A very short note on homotopy λ-calculus

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The homotopy λ-calculus is a hypothetical (at the moment) type system. To some extent one may say that $H\lambda$ is an attempt to bridge the gap between the "classical" type systems such as the ones of PVS or HOL Light and polymorphic type systems such as the one of Coq. The main problem with the polymorphic type systems lies in the properties of the equality types. As soon as we have a universe U of which $Prop$ is a member we are in trouble. In the Boolean case, $Prop$ has an automorphism of order 2 (the negation) and it is clear that this automorphism should correspond to a member of $Eq(U; Prop, Prop)$. However, as far as I understand there is no way to produce such a member in, say, Coq. A related problem looks as follows. Suppose $T, T' : U$ are two type expressions and there exists an isomorphism $T \to T'$ (the later notion of course requires the notion of equality for members of T and T'). Clearly, any proposition which is true for T should be true for T' i.e. for all functions $P : U \to Prop$ one should have $P(T) = P(T')$. Again as far as I understand this can not be proved in Coq no matter what notion of equality for members of T and T' we use.

Here is the general picture as I understand it at the moment. Let us consider the type system TS which is generated by the sequents

$$\vdash U_i : U_{i+1}$$

(for $i = -1, 0, 1, \ldots$) and the rules:

1. $$\frac{\Gamma \vdash T : U_i}{\Gamma \vdash T : U_{i+1}} \qquad \frac{\Gamma \vdash T : U_i}{\Gamma \vdash T : Type}$$

2. The usual dependent $\prod$-rules (inside each U_n)

3. The usual dependent $\sum$-rules with strong elimination (inside each U_n)[1]

The system $H\lambda$ is supposed to be an extension of TS. In $H\lambda$, U_{-1} becomes the empty type $\emptyset$ and U_0 becomes $Prop$. The natural numbers are defined (see (1) below) in terms of U_1.

Let CC be the contexts category of TS. By a model of TS with values in a category D, I mean a functor $CC \to D$ which "preserves the relevant structures". The main observation is that there is a canonical model M of TS with values in the usual homotopy category H provided that we consider homotopy types based on a sufficiently large universe of sets. To define this model one starts with a not-so-canonical model N of TS with values in the category of spaces (actually simplicial sets, but I will speak of spaces since they provide a more familiar model for homotopy types) and then sets M to be the composition of N with the projection $Spc \to H$. The main properties of N are are follows.

[1] We may also consider systems TS_X where X is any "recursive" partially ordered set such that U_x is defined for any $x \in X$ and the rules are modified accordingly. If X is just a finite set with the trivial ordering then it seems that TS_X will be just the usual typed λ-calculus with products generated by n primitive types. The first system with real dependencies is TS_X where $X = \{0, 1\}$ with the usual ordering.

1. By definition N takes a context Γ to a space $N(\Gamma)$.
2. A sequent of the form $\Gamma \vdash T : Type$ (where T is an expression) defines a morphism
$$(\Gamma, x : T) \to \Gamma$$
in CC. Morphisms of this type go to *fibrations*
$$N(\Gamma; T) : N(\Gamma, x : T) \to N(\Gamma),$$
3. A sequent of the form $\Gamma \vdash t : T$ (where T and t are expressions) defines a morphism
$$\Gamma \to (\Gamma, x : T)$$
in CC. Morphisms of this type go to *sections*
$$N(\Gamma; T, t) : N(\Gamma) \to N(\Gamma, x : T)$$
of $N(\Gamma; T)$.

Given $\Gamma \vdash P : Type$ and $\Gamma, x : P \vdash Q : Type$ we can form $\Gamma \vdash \prod x : P.Q$ and $\Gamma \vdash \sum x : P.Q$. On the model level our data defines two fibrations
$$N(\Gamma, x : P, y : Q) \xrightarrow{q} N(\Gamma, x : P) \xrightarrow{p} N(\Gamma)$$
The fibration
$$N(\Gamma, z : \prod x : P.Q) \to N(\Gamma)$$
is the "$p_*(q)$". Its fiber over $x \in N(\Gamma)$ is the space of sections (continuous ones!) of the fiber of q over x.

The fibration
$$N(\Gamma, z : \sum x : P.Q) \to N(\Gamma)$$
is the "$p_!(q)$". It is simply the composition of p and q. The meaning of term constructors associated with $\sum$ and $\prod$ is the obvious one. If we took a model with values in *Sets* where all maps are fibrations we would get the usual rules for interpretation of $\sum$ and $\prod$ but formulated in a slightly unusual way.

The rigorous description of the value of N on U_n's is complicated. Up to homotopy equivalence, the space $N(U_n)$ is the nerve of the n-groupoid of $(n-1)$-groupoids in the ZF with $n-2$ universes (see below for the explicit form in the case $n \leq 1$). Alternatively, one may say that U_n is the base of the universal fibration whose fibers are $(n-1)$-types which lie in ZF with $n-2$ universes (so that itself it lies in the ZF with $n-1$ universe. The equivalence of these two points of view follows from the fact that n-groupoids are the same as n-homotopy types.

In particular,
$$M(U_{-1}) = \emptyset$$
and
$$M(U_0) = \{0, 1\}.$$

We further have

$$M(U_1) = \coprod_{n\geq 0} BS_n$$

where BS_n is the classifying space of the permutation group on n elements (BS_0 is empty, BS_1 is one point and BS_2 is homotopy equivalent to $\mathbf{R}P^\infty = B\mathbf{Z}/2$). In particular, $\pi_0(M(U_1)) = \mathbf{N}$ and one uses U_1 to define the type of natural numbers in $H\lambda$. As far as I understand at the moment $M(U_2)$ is $\coprod_{X\in u_2} BAut(X)$ where u_2 is the set of equivalence classes of all groupoids with sets of morphisms and objects being ZF-sets. Starting with U_2 one needs ZF with universes in order for the model to be defined. The model of U_3 is the nerve of the 3-groupoid of 2-groupoids in ZF with one universe.

This model is very "incomplete" in the sense that there are many type expressions T such that $M(T)$ is non-empty while T has no terms in TS. This is of course unavoidable because of the Goedel's theorem. However, some of these incompletenesses are of a special kind. For example $M(U_{-1}) = \emptyset$ hence we may add the empty type rule

$$\frac{\Gamma \vdash c : U_{-1} \quad \Gamma \vdash T : Type}{\Gamma \vdash \iota(c, T) : T}$$

which expresses the fact that if the empty type is inhabited in a context then any other type is. It does not look provable in TS.

Other examples of such rules involve the equality types. Given a valid type expression $T : U_n$ and two term expressions $t1, t2 : T$ we get on the level of models a space $X = M(T)$ (up to homotopy) and two points $x1, x2 \in X$. One of the most important observations concerning the picture outlined so far is that it is possible to define equality (equivalence) types $Eq(T; t1, t2)$ in TS such that the model of $Eq(T; t1, t2)$ is (homotopy equivalent to) the space $P(X; x1, x2)$ of paths from $x1$ to $x2$ in X.

The definition proceeds in the following steps:

Define the contractibility on the level of TS. Set

$$true = (U_{-1} \to U_{-1}) : U_0$$

$$false = U_{-1} : U_0.$$

For $T, T' : U_0$ set

$$Equiv(T, T') = (T \to T') \times (T' \to T).$$

For $T : U_n$ set

$$Contr(T) = \prod F : U_n \to U_0.Equiv(F(T), F(true))$$

then $M(Contr(T)) \neq \emptyset$ iff $M(T)$ belongs to the same connected component of $M(U_n)$ as $M(true)$ i.e. if $M(T)$ is a contractible space. In that case $M(Contr(T))$ is itself contractible.

Define representable functors on the level of TS. Suppose $T : U_{n+1}$ is a type (expression). I want to think of its model $X = M(T)$ as of the nerve of some n-groupoid in U_{n+1}. The members of T correspond to objects. For $T = U_n$ we get the n-groupoid of all $n-1$-groupoids. Functions $T \to U_n$ correspond to functors from T to the groupoid of all groupoids. Among these functors there are representable ones i.e. we have the homotopy type $Rep(T)$ which maps to $T \to U_n$. For $F : T \to U_n$ set

$$rep(F) = Contr(\sum t : T.F(T)).$$

One verifies that on the level of models $rep(F) \neq \emptyset$ iff F is representable. Set

$$Rep(T) = \sum F : T \to U_n.rep(F).$$

then the model of $Rep(T)$ is the space of representable functors on T. By abuse of notation I will write $F(t) : U_n$ instead of the formal $(\pi F)(t)$ for $F : Rep(T)$ and $t : T$.

Define the equality types. For $T : U_n$ and $T1, t2 : T$ one sets:

$$Eq(T; t1, t2) = \prod F : Rep(T).F(t1) \to F(t2)$$

where I write $F(t)$ for $F : Rep(T)$ instead of the correct but long $(\pi F)(t)$.

Theorem 1 *There is a homotopy equivalence*

$$M(Eq(T; t1, t2)) = P(M(T); M(t1), M(t2)).$$

Once the equality types (path spaces) are defined many other constructions familiar on the model level can be formulated on the level of the type system. The first thing to define is the *level* "filtration" on type expressions or, equivalently on the types U_n. The model of U_n has a natural filtration by subspaces $U_{n,k}$, $k = 0, \dots, n$ where $U_{n,k}$ is (the nerve of) the k-groupoid of $(k-1)$-groupoids in the universe U_n. In particular $U_{n,1}$ is the (nerve of) the usual groupoid of sets in U_n and their isomorphisms. We define a (-1)-groupoid as a set where any two elements are equal i.e. one of the two sets $\emptyset$ and pt. Hence for any $n \geq 0$ the model of $U_{n,0}$ is the two point set $\{0, 1\} = \{true, false\}$.

$$\begin{array}{ccccccccc}
U_{0,0} & = & U_{1,0} & = & U_{2,0} & = & U_{3,0} & = & \dots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & U_{1,1} & \longrightarrow & U_{2,1} & \longrightarrow & U_{3,1} & \longrightarrow & \dots \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & U_{2,2} & \longrightarrow & U_{3,2} & \longrightarrow & \dots \\
 & & & & & & \downarrow & & \\
 & & & & & & U_{3,3} & \longrightarrow & \dots
\end{array}$$

All the arrows are inclusions with the image being a disjoint union of some of the connected components of the target and the usual arguments a-la Russell's paradox imply that except for the ones marked as equalities the arrows are proper inclusions e.g. $U_{2,1}$ (which is responsible for sets in U_2) is strictly larger than $U_{1,1}$ (which is responsible for sets in U_1) etc.

To get $U_{n,k}$'s as type expressions we first define type expressions $Lv_k(T) : U_0$ for $T : U_n$ which are "indicator functions" for $U_{n,k}$ setting:

$$Lv_{-1}(T) = Contr(T)$$

and for $k \geq 0$

$$Lv_k(T) = \prod t1 : T. \prod t2 : T.Lv_{k-1}(Eq(T; t1, t2)).$$

Then

$$U_{n,k} = \sum T : U_n.Lv_k(T).$$

One verifies easily that this definition is consistent with the model level definition given above. We will also use the following notations. For $F : T' \to T$ and $t : T$ set

$$Hfiber(F, t) = \sum t' : T'.Eq(T; t, F(t'))$$

one verifies easily that the model of $Hfiber$ is the homotopy fiber of F over t. Set further

$$isheq(F) = \prod t : T.Contr(hfiber(F, t)).$$

This is a truth value and the model of $isheq(f)$ is $true$ iff the model of f is a homotopy equivalence. Set further

$$Heq(T', T) = \sum F : T' \to T.isheq(F)$$

then $M(Heq(T', T))$ is the space of homotopy equivalences from T' to T. For a type expression T set:

$$\Pi_{-1}(T) = (T \to U_{-1}) \to U_{-1}$$

it is a truth value and on the model level $\Pi_{-1}(T) = true$ iff T is not empty. For $F : T' \to T$ set

$$Im(F) = \sum t : T.\Pi_{-1}(Hfiber(F, t)).$$

The model of $Im(F)$ is the union of connected components of T whose pre-image under F is non-empty. Set

$$ev(T', T) = \lambda t : T.\lambda F : T' \to T.F(t) : T' \to ((T' \to T) \to T)$$

$$\Pi_0(T) = Im(ev(T, U_0)).$$

The model of $\Pi_0(T)$ is the set of connected components of T.

We can now give more examples of things which hold on the model level but (probably) can not be proved on the level of TS or even TS with the empty type rule.

1. The natural maps $U_{n,0} \to U_{n+1,0}$ are equivalences on the model level for $n \geq 0$. It seems to be unprovable in TS. To fix it one may add the rule

$$\frac{\Gamma \vdash T : Type \quad \Gamma \vdash a : Lv_0(T)}{\Gamma \vdash T : U_0}.$$

   Alternatively one can impose stabilization together with the Boolean rule by adding a term constructor

$$\frac{\Gamma \vdash T : Type \quad \Gamma \vdash a : Lv_0(T)}{\Gamma \vdash boo(T, a) : ((T \to \emptyset) \to \emptyset) \to T}$$

2. Set
$$\mathbf{N} = \Pi_0(U_1). \tag{1}$$
The model of $\mathbf{N}$ is of course $\mathbf{N}$ – the set of natural numbers. It is not clear however to what extend $\mathbf{N}$ is a natural numbers object in the sense of type theory.

3. For $T : U_n$ we may consider T also as a member of U_{n+1}. Thus we have two definitions of $Rep(T)$ one using U_n and another one using U_{n+1}. They should agree i.e. any $F : T \to U_{n+1}$ such that $rep(F) = true$ should factor through U_n. More precisely there are two expressions $Rep_n(T)$ and $Rep_{n+1}(T)$ and it should be possible to construct a function $Rep_{n+1}(F) \to Rep_n(F)$ which is "inverse" to the obvious one going in the opposite direction.

4. Given $T, T' : U_n$ there are two different expressions which both model to the space of equivalences from T' to T. One is $Eq(U_n; T', T)$ and another one is $Heq(T', T)$. So we should have an equivalence
$$Heq(T', T) \to Eq(U_n; T', T).$$
Again it is unclear how to construct it on the level of the type system.

A subtle thing about imposing all these properties on TS is that while they all hold for M it is not clear which ones one may get on the level of N. In particular the stabilization of $U_{0,k}$'s does not hold for the version of N which I have been considering. For example $N(U_{0,2})$ is much smaller as a space then $N(U_{0,3})$ since there are many more one point sets in ZF with a universe then there are in "pure" ZF.

The context category $CH\lambda$ of $H\lambda$ has a structure reminiscent of a Quillen model structure or rather of the structure of a category with fibrations and weak equivalences considered by Baues in "Algebraic Homotopy". The associated homotopy category HH is some sort of a free homotopy category.

Originally, I was considering a different approach to $H\lambda$ where the equality types where introduced as "primitives" along with $\sum$ and $\prod$ and the universes where "defined" but it seems to me now that it is nicer to start with $\sum$, $\prod$ and universes and define the equality types later. What is left from this earlier stage is certain understanding of which properties of/structures on the equality types might be sufficient to ensure that they behave nicely (e.g. that for any $t : T$, $\pi_1(T; t) = \Pi_0(Eq(T; t, t))$ is a group or that there is a long exact sequence of "homotopy groups" associated to a fibration).

The advantage of $H\lambda$ and its homotopy-theoretic model over the less sophisticated type systems is that it better reflects the way mathematicians envision "types" corresponding to mathematical structures of higher level. For example if we fix the size of the universe n and write in the usual way the type expression for, say, the type $Gr(U_n)$ of groups in U_n then the model of $Gr(U_n)$ will be (the nerve of) the groupoid of groups in the universe U_n and their isomorphisms. Similarly, if we write down the definition of a category in a proper way then the model of $Cat(U_n)$ will be (the nerve of) the 2-groupoid of categories in U_n, their equivalences and natural isomorphisms between equivalences. Moreover, any construction on categories described in the language of $H\lambda$ is automatically "invariant" under equivalences of categories. E.g. any function we can describe in $H\lambda$ from $Cat(U_n)$ to $Gr(U_n)$ will on the model level correspond to a construction which produces a group from a category which maps equivalences between categories to isomorphisms between the corresponding groups. In the usual type systems we can do something like that for types of "level 1" i.e. sets with structures but not for higher levels (e.g. categories).

At the moment much of what I said above is at the level of conjectures. Even the definition of the model of TS in the homotopy category is non-trivial. Similarly, the definition of equality types in terms of universes is rather involved and I am not sure which of the properties of these types have to be imposed so that the rest will follow.

October 13, 2009 Thoughts about future automation/formalization of mathematical proofs.

The following structure is suggested for math assistants based on type systems. Such an assistant should consist of the following components:

1. A type system - which may be inconsistent,

2. A subset of this type system TS_{ZF} which need not be closed under any operation but which comes together with an algorithm verifying that a given sequent belongs to TS_{ZF},

3. A formally constructed model of TS_{ZF} "with values in Zermelo-Fraenkel set theory" (see below). Better yet a program ("compiler") which transfers proofs which belong to this subsystem to ZF-theorems.

4. Potentially, a machinery for the extension of the type system TS_{ZF} which is built in such a way that an attempt to extend the underlying TS will generate a "proof obligation" in ZF.

5. Provisions for the flavors of ZF.

October 18, 2009 We construct a type system $H\lambda$ which has an essentially unique model $M_{H\lambda} : FC(H\lambda) \to H$ with values in the homotopy category H corresponding to a set theory with large cardinals. Due to this uniqueness of the model, the types and terms of this system have an intrinsic semantics. We suggest that this type system provides a natural starting point for the formalization of contemporary mathematics.

We further suggest that it is important to consider not only $H\lambda$ and M but all quadruples of the form $(TS, \phi_{TS}, M_{TS}, \psi_{TS})$ where TS is a type system (or, strictly speaking any recursive set-level category), $M_{TS} : FC(TS) \to H$ is a model of TS in H, ϕ_{TS} a contextual functor $FC(H\lambda) \to FC(TS)$ and ψ_{TS} a natural isomorphism $M_{H\lambda} \to M_{TS}\phi_{TS}$.

From this perspective $H\lambda$ provides an initial formalization and different quadruples as above provide semantics-preserving enrichments of this formalization. Recognizing that the goal of constructing convenient extensions of $H\lambda$ over M is unlimited we concentrate here on the construction of the back-bone structure $(H\lambda, M_{H\lambda})$.

November 4, 2009 Notes for the Munich talk.

Main message 1: I suggest that there is "the" canonical model for polymorphic dependent type systems with intensional equality.

Main message 2: an outline of the structure of the future "proof environment".

1. Coq-like proof assistant. 2. Formalized extension of the core model to the type system of the proof assistant. An implementation of this extension in the form of a program which translates proofs in that type system into (formal) proofs in extended ZFT. 3. A tool which allow to extend

and modify the underlying type system (including the introduction of new axioms and new rewriting rules) which accepts only modifications which are supplied together with an "authorization" code which is essentially a formal (in ZFT) construction of an extension of the core model to the modified type system.

What is to be done:

A. General algebraic theory of type systems.

1. Formal "algebraic" definition of "a type system". 2. Notion of a type system defined by rules with an easy proof that any rule system defines a type system. 3. Notion of a free rule system (roughly corresponding to a rule system such that a sequent in the corresponding type system carries with it a well defined up to a well defined equivalence sequence of rules which led to it), 4. A theorem which shows that any system of pre-universes (in an lcc?) with additional structures directly corresponding to the rules defines a closed model of a free type system.

B. The core universe system in $\Delta^{op}Sets$.

1. Definition of univalent fibrations. Properties of such fibrations. 2. Construction of the universal univalent fibration corresponding to an inaccesible cardinal. 3. Construction of the "standard" Π, Σ and $Prop$ structures on this universe (straightforward).

C. Applications to pCIC-Coq.

1. Proof that pCIC is a type system defined by a free rule system. 2. Construction of the structures corresponding to the induction rules of Coq on the core universes in $\Delta^{op}Sets$.

D. Design and implementation of a seed Coq-like (Coq-based ?) system grounded through the core model in the ZFT with tools for the addition after verification (in the sense of compatibility with the core model) of new axioms and re-writing rules.

1. Several years ago I understood something new about how to construct models for dependent type systems.

2. The precise formulation of my ideas about models is most conveniently achieved through the use of Cartmell-Streicher contextual categories.

3. Pre-universe structure on a category. Category $CC(\mathcal{C}, p)$. Notion of a closed model.

November 13, 2009. To Munich lecture.

$Coq \longrightarrow Type\ \ systems \longrightarrow Universe\ \ maps \longrightarrow Univalent\ \ fibrations.$

Scenario:

1. Axiom $cls : forall\ P : Prop, P \vee \neg P$.

2. Theorem $100monkeys : exists\ n : nat, n = n + 1$. Proof. .... Qed.

Why can we be sure that if such a thing happened it would mean that the current foundations of mathematics are inconsistent?

3. Theorem (100monkeys edited) $th1 : False$. Proof. .... Qed.

Why is this impossible?

The proof will be translated by Coq into a sentence of the form

$$cls : forall\ P : Prop, P \vee \neg P \vdash \mathbf{th1} : False$$

where **th1** will be an expression (possibly a very long one) with one free variable named "cls".

Qed command gives instructions to a small independent sub-program called a proof-checker to verify that the sentence

$$cls : forall\ P : Prop, P \vee \neg P \vdash \mathbf{th1} : False$$

is correct. "Correct" from the point of view of this program means "well formed" or "grammatically correct" a condition which is very easy to verify.

To make sure that 100monkeys theorem would indeed mean that we have to reconsider all of the contemporary mathematics we have to prove, using current foundations of mathematics, that there does not exist any grammatically well formed sentence of the form

$$cls : forall\ P : Prop, P \vee \neg P \vdash \mathbf{th1} : False \tag{2}$$

All well formed sentences are of one of the two types:

$$C_n = \{x_1 : T_1; \ldots; x_n : T_n\}$$

$$J_n = \{x_1 : T_1; \ldots; x_n : T_n \vdash t : T\}$$

where $x_1, \ldots, x_n$ are names of variables and

$$T_i \in Expressions(\{x_1, \ldots, x_{i-1}\}), \qquad t, T \in Expressions(\{x_1, \ldots, x_n\})$$

They are called contexts and term sequents. Our main example is a term sequent.

The only currently known approach to proving that a 100monkeys theorem is "impossible" is based on construction of a mapping which assigns to any element $\{x_1 : T_1; \ldots; x_n : T_n\}$ of C_n a sequence of maps of sets

$$M(x_1 : T_1; \ldots; x_n : T_n) \xrightarrow{p_{T_1;\ldots;T_n}} \ldots\ldots\ldots \xrightarrow{p_{T_1;T_2}} M(x_1 : T_1) \xrightarrow{p_{T_1}} pt$$

and to any element $\{x_1 : T_1; \ldots; x_n : T_n \vdash t : T\}$ of J_n a sequence of maps of sets

$$M(x_1 : T_1; \ldots; x_n : T_n; T) \xrightarrow{p_{T_1;\ldots;T_n;T}} M(x_1 : T_1; \ldots; x_n : T_n) \xrightarrow{p_{T_1;\ldots;T_n}} \ldots$$

together with a section

$$s(t) : M(x_1 : T_1; \ldots; x_n : T_n) \to M(x_1 : T_1; \ldots; x_n : T_n; T)$$

of $p_{T_1;\ldots;T_n;T}$ and such that

1. $M(cls : forall\ P : Prop, P \vee \neg P) \neq \emptyset$
2. for any $\{x_1 : T_1; \ldots; x_n : T_n\}$ in C_n, $M(x_1 : T_1; \ldots; x_n : T_n; False) = \emptyset$.

Then (2) would map to a sequence

$$M(cls : forall\ P : Prop, P \vee \neg P; False) \xrightarrow{p} M(cls : forall\ P : Prop, P \vee \neg P)$$

$$\to pt$$

together with a section of p which is impossible since $M(cls : forall\ P : Prop, P \vee \neg P; False)$ is empty and $M(cls : forall\ P : Prop, P \vee \neg P)$ is not.

Formal languages whose valid sentences have the form as above are, somewhat informally, called ”type systems” and mappings of the form described above are called (generalized, set-theoretic, classical) models of a type system.

In order to be sure that a proof of a 100monkeys theorem in a given proof assistant should make us worried about the foundations of mathematics (which is equivalent to saying that a theorem whose proof is accepted by this proof assistant may be considered proved by the mathematical community) we should know that:

1. There is a formal specification of the type system which underlies the proof assistant,
2. There is an accepted mathematical proof that a set-theoretic model of this type system exists,
3. There are several independent implementations of the type checker for this type system.

I will add to this a strengthening of the second requirement which I think is important

(*) There should be a formalized description of a set-theoretic model which would allow one to formally verify the possibility of adding new axioms to the basic context.

The current state in the case of Coq:

1. formal specification exists but is not easy to find,

2. a mathematically acceptable construction of set-theoretical model does not exist (to the best of my knowledge),

3. independent proof checkers do not exist (to the best of my knowledge).

The Coq type system is very sophisticated. What is known about set-theoretic models of less complex type systems?

Example: ECC (extended calculus of constructions) developed by Luo in the late 80-ies early 90-ies.

This theory has constants $Type_0, Type_1, \ldots,$ (i.e. contexts of the form $\{x : Type_i\}$ where x is any name of a variable, are valid) together with "constructors" Π, λ, ev, Σ, π_1, π_2 and $pair$.

It also has rules which allow one to consider members of the types $Type_i$ to be types themselves. More precisely, for each $i \geq 0$, $\{T : Type_i, x : T\}$ is a valid context (as always x, T are names of variables).

A set-theoretic model of ECC defines for each i and any two variable names x, T a map of sets

$$p_{i,T,x} : M(T : Type_i, x : T) \to M(T : Type_i)$$

Since everything should be invariant under the re-naming of variables this map actually depends only on i. Let us denote $M(T : Type_i, x : T)$ by $\widetilde{U}_i$, $M(T : Type_i)$ by U_i and ths map by

$$p_i : \widetilde{U}_i \to U_i.$$

If we assume that the model which we consider behaves well with respect to variable substitution then it follows from the rules of ECC that there should be pull-back squares of the form

$$\begin{array}{ccc} U_i & \longrightarrow & \widetilde{U}_{i+1} \\ \downarrow & & \downarrow \\ pt & \longrightarrow & U_{i+1} \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{U}_i & \longrightarrow & \widetilde{U}_{i+1} \\ \downarrow & & \downarrow \\ U_i & \longrightarrow & U_{i+1} \end{array}$$

The maps p_i are called the universe maps corresponding to a model.

We can now ask:

Q1. Does there exist a set-theoretic model of ECC such that for at least one context $\{x_1 : T_1; \ldots ; x_n : T_n\}$ one has $M(x_1 : T_1; \ldots ; x_n : T_n) = \emptyset$?

Q2. Is there a natural choice for such a model?

The answer to the first question is known to be "yes". The answer to the second is "no". A model corresponds to the choice of the maps p_i together with some additional structures on these maps which are called Π- and Σ-structures and while one may argue that there are natural candidates for maps p_i in $Sets$ there are many non-equivalent choices of Π and Σ structures on these maps.

It turns out however that there are "universal" maps p_i in the homotopy category which carry canonical Π and Σ structures which allows one to construct a very interesting model of ECC with values in this category.

We will use simplicial sets $\Delta^{op}Sets = Funct(\Delta^{op}, Sets)$ as a model for H.

The basics of simplicial homotopy theory:

1. A morphism $p : E \to B$ is called a Kan fibration if for any commutative square of the form
$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & E \\ \downarrow & & \downarrow \\ \Delta^n & \longrightarrow & B \end{array}$$
there exists a morphism $\Delta^n \to E$ which makes the two triangles commutative. A Kan simplicial set is a simplicial set X such that $X \to pt$ is a Kan fibration.

2. A morphism $i : A \to X$ is called anodyne if for any commutative square of the form
$$\begin{array}{ccc} A & \longrightarrow & E \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

such that p is a Kan fibration, there exists a morphism $X \to E$ which makes the two triangles commutative,

3. Localization of $\Delta^{op}Sets$ by anodyne morphisms is called the homotopy category H. Morphisms which become isomorphisms in H are called weak equivalences.

There are two fundamental theorems:

1. Factorization theorem: for any $f : X \to Y$ there exists a factorization $X \stackrel{i}{\to} X' \stackrel{f'}{\to} Y$ such that i is anodyne and f' is a Kan fibration,
2. Representation theorem: H is equivalent to the category of Kan simplicial sets and homotopy classes of maps (where homotopy is $h : X \times \Delta^1 \to Y$).

Consider the diagonal $X \to X \times X$. We can factorize it as $X \to PX \to X \times X$ where the first map is anodyne and the second a fibration. If X is Kan then one can take the space of simplicial paths $\underline{Hom}(\Delta^1, X)$ as PX.

Let $p : E \to B$ be a (Kan) fibration. Then $Fp : \underline{Hom}_{B\times B}(E \times B, B \times E) \to B \times B$ is a Kan fibration whose fiber over $(b_1, b_2) \in B$ is the space of maps $\underline{Hom}(E_{b_1}, E_{b_2})$. There is a natural section of Fp over the diagonal which extends (by the definition of an anodyne map) to a morphism

$$mm0(p) : PB \to \underline{Hom}_{B\times B}(E \times B, B \times E)$$

One can easily see that for any $b_1, b_2 \in B$ the map

$$mm0(b_1, b_2) : P(B, b_1, b_2) \to \underline{Hom}(E_{b_1}, E_{b_2})$$

defined by $mm0$ takes values in the subspace

$$Heq(E_{b_1}, E_{b_2}) \subset \underline{Hom}(E_{b_1}, E_{b_2})$$

of maps which are weak equivalences and thus defines a map

$$mm(b_1, b_2) : P(B, b_1, b_2) \to Heq(E_{b_1}, E_{b_2})$$

Definition 2 *A fibration p is called univalent if for any $b_1, b_2 \in B$ the map $mm(b_1, b_2)$ is a weak equivalence.*

Univalent fibrations have many interesting properties. Among them is the property that the space of Π and Σ structures on a univalent fibration is always either empty or contractible.

Moreover, if we consider univalent fibrations which have a Π structure and at least one empty fiber it turns out that they are organized into a sequence whose starting terms are

$$\begin{array}{ccccc} \widetilde{U}_0 & \longrightarrow & \widetilde{U}_1 & \longrightarrow & \dots \\ {\scriptstyle p_0}\downarrow & & {\scriptstyle p_1}\downarrow & & \\ U_0 & \longrightarrow & U_1 & \longrightarrow & \dots \end{array}$$

where $U_0 = \{0, 1\}$ and $U_1 = \amalg_{n \geq 0} BS_n$.

Using these fibrations one obtains a canonical model of the ECC with values in H which I call the univalent model.

Moreover, it is possible, at least in the ECC extended by the Martin-Lof "intensional identity types" to formulate an axiom whose validity implies the univalence of the universe maps p_i. It appears that the ECC together with identity types and this "equivalence axiom" has an essentially unique well behaved model which necessarily takes vales in the homotopy category and which provide unambiguous "meanings" to its sentences.

Next steps:

1. Formulate and proof the above assertions as mathematical theorems,

2. Try to extend it the univalent model to the type system of Coq (including inductive types).

Q2. Which maps p_i can appear as universe maps of such a model?

Q3. Given a choice of the maps p_i, is a model corresponding to this choice unique (up to an equivalence)?

To explain the third question:

Definition 3 *Two models M and N are called (logically) equivalent if for any $\{x_1 : T_1, \ldots, x_n : T_n\} \in C_n$ one has*

$$(M(T_1, \ldots, T_n) = \emptyset) \Leftrightarrow (N(T_1, \ldots, T_n) = \emptyset)$$

The answers to the questions Q1-Q3 are as follows:

1. Yes.
2. Assuming some natural additional conditions on the model, one can take any sequence of p_i's such that all isomorphism classes of sets of cardinality $< \alpha_i$ appear as fibers of p_i (where $\alpha_0 = 2$, $\alpha_1 = \aleph_0$ and α_i is the "i-th inaccessible cardinal").
3. No. Moreover, I believe that one can prove that for any choice of p_i's such that there exist at least one model with these universe maps there exist many non-equivalent ones.

The negative answer to Q3 and the fact that many non-equivalent choices of p_i's are possible means that ECC is not rigid enough - there is no way to unambiguously assign "meaning" to its sentences even after the sizes of the type universes are choosen.

This is a well known issue in type theory to which I want to offer a solution:

1. A new axiom expressible in ECC which is called an equivalence axiom,

2. A (generalized) set-theoretic model satisfying this axiom,

3. A pre-theorem that suggests that any two models M_1, M_2 of (ECC + the equivalence axiom) such that the cardinality of $M_1(Type_i)$ is equal to the cardinality of $M_2(Type_i)$ are equivalent.

The precise formulation of the equivalence axiom in the ECC is rather technical (see my "Short note on homotopy λ-calculus" from 2006. There is an easier way to formulate this axiom if we first add to the ECC the (constructors for) the Martin-Lof intensional identity type. See ....

Let me concentrate on the model which explains the axiom. This model, by its nature, takes values in the homotopy category rather than in the category of sets. The set theoretic model is obtained from

While I think it might be possible to express