \in S$ are inert, i.e., the ring $O_\gamma/q$ is isomorphic to the field $\mathbb{F}_{q^2}$ with $q^2$ elements.

For each $q \in S$, the set of elements whose norm is divisible by $q$ is a maximal ideal in $R$, whose associated quotient is $\mathbb{F}_{q^2}$. After choosing a reduction map $\nu_q : R \rightarrow \mathbb{F}_{q^2}$ for each $q \in S$, the collection

$$\eta_q(\gamma) := \nu_q(\sqrt{D}) \in \mathbb{F}_{q^2}^\times,$$

where $\sqrt{D} \in O_\gamma$ is the “positive” square root of $D$ relative to the chosen real embedding, is an invariant of $\gamma$, called the orientation of $\gamma$ at $q \in S$. After fixing, for each $q \in S$, an element $\delta_q \in \mathbb{F}_{q^2}^\times$ satisfying $\delta_q^2 = D$, we can then consider the set $\Pi_D$ of geodesics of discriminant $D$ in $\Gamma \backslash \mathcal{H}$ having orientation $\delta_q$ at $q$, for each $q \in S$. The group $\Gamma$ acts naturally on $\Pi_D$.

**Lemma 4.** The quotient $\Gamma \backslash \Pi_D$ has cardinality $h(D) := \# \text{Pic}^+(O_D)$.

**Sketch of proof.** The set $\Pi_D$ is in bijection with the set of oriented optimal embeddings of $O_D$ into $R$. When $D$ is a negative discriminant and $R$ is a maximal order in a definite quaternion algebra, the set of such embeddings is endowed with a simply transitive action of the class group of $O_D$, as described in [Gr1, §3]. These definitions adapt to our situation, with the following differences:

1. Since $B$ is an indefinite quaternion algebra, it satisfies the so-called Eichler condition and there is a single $B^\times$ conjugacy class of maximal orders in $B$. Hence any embedding of $O_D$ into some maximal order can be conjugated into an embedding in $R$.

2. The orientation on a geodesic is reversed when it is translated by a principal ideal of $O_D$ having a generator of negative norm, and hence it is the class group in the narrow sense that acts simply transitively on $\Gamma \backslash \Pi_D$.

Let $\gamma_1$ and $\gamma_2$ be two geodesics on $\Gamma \backslash \mathcal{H}$ attached to $(\tau_1, \tau_1')$ and $(\tau_2, \tau_2')$. The topological intersection of $\gamma_1$ and $\gamma_2$ on $\Gamma \backslash \mathcal{H}$ is given by formula (1) of the introduction:

$$\langle \gamma_1 \cdot \gamma_2 \rangle^\Gamma = \sum_{g \in \Sigma_{12}} \gamma_1 \cdot g \gamma_2.$$

The topological interpretation of this sum reveals that it involves only finitely many non-zero terms.

For each $q \in S$, the pair $(\gamma_1, \gamma_2)$ of geodesics gives rise to a canonical invariant

$$\eta_q(\gamma_1, \gamma_2) := \eta_q(\gamma_1) \eta_q(\gamma_2) \in \mathbb{F}_{q^2}^\times.$$

Unlike the invariants $\eta_q(\gamma_1)$, it does not depend on the choice of the reduction map $\nu_q : R \rightarrow \mathbb{F}_{q^2}$ that was made in order to define it. It determines a distinguished ideal of $\mathbb{Q}(\sqrt{D_1 D_2})$ above each prime $q \in S$. Note that these primes are split since they are assumed to be inert in both $\mathbb{Q}(\sqrt{D_1})$ and $\mathbb{Q}(\sqrt{D_2})$.

### 2. Arithmetic Intersections

The definition of the arithmetic intersection rests on the following explicit expression for the cross-ratio attached to the pair $(\gamma_1, \gamma_2)$ of modular geodesics of relatively prime discriminants $D_1$ and $D_2$. Let $b_1, b_2 \in R_0$ denote the images of the positive square roots $\sqrt{D_1} \in O_{\gamma_1}$ and $\sqrt{D_2} \in O_{\gamma_2}$ respectively.
Lemma 5. The cross-ratio attached to \((\gamma_1, \gamma_2)\) is given by
\[
(\gamma_1; \gamma_2) := (\tau_1 \tau_1'; \tau_2, \tau_2') = \frac{\langle b_1, b_2 \rangle + \sqrt{D_1 D_2}}{\langle b_1, b_2 \rangle - \sqrt{D_1 D_2}}.
\]
The two geodesics intersect if and only if \(\langle b_1, b_2 \rangle^2 \leq D_1 D_2\), i.e., the elements \(b_1\) and \(b_2\) generate a positive definite subspace of \(B_0\) relative to the trace form.

Proof. This lemma follows from a direct calculation, as in \(\mathbb{R}\) for example. For instance, the second assertion can be seen by noting that two geodesics on \(\mathcal{H}\) intersect if and only if their associated cross-ratio is negative, by exploiting the \(\text{SL}_2(\mathbb{R})\)-invariance of the cross-ratio to reduce to the case where \(\gamma_1 = (0, \infty)\). \(\Box\)

Thanks to Lemma 5, the arithmetic intersection \((\gamma_1 \ast \gamma_2)\Gamma\) generalises directly to the case where \(\Gamma\) is the group of norm one elements of the multiplicative group of a maximal order in an indefinite quaternion algebra over \(\mathbb{Q}\), as in the previous section. Namely, with notations as in (7), we extend (2) and (3) by setting
\[
(\gamma_1 \ast \gamma_2) := (\tau_1 \tau_1'; \tau_2, \tau_2')^{(\gamma_1 \gamma_2)}, \quad (\gamma_1 \ast \gamma_2)\Gamma = \prod_{g \in \Sigma_{12}} (\gamma_1 \ast g \gamma_2).
\]
Lemma 5 shows that \((\gamma_1 \ast \gamma_2)\Gamma\) belongs to the real quadratic field \(\mathbb{Q}(\sqrt{D_1 D_2})\), and is of norm one in this field.

Let \(\tilde{\Gamma} := R^\times\), which contains \(\Gamma\) with index two.

Lemma 6. The arithmetic intersection \((\gamma_1 \ast \gamma_2)\Gamma\) satisfies the identities
\[
\begin{align*}
(g \gamma_1 \ast g \gamma_2)\Gamma &= (\gamma_1 \ast \gamma_2)\Gamma^{\text{sgn} (\det (g))} \quad \text{for all } g \in \tilde{\Gamma}, \\
(\gamma_2 \ast \gamma_1)\Gamma &= (\gamma_1 \ast \gamma_2)\Gamma^1, \\
(\gamma_1' \ast \gamma_2)\Gamma &= (\gamma_1 \ast \gamma_2)\Gamma.
\end{align*}
\]

Proof. The first identity follows from the observation that the cross-ratio of any quadruple is invariant under the action of \(\text{GL}_2(\mathbb{R})\), while \(\gamma_1 \ast \gamma_2\) is only invariant under the action of the orientation-preserving group \(\text{SL}_2(\mathbb{R})\), and is negated by orientation-reversing elements. The second identity follows from the symmetry of the cross-ratio and the antisymmetry of the topological intersection, and the last follows from the fact that replacing \(\gamma_1\) by \(\gamma_1'\) sends the resulting cross-ratio to its inverse, and negates the topological intersection of the modular geodesics. \(\Box\)

3. \(p\)-arithmetic intersections

As explained in the introduction, the \(p\)-arithmetic intersection is obtained by replacing the arithmetic group \(\Gamma\) by its \(p\)-arithmetic counterpart \(\Gamma_p\). With notations as in the previous section, if \(p\) is a prime which does not divide the discriminant of the indefinite quaternion algebra \(B\), and \(\Gamma = R_p^\times\) where \(R\) is an order of \(B\) of discriminant prime to \(p\), then \(\Gamma_p := (R[1/p])^\times\). The groups \(\Gamma\) and \(\Gamma_p\) are both viewed as subgroups of \(\text{SL}_2(\mathbb{R})\) by fixing an identification of \(B \otimes \mathbb{R}\) with \(M_2(\mathbb{R})\), as before.

Let
\[
\Gamma_1^p := \text{Stab}_{\Gamma_p}(\gamma_1), \quad \Gamma_2^p := \text{Stab}_{\Gamma_p}(\gamma_2), \quad \Sigma_{12}^p := \Gamma_1^p \Gamma_2^p / \Gamma_2^p
\]
be the \(p\)-arithmetic analogues of \(\Gamma_1\), \(\Gamma_2\), and \(\Sigma_{12}\) attached to the geodesics \(\gamma_1\) and \(\gamma_2\). The \(p\)-arithmetic intersection of \(\gamma_1\) and \(\gamma_2\) is obtained by considering the infinite product
\[
(\gamma_1 \ast \gamma_2)\Gamma_p := \prod_{g \in \Sigma_{12}^p} (\gamma_1 \ast g \gamma_2).
\]
In order to make sense of this infinite product, it is essential to view it as a \( p \)-adic quantity: to this end, fix an embedding of the real quadratic field \( \mathbb{Q}(\sqrt{D_1D_2}) \) into \( \mathbb{Q}_p \).

**Theorem 7.** The infinite product (10) converges absolutely \( p \)-adically.

The proof of Theorem 7 rests on the study of the quantity
\[
n_g := \langle b_1, gb_2 \rangle \in \mathbb{Z}[1/p]
\]
attached to any \( g \in \Sigma_{12}^p \). Let \( v_p(n) := p^{-\text{ord}_p(n)} \) denote the normalised \( p \)-adic norm of \( n \in \mathbb{Q} \). We claim that the set
\[
\Sigma_{12}^{\leq N} := \{ g \in \Sigma_{12}^p \text{ such that } (\gamma_1; g\gamma_2) < 0 \text{ and } v_p(n_g) \leq p^N \}
\]
is finite. Indeed, the inequality \((\gamma_1; g\gamma_2) < 0\) holds if and only if \( n_g^2 < D_1D_2 \). But there are finitely many such \( n_g \in \mathbb{Z}[1/p] \) with \( v_p(n_g) \leq p^N \), and the finiteness of \( \Sigma_{12}^{\leq N} \) is a consequence of the following lemma:

**Lemma 8.** Let \( n \in \mathbb{Z}[1/p] \) be an element satisfying \( n^2 \leq D_1D_2 \). Then the set
\[
\Pi(D_1, D_2, n) := \{(a_1, a_2) \in R[1/p]^2 \text{ with } \langle a_1, a_1 \rangle = D_1, \langle a_2, a_2 \rangle = D_2, \langle a_1, a_2 \rangle = n \}
\]
is preserved by the conjugation action of \( \Gamma_p \) and is a union of finitely many orbits for this action.

**Proof of lemma.** Consider the definite quadratic form attached to \((a_1, a_2) \in \Pi(D_1, D_2, n)\):
\[
F(X, Y) = \text{disc}(Xa_1 + Ya_2) = D_1X^2 + 2nXY + D_2Y^2.
\]
Let \( B_{12} \) be the Clifford algebra of the quadratic space \((\mathbb{Q}^2, F)\). It is a quaternion algebra over \( \mathbb{Q} \) with a basis \( \{1, e_1, e_2, e_1e_2\} \) satisfying
\[
e_1^2 = D_1, \quad e_2^2 = D_2, \quad \text{Tr}(e_1e_2) = 2n.
\]
Let \( R_{12} \) be the \( \mathbb{Z}[1/p] \)-order in \( B_{12} \) generated by \( e_1 \) and \( e_2 \). The map sending \( e_i \) to \( a_i \) is an embedding of \( \mathbb{Z}[1/p] \)-orders
\[
t : R_{12} \hookrightarrow R[1/p],
\]
and simultaneous conjugation of \((a_1, a_2) \) by \( \Gamma_p = R[1/p]_1^\times \) corresponds to conjugation of the embedding. The index of any such embedding is bounded by \( D_1D_2 - n^2 \), and hence there are at most finitely many \( \Gamma_p \)-conjugacy classes of such embeddings. In particular, there are only finitely many possible pairs \((a_1, a_2) \in \Pi(D_1, D_2, n) \) up to \( \Gamma_p \)-equivalence. The result follows. \( \square \)

**End of proof of Theorem 7.** Lemma 8 and the discussion preceding it implies that \( \Sigma_{12}^{\leq N} \) is finite. Furthermore, if \( g \) belongs to the complement of \( \Sigma_{12}^{\leq N} \), then \( v_p(n_g) > p^N \), and hence
\[
(\gamma_1; g\gamma_2) = \frac{n_g + \sqrt{D_1D_2}}{n_g - \sqrt{D_1D_2}} \equiv 1 \pmod{p^N}.
\]
It follows that for any given \( N \geq 1 \), all but finitely many factors in the infinite product (10) are congruent to 1 modulo \( p^N \). This infinite product therefore converges absolutely \( p \)-adically, as was to be shown. \( \square \)
We now turn to the algebraicity properties of the quantity $(\gamma_1 \ast \gamma_2)_{\Gamma_p}$ of (10). Assume for simplicity that the discriminants $D_1$ and $D_2$ are fundamental and relatively prime, and let

$$F_1 = \mathbb{Q}(\sqrt{D_1}), \quad F_2 = \mathbb{Q}(\sqrt{D_2}), \quad F_{12} := \mathbb{Q}(\sqrt{D_1D_2})$$

be the real quadratic fields of discriminants $D_1$, $D_2$, and $D_1D_2$. Let $H_1$ and $H_2$ denote the associated narrow Hilbert class fields, and let $H_{12}$ be their compositum. Let $\mathbb{Q}_{p^2}$ be the quadratic unramified extension of $\mathbb{Q}_p$, and let

$$\iota_p : H_{12} \longrightarrow \mathbb{Q}_{p^2}$$

be a $p$-adic embedding extending the $p$-adic embedding of $F_{12}$ that was made in defining $(\gamma_1 \ast \gamma_2)_{\Gamma_p}$. Conjecture 2 of the introduction predicts that the $p$-arithmetical intersection belongs to $\iota_p(H_{12})$, at least in some particular settings. In general, it need not be algebraic, but the extent to which it may fail to be is well understood, at least conjecturally. Recall that $S$ is the discriminant of the quaternion algebra that was used to define the arithmetic group $\Gamma$. Let $T$ be the “good Hecke algebra” of level $Sp$, generated by operators $T_n$ with index $n$ prime to $pS$. It acts as correspondences on the formal $\mathbb{Z}$-module generated by the modular geodesics, and the symbol $(\gamma_1 \ast \gamma_2)_{\Gamma_p}$ can be extended to such formal linear combinations by multiplicativity.

**Conjecture 9.** Let $T \in \mathbb{Z}$ be any Hecke operator which annihilates the space of cusp forms of weight 2 on $\Gamma_0(Sp)$. Then the quantity $(\gamma_1 \ast T\gamma_2)_{\Gamma_p}$ belongs to $\iota_p(H_{12})$.

**Remark 10.** In particular, $(\gamma_1 \ast \gamma_2)_{\Gamma_p}$ is predicted to always be algebraic when there are no cusp forms of weight two and level $Sp$, which occurs precisely when the set $S$ (of even cardinality) is empty and $p = 2, 3, 5, 7$, or 13. This is the setting that arises in Conjecture 2 and that was studied extensively in [DV]. Conjecture 9 has been explored in more general quaternionic settings in [GMX].

**Remark 11.** The hypothesis on the Hecke operator $T$ implies that $(\gamma_1 \ast T\gamma_2)_{\Gamma_p} = 0$. The $p$-arithmetical intersection in $H_{12}$ might best be envisioned as a higher operation in cohomology, akin to a linking number or a Massey product.

Assuming Conjecture 9, the quantity $(\gamma_1 \ast \gamma_2)_{\Gamma_p}$ can be viewed as an element of $H_{12}$, depending on the choice of a prime of $H_{12}$ that lies above $m_p$. To suppress the dependence on this choice, it is convenient to think of $(\gamma_1 \ast \gamma_2)_{\Gamma_p}$ as an element of $H_{12}/G_{12}$, where $G_{12} := \text{Gal}(H_{12}/F_1F_2)$. This is what shall be done from now on:

**Definition 12.** The $p$-arithmetical intersection of $\gamma_1$ and $\gamma_2$ is the element

$$(\gamma_1 \ast \gamma_2)_{\Gamma_p} \in H_{12}/G_{12},$$

whose image under a suitable embedding of $H_{12}$ into $\mathbb{Q}_{p^2}$ coincides with the infinite product of (10).

An appealing variant of Conjecture 9 occurs when $\Gamma_p = \text{SL}_2(\mathbb{Z}[1/p])$ arises from the globally split quaternion algebra, and $\gamma_1$ is a geodesic joining a pair of cusps rather than a pair of conjugate real irrationalities. Suppose for instance that $\gamma_1 = \gamma_\omega := (0, \infty)$ is the so-called “winding element”: the geodesic corresponding to the imaginary axis in $\mathbb{H}$, whose endpoints are the roots of the binary quadratic form $F_1 = XY$ of discriminant 1. Assume that $\gamma_2 = \gamma = (\tau, \tau')$ is the geodesic joining the roots of a binary quadratic form $F$ whose discriminant $D > 0$ satisfies $(\frac{D}{p}) = -1$. Letting

$$\Pi_\tau := \{z \in \Gamma \tau \text{ satisfying } zz' < 0 \text{ and } 0 \leq \text{ord}_p(z) < 2\},$$

...
one has

\[(\gamma_w \ast \gamma)_{\Gamma_p} = \prod_{z \in \Omega_p} \frac{z}{\tau} = \prod_{-b + \sqrt{D}} -b - \sqrt{D},\]

where the second product ranges over the binary quadratic forms \(ax^2 + bxy + cy^2\) of discriminant \(D\) with coefficients in \(\mathbb{Z}[1/p]\) that are \(\Gamma_p\)-equivalent to \(F(x, y)\) and satisfy \(0 \leq \text{ord}_p(a) < 2\).

For instance, when \(\tau = -1 + \sqrt{3}\), a real quadratic irrationality of discriminant 12 with ring class field \(\mathbb{Q}(\sqrt{3}, \sqrt{-3})\). One finds experimentally that

\[\begin{align*}
(\gamma_w \ast \gamma)_{\Gamma_3} &\equiv -1 + 2i, \\
(\gamma_w \ast \gamma)_{\Gamma_5} &\equiv ((-13 + 3\sqrt{-3})/2)^{1/3}.
\end{align*}\]

Note that the latter is not contained in the ring class field of \(\tau\), though its third power is. In the language of rigid meromorphic cocycles used in [DV], which we have avoided in this paper, this phenomenon may be explained due to the presence of cohomological torsion.

When \(\tau = (-7 + \sqrt{77})/2\) of discriminant 77, we find

\[\begin{align*}
(\gamma_w \ast \gamma)_{\Gamma_2} &\equiv (-5983 + 2115\sqrt{-7})/2, \\
(\gamma_w \ast \gamma)_{\Gamma_3} &\equiv -679 + 80\sqrt{-11}, \\
(\gamma_w \ast \gamma)_{\Gamma_5} &\equiv 3 + 36\sqrt{-11}.
\end{align*}\]

Finally, we consider the discriminant 321 of narrow class number 6, also contained in the introduction. Here we compute that \((\gamma_w \ast \gamma)_{\Gamma_7}\) satisfies

\[7^4x^6 - 20976x^5 - 270624x^4 + 526859689x^3 - 649768224x^2 - 12092465776x + 7^{16},\]

In conclusion, it appears that whenever \(p - 1\) divides 12, the quantities \((\gamma_w, \gamma)\) are algebraic, and more precisely, that up to a small power they are \(p\)-units in the ring class field of discriminant \(D\). This somewhat degenerate variant of Conjecture 9 has been proved in [DPV1, DPV2] by studying the diagonal restrictions of \(p\)-adic deformations of Hilbert modular Eisenstein series.

When \(p = 11\) or \(p > 13\), i.e., when the modular curve \(X_0(p)\) has nonzero genus, the \(p\)-arithmetic intersections are arguably even more interesting, because of the following conjecture:

**Conjecture 13.** The \(p\)-adic logarithm of \((\gamma_w \ast \gamma)_{\Gamma_p}\) is a finite linear combination of \(p\)-adic logarithms of a \(p\)-unit of \(H\) and of formal group logarithms of global points on \(J_0(p)(H)\).

For instance, when \(\tau := \frac{3 + \sqrt{21}}{6}\) is a real quadratic irrationality of discriminant 21, it was verified to 100 digits of 11-adic precision that:

\[\log_{11} \left(\frac{3 + 4\sqrt{-7}}{2}\right) = \frac{1}{3} \log_{11} \left(3 + 4\sqrt{-7}\right) - \frac{1}{5} \log_{E} \left(\frac{-3 - \sqrt{-7}}{2}, \frac{-3 - \sqrt{-7}}{2}\right),\]

where \(\log_{E}\) is the 11-adic formal group logarithm on the elliptic curve \(E\) of conductor 11 with Weierstraß equation

\[E : y^2 + y = x^3 - x^2 - 10x - 20.\]
4. Incoherent intersections

Let \( D_1 \) and \( D_2 \) be a pair of positive discriminants. For ease of exposition, assume that they are fundamental and co-prime. As in earlier sections, write

\[
F_1 := \mathbb{Q}(\sqrt{D_1}), \quad F_2 := \mathbb{Q}(\sqrt{D_2}), \quad F_{12} := \mathbb{Q}(\sqrt{D_1D_2}),
\]

and let \( O_1, O_2, \) and \( O_{12} \) denote their respective rings of integers. Let

\[
S := \left\{ \text{rational primes } p \text{ such that } \left( \frac{D_1}{p} \right) = \left( \frac{D_2}{p} \right) = -1 \right\}.
\]

The primes in \( S \) are split in the intermediate field \( F_{12} \). For each \( p \in S \), fix an ideal \( m_p \) of \( F_1F_2 \) above \( p \). This determines, for each such \( p \), an element \( \delta_p \in \mathbb{F}_p \) satisfying

\[
\delta_p^2 = D_1D_2.
\]

Let \( S \) be a finite subset of \( S \).

When \( S \) has even cardinality, it determines an arithmetic group \( \Gamma(S) \subset \text{SL}_2(\mathbb{R}) \) consisting of the norm one elements in a maximal order

\[
R(S) \subset B(S) \subset M_2(\mathbb{R})
\]

of the quaternion algebra \( B(S) \) ramified exactly at the primes of \( S \). The group \( \Gamma := \Gamma(S) \) depends only on \( S \), up to conjugation in \( \text{SL}_2(\mathbb{R}) \). Let \( (\gamma_1, \gamma_2) \) be any pair of modular geodesics on \( \Gamma(S) \setminus \mathcal{H} \). One may then consider the topological and arithmetic intersection numbers

\[
(\gamma_1 \cdot \gamma_2)_S := (\gamma_1 \cdot \gamma_2)_\Gamma(S), \quad (\gamma_1 \ast \gamma_2)_S := (\gamma_1 \ast \gamma_2)_\Gamma(S) \in F_{12}.
\]

Assume now that \( S \subset S \) has odd cardinality. For each prime \( p \in S \), we may choose a pair \( (\gamma_1, \gamma_2) \) of modular geodesics on \( \Gamma(S \setminus \{p\}) \setminus \mathcal{H} \) satisfying the conditions

\[
O_{\gamma_1} = O_1, \quad O_{\gamma_2} = O_2, \quad \eta_q(\gamma_1, \gamma_2) = \delta_q, \text{ for all } q \in S \setminus \{p\}.
\]

Let \( T \) be a good Hecke operator that annihilates the space of cuspidal newforms of weight 2 and level \( S \). Assuming Conjecture 9, one may then consider the \( p \)-arithmetic intersection numbers

\[
(\gamma_1 \ast T\gamma_2)_{S\setminus\{p\},p} := (\gamma_1 \ast T\gamma_2)_{\Gamma(S\setminus\{p\})},
\]

as global invariants in \( H_{12}/G_{12} \). It appears that these quantities depend only on \( S \) and not on the choice of \( p \in S \) that was made to compute them \( p \)-adically:

**Conjecture 14.** There is an element \( (\gamma_1 \ast T\gamma_2)_S \in H_{12}/G_{12} \) such that

\[
(\gamma_1 \ast T\gamma_2)_S = (\gamma_1 \ast T\gamma_2)_{S\setminus\{p\},p}, \quad \text{for all } p \in S.
\]

The odd set \( S \) of primes can be viewed as the data for an incoherent indefinite quaternion algebra over \( \mathbb{Q} \) in the sense of \([\text{Gr}2]\) and \([\text{Gr}3]\). These references focus on incoherent definite data, where \( S \) contains the archimedean place, and explain how such data naturally corresponds to a Shimura curve over \( \mathbb{Q} \). When \( \infty \not\in S \), the arithmetic quotients \( \Gamma(S \setminus \{p\}) \setminus \mathcal{H} \) for \( p \in S \) can be envisaged as an “incoherent collection of Shimura curves”. Conjecture 14 rests on the strong analogy between compatible systems of geodesics like \( \gamma_1 \) and \( T\gamma_2 \) on \( \Gamma(S \setminus \{p\}) \setminus \mathcal{H} \) (with \( \infty \not\in S \)) and CM points on the Shimura curve attached to an odd set \( S \) containing \( \infty \).
Definition 15. The conjectural quantity
\[(\gamma_1 \ast T\gamma_2)_S \in H_{12}/G_{12}\]
of Conjecture 14 is called the \textit{incoherent intersection number} attached to the pair \((\gamma_1, T\gamma_2)\).

Remark 16. Conjecture 14 adds nothing to Conjecture 9 when \(S = \{p\}\) is a singleton, since \((\gamma_1 \ast \gamma_2)_p\) can then only be computed one way, as a \(p\)-arithmetic intersection number for \(\text{SL}_2(\mathbb{Z}[1/p])\). This is the case that was considered in [DV]. The recent work [GMX] of Guitart, Masdeu and Xarles gives striking experimental confirmation of (a more general version of) Conjecture 14 in the case where \(S = \{2, 3, 5\}\), observing that, for a few pairs \((D_1, D_2)\) of discriminants in which 2, 3 and 5 are inert, and for suitable Hecke operators \(T\) annihilating \(S_2(\Gamma_0(30))\), one indeed has a global invariant
\[(\gamma_1 \ast T\gamma_2)_{2, 3, 5} \in H_{12}/G_{12}\]
which coincides with
\[(\gamma_1 \ast T\gamma_2)_{\Gamma(15)} \cong (\gamma_1 \ast T\gamma_2)_{\Gamma(10)} \cong (\gamma_1 \ast T\gamma_2)_{\Gamma(6)},\]
at least up to the high level of 2-adic, 3-adic and 5-adic accuracies respectively with which these three quantities were computed numerically. The calculations in [GMX] involve the three neighbouring quaternion algebras, of discriminants 15, 10 and 6 respectively, in a way that evokes the Cerednik Drinfeld “interchange of invariants” occuring in the \(p\)-adic uniformisation theory of Shimura curves. What mathematical structure might underly the incoherent collections of Shimura curves and their compatible systems of modular geodesics in accounting for the global nature of \((\gamma_1 \ast T\gamma_2)_{2, 3, 5} \in H_{12}/G_{12}\), remains to be elucidated.

5. Factorisation conjectures

To each finite subset \(S \subset \mathcal{S}\), we have associated quantities \((\gamma_1 \ast \gamma_2)_S\) whose arithmetic nature depends crucially on the parity of the cardinality of \(S\).

- When \(S\) has even cardinality,
\[(\gamma_1 \ast \gamma_2)_S := (\gamma_1 \ast \gamma_2)_{\Gamma(S)} \in F_{12}\]
is an arithmetic intersection number on the indefinite quaternion algebra ramified at the places of \(S\).

- When \(S\) has odd cardinality, the incoherent intersection number
\[(\gamma_1 \ast \gamma_2)_S \in H_{12}/G_{12}\]
lies significantly deeper, and is defined analytically as a \(p\)-arithmetic intersection number on the indefinite quaternion algebra ramified at \(S - \{p\}\), for any \(p \in S\). Conjecturally, it belongs to \(H_{12}\) and is independent (up to Galois conjugacy) on the choice of \(p \in S\) that is made to compute it.

For any \(q \in \mathcal{S}\), the prime \(m_q\) of \(F_1F_2\) splits completely in \(H_{12}\), and the set of primes \(\mathcal{Q}\) of \(H_{12}\) lying above \(m_q\) is therefore a principal homogeneous space for \(G_{12}\). Define a map
\[
\text{Ord}_{m_q} : H_{12}^\times/G_{12} \rightarrow \mathbb{Z}[G_{12}]/G_{12}
\]
by choosing a fixed \(\mathcal{Q}\) above \(m_q\) and setting
\[
\text{Ord}_{m_q}(x) = \sum_{\sigma \in G_{12}} \text{ord}_\mathcal{Q}(x^\sigma)\sigma^{-1}.
\]
The value of \(\text{Ord}_{m_q}\) does not depend on the choice of prime \(\Omega\) that was made to define it, since replacing \(\Omega\) by another prime above \(m_q\) merely has the effect of multiplying the right hand side of \([14]\) by a group-like element in \(G_{12} \subset \mathbb{Z}[G_{12}]^\times\).

The goal of Conjecture 17 below is to relate \(\text{Ord}_{m_q}((\gamma_1 \star \gamma_2)_S)\), when \(S\) is of odd cardinality, to the \(m_q\)-adic valuations of certain arithmetic intersection numbers.

Given \(q \in S\), let
\[
S_q := \begin{cases} S & \text{if } q \in S, \\ S \cup \{q\} & \text{if } q \notin S. \end{cases}
\]
The set \(S_q\) then has even cardinality, and the indefinite quaternion algebra \(B(S_q)\) represents a *nearby quaternion algebra* for the incoherent datum \(S\).

Recall that the narrow class groups \(G_1 := \text{Pic}^+(O_{\gamma_1})\) and \(G_2 := \text{Pic}^+(O_{\gamma_2})\) act simply transitively on the set of geodesics on \(\Gamma(S_q)\) of discriminants \(D_1\) and \(D_2\) with the given orientations at \(q \in S\). Since the extensions \(H_1\) and \(H_2\) are linearly disjoint over \(F_1F_2\), one has a canonical identification of \(G_1 \times G_2 = G_{12}\). For each \(q \in S\), let
\[
\text{Ord}_{m_q}((\gamma_1 \star \gamma_2)_S) := \sum_{\sigma \in G_{12}} (\gamma^\sigma_1 \star \gamma^\sigma_2)_S \cdot \sigma^{-1} \in \mathbb{Z}[G_{12}]/G_{12}.
\]
Viewing the target as \(\mathbb{Z}[G_{12}]\) modulo the action of the group-like elements \(G_{12} \subset \mathbb{Z}[G_{12}]^\times\) means that \(\text{Ord}_{m_q}((\gamma_1 \star \gamma_2)_S)\) is independent of the choice of geodesics \(\gamma_1\) and \(\gamma_2\), but depends only on their discriminants and on the orientations \(\eta_r(\gamma_1, \gamma_2)\) for \(r \in S_q\).

**Conjecture 17.** Let \(S \subset C\) be a set of odd cardinality, and let \(T\) be a Hecke operator that annihilates the space of modular forms of weight two and level \(S\). If \(q \notin S\) and \(q \nmid D_1D_2\), then
\[
\text{ord}_\Omega((\gamma_1 \star T\gamma_2)_S) = 0,
\]
for any prime \(\Omega\) of \(H_{12}\) above \(q\). For all \(q \in S\),
\[
\text{Ord}_{m_q}((\gamma_1 \star T\gamma_2)_S) = \text{Ord}_{m_q}((\gamma_1 \star T\gamma_2)_{S_q}).
\]

Conjecture 17 suggests in particular that, for all \(p \in S\),
\[
\text{ord}_p((\gamma_1 \star T\gamma_2)_{\Gamma(S-\{p\})}) = \text{ord}_{m_q}((\gamma_1 \star T\gamma_2)_{\Gamma(S-\{p\})}),
\]
where the \(p\)-arithmetic intersection number \((\gamma_1 \star \gamma_2)_{\Gamma(S-\{p\})}\) of the left hand side is viewed as an element of \(\mathbb{Q}_p\). The methods used to prove \([DV\] Thm. 3.26\) allow one to establish this result unconditionally by elementary means. The details are left to the assiduous reader.

**References**


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