BRYAN-STEINBERG PAIRS AND A NON-CALABI-YAU WALL CROSSING

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Contents

1. Introduction

Let X be a projective threefold with rational, Gorenstein singularities with a resolution of singularities $f : Y \to X$ of relative dimension at most 1. When Y is a Calabi-Yau threefold, Bryan-Steinberg defined enumerative invariants associated to such resolutions which then they compared with the Donaldson-Thomas, or equivalently Pandharipande-Thomas invariants of Y. In this article, we extend their definition to the case when Y is not necessarily Calabi-Yau, and we conjecture a relation between the generating functions of these invariants and of Pandharipande-Thomas invariants of Y. We check the conjecture for the contraction $Y \to X$ of a rational curve C with normal bundle $N_{C/Y} = \mathcal{O}(-1)^{\oplus 2}$ using degeneration and localization techniques to reduce to a Calabi-Yau situation, which we then treat using the Joyce's motivic Hall algebra.

1.1. **Bryan-Steinberg pairs.** Let X be a projective threefold with rational, Gorenstein singularities with a resolution of singularities $f : Y \to X$ of relative dimension at most 1. The Gorenstein singularities condition means that X has a dualizing sheaf ω_X, and the rational singularities condition means that $Rf_*\mathcal{O}_Y = \mathcal{O}_X$.

Bryan-Steinberg defined invariants which "count" certain two term complexes, which we call BS pairs, whose generating series equals the ratio of two PT generating series, see the statement of Theorem 1.2. This ratio of PT series, or equivalently the analogous ratio of DT series [7], appears in the statement of the DT crepant resolution conjecture of Bryan-Cadman-Young [4]. The introduction of BS pairs was a first step towards proving the DT crepant resolution conjecture by giving a geometric interpretation of the ratio of PT series. The conjecture was eventually proved by Beentjes-Calabrese-Rennemo [3] using motivic Hall algebra techniques.

Let $\mathrm{Coh}_{\leqslant 1}(Y)$ be the abelian category of coherent sheaves on Y with support of dimension at most 1. Consider the torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathrm{Coh}_{\leqslant 1}(Y)$ defined as follows: $\mathcal{T} \subset \mathrm{Coh}_{\leqslant 1}(Y)$ is the full subcategory of sheaves T such that

$$Rf_*T \in \mathrm{Coh}_{\leqslant 0}(X)$$

is a sheaf on X with support of dimension at most zero, and $\mathcal{F} \subset \mathrm{Coh}_{\leqslant 1}(Y)$ is its complement. A two-term complex

$$\mathcal{O}_Y \xrightarrow{s} F$$

for F a sheaf on Y is called a *Bryan-Steinberg pair* if $F \in \mathcal{F}$ and $\mathrm{coker}\,(s) \in \mathcal{T}$. When X is smooth and f is the identity, Bryan-Steinberg (BS) pairs are the same as Pandharipande-Thomas (PT) pairs.

Denote by $N_1(Y)$ the abelian group of curves in Y up to numerical equivalence, by $N_1^{\mathrm{exc}}(Y) \subset N_1(Y)$ the subspace of curves supported on the exceptional locus, and let $N_{\leqslant 1}(Y) = N_1(Y) \oplus \mathbb{Z}$ and $N_{\leqslant 1}^{\mathrm{exc}}(Y) = N_1^{\mathrm{exc}}(Y) \oplus \mathbb{Z}$. Consider the moduli stack $\mathfrak{M}(\mathcal{O})$ of sheaves on Y with a section. Bryan-Steinberg showed that the locus

$$\mathrm{BS}_n(Y, \beta) \subset \mathfrak{M}(\mathcal{O})$$

of BS pairs $\mathcal{O}_Y \to F$ with $\mathrm{ch}\,(F) = (\beta, n) \in N_{\leqslant 1}(Y)$ is a finite-type constructible set [9]. When Y is Calabi-Yau, the BS invariants are defined using the Behrend function $\nu : \mathfrak{M}(\mathcal{O}) \to \mathbb{Z}$:

$$\mathrm{BS}(Y, \beta, n) := \sum_{n \in \mathbb{Z}} n\, \chi(\nu^{-1}(n)).$$

In Section 3, we define analogous enumerative invariants when Y is not necessarily Calabi-Yau by integrating against some natural virtual fundamental classes.

Theorem 1.1. *Let X be a projective threefold with rational, Gorenstein singularities with a resolution of singularities $f : Y \to X$ of relative dimension at most* 1*, and let $(\beta, n) \in N_{\leqslant 1}(Y)$. The functor Φ_{BS} of Bryan-Steinberg pairs $\mathcal{O}_Y \to F$ with $ch\,(F) = (\beta, n) \in N_{\leqslant 1}(Y)$ is representable by a proper algebraic space $BS_n(Y, \beta)$ with a natural virtual fundamental class.*

By the Artin representability criterion, the functor Φ_{BS} is representable by a proper algebraic space if Φ_{BS} is open (Proposition 3.4, follows from work of Beentjes-Calabrese-Rennemo [3]), bounded (already proved by Bryan-Steinberg), separated (Proposition 3.5), complete (Proposition 3.6), and has trivial automorphisms (Proposition 3.2). The proof of separatedness and completeness is similar to Langton's argument that the moduli of semistable sheaves on a projective variety has the same properties [20]. To obtain a virtual fundamental class on the space $\mathrm{BS}_n(Y, \beta)$, it is enough to show, by a theorem of Huybrechts-Thomas [13], that BS pairs are open in the derived category of complexes on Y, see Theorem 3.7. This follows from work of Beentjes-Calabrese-Rennemo [3].

1.2. **Wall-crossing between BS and PT invariants.** When Y is Calabi-Yau, consider the generating series of BS invariants [9]:

$$\mathrm{BS}(q) := \sum_{(\beta,n)\in N_{\leqslant 1}(Y)} \mathrm{BS}(Y,\beta,n)q^{\beta+n}.$$

The generating series $\mathrm{PT}(q)$ of PT invariants is defined similarly; we also consider the generating series

$$\mathrm{PT}^{\,\mathrm{exc}}(q) := \sum_{(\beta,n)\in N^{\mathrm{exc}}_{\leqslant 1}(Y)} \mathrm{PT}(Y,\beta,n)q^{\beta+n}.$$

Both generating series are elements of the Laurent ring $\mathbb{C}[\Delta]_\Phi$, see Definition 2.4. Bryan-Steinberg proved the following wall-crossing theorem using identities in the motivic Hall algebra of $\mathfrak{M}(\mathcal{O})$ and the Joyce-Song integration map:

Theorem 1.2. *Let $f : Y \to X$ be as in Theorem 1.1 and let Y be Calabi-Yau. The generating series for BS and PT invariants are related by:*

$$BS(q) = \frac{PT(q)}{PT^{exc}(q)}.$$

For Y not necessarily Calabi-Yau, we can define BS and PT invariants with insertions, see Subsections 3.3 and 2.6, respectively. Let $(\beta,n) \in N_{\leqslant 1}(Y)$ and consider insertions $\gamma_1,\cdots,\gamma_r \in f^*H^{\cdot}(X,\mathbb{Z})$ and descendent levels $k_1,\cdots,k_r \geqslant 0$. We define BS invariants with insertions $\langle \tau_{k_1}(\gamma_1),\cdots,\tau_{k_r}(\gamma_r)\rangle_{\beta,n}$, see Subsection 3.3. Consider the generating series of BS invariants with insertions $\gamma_1,\cdots,\gamma_r$ and descendant levels $k_1,\cdots,k_r \geqslant 0$:

$$\mathrm{BS}_\gamma(q) := \sum_{(\beta,n)\in N_{\leqslant 1}K(Y)} \langle \tau_{k_1}(\gamma_1),\cdots,\tau_{k_r}(\gamma_r)\rangle_{\beta,n}\, q^{\beta+n}.$$

The definitions of the generating series $\mathrm{PT}_\gamma(q)$ and $\mathrm{PT}^{\mathrm{exc}}_\gamma(q)$ are similar.

Conjecture 1.3. *Let $f : Y \to X$ be as in Theorem 1.1, and consider insertions $\gamma_1,\cdots,\gamma_r \in f^*H^{\cdot}(X,\mathbb{Z})$ and arbitrary descendant levels $k_1,\cdots,k_r \geqslant 0$. The generating series for BS and PT invariants with these corresponding insertions and descendant levels are related by:*

$$BS_\gamma(q) = \frac{PT_\gamma(q)}{PT^{exc}_\gamma(q)}.$$

This conjecture is similar in spirit to the DT/PT correspondence conjectured by Pandharipande-Thomas and to the DT crepant resolution conjecture of Bryan-Cadman-Young. The DT/PT correspondence was proved in the Calabi-Yau case by Bridgeland [7], Toda [28] using the machinery of motivic Hall algebras developed by Joyce [14], [15], [16], and Joyce-Song [17]. When Y is Calabi-Yau, Conjecture 1.3 follows from Theorem 1.2. In section 5, we check the above conjecture in a particular case without assuming that Y is Calabi-Yau:

Theorem 1.4. *Conjecture 1.3 holds for* $f : Y \to X$ *the contraction of a curve* $C \cong \mathbb{P}^1$ *with normal bundle* $N_{C/Y} = \mathcal{O}(-1)^{\oplus 2}$.

We next explain the main steps that of the proof of Theorem 1.4:

Step 1. In Section 4, we define relative BS pairs and prove a degeneration formula for BS pairs similar to the degeneration formula for DT, PT pairs proved by Li-Wu [21].

Step 2. In Section 5, we use the degeneration formulas for BS and PT invariants for the family

$$\mathrm{Bl}_{C\times 0}(Y \times \mathbb{A}^1) \to \mathrm{Bl}_{p\times 0}(X \times \mathbb{A}^1)$$

over $\mathbb{A}^1$, where $p \in X$ is the singular point. The fiber over zero is

$$\mathrm{Bl}_C Y \cup_S \mathbb{P},$$

where we use the notations $\mathbb{P} = \mathbb{P}_C(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O})$ and $S = \mathbb{P}_C(\mathcal{O}(-1)^{\oplus 2})$. When using the degeneration formulas for PT and BS invariants for this family, the insertions will be in the $\mathrm{Bl}_C Y$ part. Theorem 1.4 follows from a correspondence between generating series of relative invariants $\mathrm{BS}(\mathbb{P}/S)$ and $\mathrm{PT}(\mathbb{P}/S)$ with no insertions.

Step 3. Also in Section 5, we use the localization theorem for the Calabi-Yau torus $T \subset \mathbb{G}_m^3$ acting on $\mathbb{P}$. The T-fixed BS or PT complexes $\mathcal{O}_Y \to F$ restricted to $Y - C$ will be ideal sheaves of a T-fixed curve which intersects S transversely, so they will have certain partition profiles π on the four legs of $\mathbb{P}$. By the localization theorem, the correspondence between relative BS and PT invariants for $\mathbb{P}/S$ follows from a correspondence between BS and PT invariants for the moduli spaces of virtual dimension zero $\mathrm{BS}_n(\pi, m)$ and $\mathrm{PT}_n(\pi, m)$ of T-fixed complexes $\mathcal{O}_Y \to F$ with fixed partition profile π and with $\mathrm{ch}(F) = ([\pi] + m[C], n) \in N_{\leqslant 1}(Y)$.

Step 4. In Section 6, we prove the correspondence between the generating series $\mathrm{BS}_{\pi,m}(q)$ and $\mathrm{PT}_{\pi,m}(q)$ using identities in the motivic Hall algebra of $\mathfrak{M}_T(\mathcal{O})$, the moduli of T-fixed sheaves on $\mathrm{Tot}_C\left(\mathcal{O}(-1)^{\oplus 2}\right)$ with a section.

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1.4. **Notations and conventions.** All the schemes considered are defined over the complex numbers $\mathbb{C}$. For X a variety, denote by $\mathrm{Coh}\,(X)$ the abelian category of coherent sheaves on X and by $D^b(X)$ the derived category of bounded complexes of coherent sheaves on X.

In the definition of BS and PT generating series, we will not specify the variety or the descendant levels.

2. Preliminaries

2.1. **Perfect obstruction theories.** Let X be a proper algebraic space. A two-term complex

$$E^{\cdot} = [E^{-1} \to E^0]$$

of vector bundles on X is called a *perfect obstruction theory* for X is there exists a morphism

$$E^{\cdot} \to \mathbb{L}^{\cdot}_X$$

in the derived category $D^b(X)$ which induces an isomorphism on h^0 and a surjection on h^{-1}. Let $d = \operatorname{rk} E^0 - \operatorname{rk} E^{-1}$ be the rank of $E^{\cdot}$. A perfect obstruction theory induces a *virtual fundamental class* $[X]^{\text{vir}} \in A_d(X), H_{2d}(X, \mathbb{Z})$, see [6].

2.2. **Symmetric perfect obstruction theories.** Let X be a proper algebraic space with a *symmetric perfect obstruction theory*

$$E^{\cdot} \to \mathbb{L}_X,$$

which means that $E^{\cdot} \to \mathbb{L}_X$ is a perfect obstruction theory and there exists a non-degenerate symmetric bilinear form $\theta : E^{\cdot\vee}[1] \to E^{\cdot}$. In this case the virtual dimension is zero, so

$$[X]^{\text{vir}} \in H_0(X, \mathbb{Z}) = \mathbb{Z}.$$

Let $\nu : X \to \mathbb{Z}$ be the associated Behrend function [5]. In loc. cit., Behrend proved that

$$[X]^{\text{vir}} = \sum_{n \in \mathbb{Z}} n\chi(\nu^{-1}(n)).$$

Let M be a smooth algebraic space and $f : M \to \mathbb{C}$ a regular function with zero the only critical value. Let X be the critical locus of f, and let $P \in X$. Then

$$\nu_X(P) = (-1)^{\dim M}(1 - \chi(M_P)),$$

where M_P is the Milnor fiber of f at P, see [5].

2.3. **Localization.** Let X be a proper algebraic space with an action of a torus T, and assume it has a T-equivariant perfect obstruction theory $E^{\cdot}$, meaning that $E \in D^b_T(X)$ and the morphism $E \to \mathbb{L}_X$ is T-equivariant. Assume that X admits an action of a torus T. Let $Z \subset X$ be a T-fixed algebraic space. Every vector bundle V on X splits over Z as $V|_Z = V^f|_Z \oplus V^m|_Z$, where $V^f|_Z \subset V|_Z$ is the sub-bundle where T acts trivially, and $V^m|_Z \subset V|_Z$ is the sub-bundle where T acts with nonzero weight.

Assume that the perfect obstruction theory on X is T-equivariant. Then the T-fixed loci $Z \subset X$ admit perfect obstruction theories

$$E^T|_Z := [E^{-1,f}|_Z \to E^{0,f}|_Z].$$

The virtual normal bundle of the inclusion $Z \subset X$ is defined by

$$N^{\text{vir}} = [E^{-1,m}|_Z \to E^{0,m}|_Z].$$

Let T act on X with fixed connected components $X_1, \cdots, X_n$. The localization formula of Graber-Pandharipande [10], [19] says that:

$$[X]^{\mathrm{vir}} = \sum_{k=1}^{n} \frac{[X_k]^{\mathrm{vir}}}{e(N_k^{\mathrm{vir}})} \in H_\cdot^T(X) \otimes \operatorname{Frac} H_T(\mathrm{pt}).$$

2.4. **Laurent rings.** Let Y be a smooth variety. Then $N_{\leqslant 1}(Y)$ is a finite dimensional abelian group. Consider $\Delta \subset N_{\leqslant 1}(Y)$ the monoid of classes $\mathrm{ch}(F)$, where F is a sheaf on Y with support of dimension at most 1. Let $\mathbb{C}[\Delta]$ be the algebra with underlying $\mathbb{C}$-vector space generated by elements $q^{\beta+n}$ and with multiplication defined by

$$q^{\beta+n}q^{\gamma+m} = q^{(\beta+\gamma)+(n+m)}.$$

The algebra $\mathbb{C}[\Delta]$ is Δ-graded.

Definition 2.1. For a Δ-graded algebra A, define the Laurent completion A_Φ as follows: as a $\mathbb{C}$-vector space, it has elements infinite series

$$x = \sum_{(\beta,n)\in\Delta} x_{\beta,n}$$

such that for every $\beta \in N_1(Y)$, the set $\{n|\, x_{\beta,n} \neq 0\}$ is bounded from below. We call such infinite series *Laurent*. For $(\beta, n) \in N_{\leqslant 1}(Y)$ and for x a Laurent series as above, let

$$\pi_{\beta,n}(x) := x_{\beta,n}.$$

The multiplication on A_Φ is defined as follows: for x and y Laurent series, xy is the Laurent series such that for every $(\beta, n) \in N_{\leqslant 1}(Y)$, we have

$$\pi_{\beta,n}(xy) = \sum_{(\beta,n)=(\gamma,m)+(\delta,p)} \pi_{\gamma,m}(x)\pi_{\delta,p}(y).$$

Such a Laurent algebra comes with a natural topology by imposing that a sequence $(x_n)_{n\geqslant 0}$ of elements in A_Φ converges if for every $(\beta, n) \in N_{\leqslant 1}(Y)$, there exists K such that for every $i, j \geqslant K$ and $n \geqslant m$, and for all large enough i and j, we have that

$$\pi_{\beta,m}(y_i) = \pi_{\beta,m}(y_j).$$

The algebra A_Φ is a topological algebra.

2.5. **Torsion pairs.** Let $\mathcal{C}$ be an abelian category with subcategories $\mathcal{T}, \mathcal{F} \subset \mathcal{C}$. The pair $(\mathcal{T}, \mathcal{F})$ is called *a torsion pair* if:

- For $T \in \mathcal{T}$ and $F \in \mathcal{F}$, we have that $\mathrm{Hom}(T, F) = 0$,
- For every $C \in \mathcal{C}$, there exist $T \in \mathcal{T}$ and $F \in \mathcal{F}$ such that

$$0 \to T \to C \to F \to 0.$$

2.6. **Pandharipande-Thomas pairs.** Let Y be a smooth projective threefold. Consider the abelian category $\mathrm{Coh}_{\leqslant 1}(Y)$ of coherent sheaves with support of dimension at most 1. Let $\mathcal{T} = \mathrm{Coh}_{\leqslant 0}(Y)$ be the category of sheaves with support of dimension at most zero, and let

$$\mathcal{F} = \{F \in \mathrm{Coh}_{\leqslant 1}(Y) | \, \mathrm{Hom}\,(\mathcal{T}, F) = 0\}$$

be the orthogonal of $\mathcal{T}$ in $\mathrm{Coh}_{\leqslant 1}(Y)$. The subcategories $(\mathcal{T}, \mathcal{F})$ form a torsion pair of $\mathrm{Coh}_{\leqslant 1}(Y)$.

Let $(\beta, n) \in N_{\leqslant 1}(Y)$. Denote by $\mathrm{PT}_n(Y, \beta)$ the (fine) moduli space of pairs

$$\mathcal{O}_Y \xrightarrow{s} F$$

such that $\mathrm{ch}\,(F) = (\beta, n) \in N_{\leqslant 1}(Y)$, F is in $\mathcal{F}$, and the cokernel of the section $\mathrm{coker}\,(s)$ in in $\mathcal{T}$. The space $\mathrm{PT}_n(Y, \beta)$ is projective. Consider the universal stable pair

$$\mathbb{I} = [\mathcal{O}_{Y \times \mathrm{PT}_n(Y,\beta)} \to \mathbb{F}] \in D^b(Y \times \mathrm{PT}_n(Y, \beta)).$$

Let $R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0$ be the kernel of the trace map

$$\mathrm{Tr} : R\mathcal{H}om(\mathbb{I}, \mathbb{I}) \to \mathcal{O}_{Y \times \mathrm{PT}_n(Y,\beta)}.$$

The Atiyah class $\mathrm{At} \in \mathrm{Ext}^1(\mathbb{I}, \mathbb{I} \otimes \mathbb{L}_{Y \times \mathrm{PT}_n(Y,\beta)})$ induces a perfect obstruction theory:

$$E^{\cdot} = R\pi_{2*}(R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \otimes \pi_1^* \omega_Y[2]) \to \mathbb{L}_{\mathrm{PT}_n(Y,\beta)}$$

using the results of Huybrechts-Thomas [13], and thus a virtual fundamental class of dimension

$$\dim_{\mathbb{C}} [\mathrm{PT}_n(Y, \beta)]^{\mathrm{vir}} = -\chi(\mathbb{I}, \mathbb{I}).$$

2.7. **Generating series of PT invariants.** Next, we define PT invariants with insertions. Recall the universal stable pair $\mathbb{I} \in D^b(Y \times \mathrm{PT}_n(Y, \beta))$. Consider a cohomology class $\gamma \in H^l(Y, \mathbb{Z})$ and an integer $k \geqslant 0$. Define

$$\mathrm{ch}_{2+k}(\gamma) : H_*(\mathrm{PT}_n(Y, \beta), \mathbb{Q}) \to H_{*-2k+2-l}(\mathrm{PT}_n(Y, \beta), \mathbb{Q})$$

by the formula

$$\mathrm{ch}_{2+k}(\gamma)(-) = \pi_{2*}(\mathrm{ch}_{2+k}(\mathbb{I}) \pi_1^*(\gamma) \cap \pi_2^*(-)).$$

The PT invariants with insertions $\gamma_1, \cdots, \gamma_r \in H^{\cdot}(Y, \mathbb{Z})$ and descendant levels $k_1, \cdots, k_r \geqslant 0$ are defined by:

$$\langle \tau_{k_1}(\gamma_1), \cdots, \tau_{k_r}(\gamma_r) \rangle_{\beta,n} = \int_{[\mathrm{PT}_n(Y,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{2+k_1}(\gamma_1) \cdots \mathrm{ch}_{2+k_r}(\gamma_r).$$

The generating series for PT invariants of class β with the given insertions and descendant levels is defined by:

$$\mathrm{PT}_{\beta,\gamma}(q) = \sum_{n \in \mathbb{Z}} \langle \tau_{k_1}(\gamma_1), \cdots, \tau_{k_r}(\gamma_r) \rangle_{\beta,n} \, q^n.$$

Observe that $\mathrm{PT}_{\beta,\gamma}(q)$ is a Laurent series in q, and thus the generating series

$$\mathrm{PT}_{\gamma}(q) = \sum_{\beta \in N_1(Y)} \mathrm{PT}_{\beta,\gamma}(q) q^{\beta} \text{ and } \mathrm{PT}_{\gamma}^{\mathrm{exc}}(q) = \sum_{\beta \in N_1^{\mathrm{exc}}(Y)} \mathrm{PT}_{\beta,\gamma}(q) q^{\beta}$$

are both elements of the Laurent ring $\mathbb{C}[\Delta]_{\Phi}$.

2.8. **Relative PT invariants.** If $S \subset Y$ is a smooth divisor, there is a Deligne-Mumford stack $\mathrm{PT}_n(Y/S, \beta)$ parametrizing pairs

$$\mathcal{O}_{Y[k]} \xrightarrow{s} F,$$

where $\pi : Y[k] \to Y$ is a k-step degeneration of Y, see [26] for a definition, and F is a sheaf on $Y[k]$ whose support pushes to $\pi_*[F] = \beta \in N_1(Y)$ and $\chi(F) = n$, and such that:

- F is pure and has a finite locally free resolution,
- F intersects the singular loci of $Y[k]$ and the relative divisor $S_\infty \subset Y[k]$ transversally,
- coker(s) has a zero dimensional support and is supported away from the singular loci of $Y[k]$,
- the pair has only finitely many automorphisms covering the automorphisms of $Y[k]/Y$.

The space $\mathrm{PT}_n(Y/S, \beta)$ admits a perfect obstruction theory of the same dimension as that of $\mathrm{PT}_n(Y, \beta)$. The complexes

$$\mathcal{O}_Y \xrightarrow{s} F$$

where F intersects S transversely form an open set of $\mathrm{PT}_n(Y/S, \beta)$, and the restriction of the perfect obstruction theory for this locus is the same as the perfect obstruction theory of $\mathrm{PT}_n(Y, \beta)$. Further, sending a sheaf F to its restriction to S defines a morphism

$$\varepsilon : \mathrm{PT}_n(Y/S, \beta) \to \mathrm{Hilb}\,(S, d),$$

where $d = \beta.S \in H_0(Y, \mathbb{Z}) = \mathbb{Z}$.

Fix a basis $\beta_1, \cdots, \beta_m$ of $H^{\cdot}(S, \mathbb{Q})$. A *cohomologically weighted partition* with respect to β_i is a set of pairs

$$\{(\eta_1, \beta_{l_1}), \cdots, (\eta_s, \beta_{l_s})\},$$

where η_i are non-negative integers and $\sum_{i=1}^{s} \eta_i$ is an unordered partition of $\beta.S = d$. For a cohomologically weighted partition as above, define:

- $l(\eta) = s$,
- $|\eta| = \sum_{i=1}^{s} \eta_i = d$,
- $\mathrm{Aut}\,(\eta)$ the group of permutation symmetries of η, and
- $\xi(\eta) = \prod_{i=1}^{s} \eta_i |\mathrm{Aut}(\eta)|$.

Let $\{C_\eta\}_{|\eta|=d}$ be the basis of $H^\cdot(\mathrm{Hilb}(S,d),\mathbb{Q})$ introduced in [24, Section 3.2.2]. For insertions and descendant levels as above, the relative PT invariants are defined as follows:

$$\langle \tau_{k_1}(\gamma_1),\cdots,\tau_{k_r}(\gamma_r)|\,\eta\rangle_{\beta,n} := \int_{[\mathrm{PT}_n(Y/S,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{2+k_1}(\gamma_1)\cdots \mathrm{ch}_{2+k_r}(\gamma_r)\cap \varepsilon^*(C_\eta).$$

The generating series for the relative PT invariants for fixed class β and insertions $\gamma_1,\cdots,\gamma_k$ is defined by the Laurent series in q:

$$\mathrm{PT}^{\mathrm{rel}}_{\beta,\eta,\gamma}(q) = \sum_{n\in\mathbb{Z}} \langle \tau_{k_1}(\gamma_1),\cdots,\tau_{k_r}(\gamma_r)|\,\eta\rangle_{\beta,n} q^n.$$

2.9. **The degeneration formula for PT invariants.** Let $\mathcal{Y}\to C$ be a smooth fourfold fibered over a smooth curve. Let Y be a nonsingular fiber, and assume that there is a singular fiber of the form

$$\mathcal{Y}_o = Y_1\cup_S Y_2,$$

where Y_1 and Y_2 are two smooth threefolds intersecting in a smooth surface S. Consider the natural inclusions $i: Y\to\mathcal{Y}$, $i_0:\mathcal{Y}_0\to\mathcal{Y}$, $i_1: Y_1\to\mathcal{Y}_0$, $i_2: Y_2\to\mathcal{Y}_0$. Further, consider the morphisms

$$H_2(Y)\xrightarrow{i_*} H_2(\mathcal{Y}) \xleftarrow{i_{0*}} H_2(\mathcal{Y}_0) \xleftarrow{i_{1*}+i_{2*}} H_2(Y_1)\oplus H_2(Y_2).$$

Here, i_{0*} is an isomorphism. Let $\beta\in H_2(Y)$ be a curve class We say that $\beta=\beta_1+\beta_2$ for classes $\beta_1\in H_2(Y_1)$, $\beta_2\in H_2(Y_2)$ if

$$i_*(\beta) = i_{0*}(i_{1*}(\beta_1)+i_{2*}(\beta_2)).$$

Consider the insertions $\gamma_1,\cdots,\gamma_r\in H^\cdot(Y,\mathbb{Z})$ and descendants levels $k_1,\cdots,k_r\geqslant 0$. The Li-Wu degeneration formula [21] relates the PT invariants of Y to the relative PT invariants of Y_1/S and Y_2/S:

$$\mathrm{PT}_{\beta,\gamma}(q) = \sum \mathrm{PT}^{\mathrm{rel},1}_{\beta_1,\eta,\gamma_A}(q)\,\mathrm{PT}^{\mathrm{rel},2}_{\beta_2,\eta^\vee,\gamma_B}(q)\frac{(-1)^{|\eta|-l(\eta)}\xi(\eta)}{q^{|\eta|}}.$$

The sum on the right hand side is taken after all splittings $\beta=\beta_1+\beta_2$ of the curve class, all partitions $A\cup B=\{1,\cdots,r\}$ of the insertions, and after all cohomologically weighted partitions η.

2.10. **Stability conditions.** Let Y be a smooth projective variety. *A stability condition* on $\mathrm{Coh}_{\leqslant 1}(Y)$ consists of a slope function $\mu:\mathrm{Coh}_{\leqslant 1}(Y)\to S$ where $(S,\leqslant)$ is a totally ordered set, such that:

(1) for any exact sequence

$$0\to A\to B\to C\to 0$$

of objects in $\mathrm{Coh}_{\leqslant 1}(Y)$, we have that either

$$\mu(A)<\mu(B)<\mu(C) \text{ or } \mu(A)=\mu(B)=\mu(C) \text{ or } \mu(A)>\mu(B)>\mu(C).$$

(2) any sheaf $F \in \mathrm{Coh}_{\leqslant 1}(Y)$ has a Harder-Narasimhan filtration
$$0 \subset F_1 \subset \cdots \subset F_{n-1} \subset F_n = F$$
such that the factors $\mathrm{gr}^i F = F_i/F_{i-1}$ are semistable and the slopes of the factors satisfy $\mu(\mathrm{gr}^1 F) > \cdots > \mu(\mathrm{gr}^n F)$.

A sheaf $F \in \mathrm{Coh}_{\leqslant 1}(Y)$ is called *(semi)stable* if for any proper subsheaf $0 \neq E \subset F$, we have $\mu(E)(\leqslant) < \mu(F)$.

Let $s \in S$, and define:
$$\mathcal{T}_s = \{T \in \mathrm{Coh}_{\leqslant 1}(Y) | \, T \twoheadrightarrow Q \neq 0, \text{ then } \mu(Q) \geqslant s\}$$
$$\mathcal{F}_s = \{F \in \mathrm{Coh}_{\leqslant 1}(Y) | \, 0 \neq S \hookrightarrow F, \text{ then } \mu(S) < s\}.$$
Then the categories $(\mathcal{T}_s, \mathcal{F}_s)$ form a torsion pair of $\mathrm{Coh}_{\leqslant 1}(Y)$.

The categories $\mathcal{T}_s$ and $\mathcal{F}_s$ can be also described as follows. For a set $A \subset S$, let $SS(A) \subset \mathrm{Coh}_{\leqslant 1}(Y)$ be the subcategory generated by semistable sheaves of slope in A. For $s \in S$, we have that
$$\mathcal{T}_s = SS(\geqslant s) \text{ and } \mathcal{F}_s = SS(< s).$$

2.11. **Moduli stacks and torsion pairs.** Let Y be a smooth projective variety. Lieblich [22] constructed an Artin stack $\mathfrak{LM}$ locally of finite type parametrizing complexes $I \in D^b(Y)$ such that $\mathrm{Ext}^{\leqslant -1}(I, I) = 0$.

Consider the abelian category defined by Toda [28]
$$\mathcal{A} = \langle \mathcal{O}_Y[1], \mathrm{Coh}_{\leqslant 1}(Y) \rangle_{\mathrm{exc}} \subset D^b(Y).$$
For a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathrm{Coh}_{\leqslant 1}(Y)$, we call an object $I \in \mathcal{A}$ *a $(\mathcal{T}, \mathcal{F})$-pair* if it is of the form
$$I \cong [\mathcal{O}_Y \xrightarrow{s} F],$$
where $F \in \mathcal{F}$ and $\mathrm{coker}(s) \in \mathcal{T}$. Further, a torsion pair $(\mathcal{T}, \mathcal{F})$ of $\mathrm{Coh}_{\leqslant 1}(Y)$ is called *open* if the categories $\mathcal{T}, \mathcal{F} \subset \mathrm{Coh}_{\leqslant 1}(Y)$ are open.

The following result is proved in [3, Lemma 4.6]; it is originally stated for some particular cases of Calabi-Yau orbifolds $\mathcal{X}$, but the proof in loc. cit. works also for smooth projective varieties Y.

Lemma 2.2. *Let $(\mathcal{T}, \mathcal{F})$ be an open torsion pair for $Coh_{\leqslant 1}(Y)$ and assume that $Coh_0(Y) \subset \mathcal{T}$. The substack of $\mathfrak{LM}$ parametrizing $(\mathcal{T}, \mathcal{F})$-pairs is open.*

2.12. **The motivic Hall algebra.** We recall the construction of Joyce's motivic Hall algebras from [16], see also [7]. Let $\mathfrak{M}$ be an Artin stack locally of finite type over $\mathbb{C}$ with affine stabilizers.

Definition 2.3. *The relative Grothendieck group over* $\mathfrak{M}$, denoted by $K(\mathrm{St}/\mathfrak{M})$, is the complex vector space of equivalence classes of symbols $[T \xrightarrow{f_T} \mathfrak{M}]$, where T is a locally finite-type stack with affine stabilizers and f_T a morphism of stacks, modulo the relations:

- for T and S Artin stacks over $\mathfrak{M}$ with affine stabilizers, we have

$$[T \cup S \xrightarrow{f_T \cup f_s} \mathfrak{M}] = [T \xrightarrow{f_T} \mathfrak{M}] + [S \xrightarrow{f_S} \mathfrak{M}],$$

- for $f : T \to S$ a map of stacks over $\mathfrak{M}$ with affine stabilizers such that $f : T(\mathbb{C}) \to S(\mathbb{C})$ is an equivalence of groupoids, we have

$$[T \xrightarrow{f_T} \mathfrak{M}] = [S \xrightarrow{f_S} \mathfrak{M}],$$

- and for $h_T : T \to U$ and $h_S : S \to U$ Zariski fibrations of the same dimension over $\mathfrak{M}$, we have

$$[T \xrightarrow{f_U h_T} \mathfrak{M}] = [S \xrightarrow{f_U h_S} \mathfrak{M}].$$

Let $\mathfrak{M}^{(2)}$ the stack of short exact sequences with elements in $\mathcal{A}$, it comes with natural maps

$$\mathfrak{M} \times \mathfrak{M} \xleftarrow{q} \mathfrak{M}^{(2)} \xrightarrow{p} \mathfrak{M},$$

where $p(0 \to A \to B \to C \to 0) = (A, C)$ and $q(0 \to A \to B \to C \to 0) = B$.

Definition 2.4. The *motivic Hall algebra* is the complex vector space $K(\mathrm{St}/\mathfrak{M})$ with the product

$$[S \to \mathfrak{M}] * [T \to \mathfrak{M}] = [E(S,T) \to \mathfrak{M}],$$

where $E(S,T) \to \mathfrak{M}$ is defined by the diagram

$$\begin{array}{ccccc} E(S,T) & \longrightarrow & \mathfrak{M}^{(2)} & \longrightarrow & \mathfrak{M} \\ \downarrow & & \downarrow & & \\ S \times T & \longrightarrow & \mathfrak{M} \times \mathfrak{M}. & & \end{array}$$

$K(\mathrm{St}/\mathfrak{M})$ is an algebra over $K(\mathrm{St}/k)$ by letting

$$[S \to k] \times [T \to \mathfrak{M}] = [S \times T \to \mathfrak{M}].$$

Definition 2.5. The *Grothendieck ring of varieties* $K(\mathrm{Var}/\mathbb{C})$ is the complex vector space of equivalence classes of symbols $[V \to \mathbb{C}]$ with relations

$$[X \to \mathbb{C}] = [U \to \mathbb{C}] + [Z \to \mathbb{C}],$$

where X is a variety over $\mathbb{C}$, $Z \subset X$ is closed, and $U = X - Z$ is its complement.

3. The virtual fundamental class for BS pairs

3.1. **Definition of Bryan-Steinberg pairs.** Let X be a projective threefold with rational, Gorenstein singularities with a resolution of singularities $f : Y \to X$ of relative dimension at most 1.

Such a map $f : Y \to X$ determines a torsion pair $(\mathcal{T}, \mathcal{F})$ for $\mathrm{Coh}_{\leqslant 1}(Y)$ [9], defined as follows: $\mathcal{T} \subset \mathrm{Coh}_{\leqslant 1}(Y)$ is the subcategory of sheaves T such that

$$Rf_* T \in \mathrm{Coh}_{\leqslant 0}(X)$$

and $\mathcal{F}$ is its complement

$$\mathcal{F} = \{F \in \mathrm{Coh}_{\leqslant 1}(Y) | \operatorname{Hom}(\mathcal{T}, F) = 0\}.$$

A complex $I = [\mathcal{O}_Y \xrightarrow{s} F]$ is called a *Bryan-Steinberg (BS) pair* if $F \in \mathcal{F}$ and $\operatorname{coker}(s) \in \mathcal{T}$.

The torsion pair $(\mathcal{T}, \mathcal{F})$ can be obtained as a torsion pair for a stability condition $\mu : \operatorname{Coh}_{\leqslant 1}(Y) \to S$, see Subsection 2.10. Consider the set $S = (-\infty, \infty] \times (-\infty, \infty]$, ordered lexicographically. Fix L an ample line bundle on X, and fix $H > f^*L$ an ample line bundle on Y. Consider the slope map

$$\mu : \operatorname{Coh}_{\leqslant 1}(Y) \to S$$

defined by the formula

$$\mu(F) = \left(\frac{\chi(F)}{\beta . f^* L}, \frac{\chi(F)}{\beta . H} \right),$$

where $\beta \in N_1(Y)$ is the (curve) support of F. This slope defines a stability condition on $\operatorname{Coh}_{\leqslant 1}(Y)$, see [9]. For $a = (\infty, 0)$, the pair $(\mathcal{T}_a, \mathcal{F}_a)$ is the torsion pair used in the definition of BS pairs; for $b = (\infty, \infty)$, the pair $(\mathcal{T}_b, \mathcal{F}_b)$ is the torsion pair used in the definition of PT pairs.

3.2. **The moduli space of BS pairs.** Let I be a BS pair as above. Consider the distinguished triangle

$$I \to \mathcal{O}_Y \to F \xrightarrow{[1]} . \tag{1}$$

Then $h^0(I) = \mathcal{I}_C$ for a one dimensional subscheme $C \subset Y$; let $h^1(I) = Q$. The BS pair I also fits in a distinguished triangle:

$$\mathcal{I}_C \to I \to Q[-1] \xrightarrow{[1]} . \tag{2}$$

Consider the functor $\Phi : (\text{Schemes}/k)^{\text{op}} \to \text{Sets}$ parametrizing BS pairs

$$\Phi(B) = \{\mathcal{O}_{Y \times B} \to \mathcal{F} \text{ such that } \mathcal{O}_{Y \times b} \to \mathcal{F}|_{Y \times b} \text{ is a BS pair for every } b \in B\}/\text{eq}.$$

where two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there exists a line bundle $\mathcal{L}$ on B such that $\mathcal{F} \cong \mathcal{F}' \otimes \pi_2^* \mathcal{L}$.

As explained in the introduction, Theorem 1.1 follows once we show that the functor Φ is bounded (proved in [9], see also [3, Section 8.3]), open, separated, complete, has trivial automorphisms, and that BS pairs are open in the Lieblich stack $\mathfrak{LM}$. We first check that the locus of BS pairs is in the Lieblich stack:

Proposition 3.1. *Let $I = [\mathcal{O}_Y \xrightarrow{s} F]$ be a BS pair. Then $\operatorname{Ext}^{\leqslant -1}(I, I) = 0$.*

Proof. Apply $\operatorname{Hom}(I, -)$ to the triangle (1) to get a long exact sequence

$$\cdots \to \operatorname{Ext}^{i-1}(I, F) \to \operatorname{Ext}^i(I, I) \to \operatorname{Ext}^i(I, \mathcal{O}_Y) \to \cdots .$$

It thus suffices to show that $\operatorname{Ext}^{\leqslant -2}(I, F) = \operatorname{Ext}^{\leqslant -1}(I, F) = 0$. For this, apply $\operatorname{Hom}(-, \mathcal{O}_Y)$ and $\operatorname{Hom}(-, F)$ to the triangle (2) to get

$$\cdots \to \operatorname{Ext}^{i+1}(Q, \mathcal{O}_Y) \to \operatorname{Ext}^i(I, \mathcal{O}_Y) \to \operatorname{Ext}^i(\mathcal{I}_C, \mathcal{O}_Y) \to \cdots$$

$$\cdots \to \operatorname{Ext}^{i+1}(Q, F) \to \operatorname{Ext}^i(I, F) \to \operatorname{Ext}^i(\mathcal{I}_C, F) \to \cdots$$

We have that $\mathrm{Ext}^{\leqslant -1}$ between any those sheaves is zero, and thus the conclusion follows. □

We next check that BS pairs have trivial automorphisms:

Proposition 3.2. *Consider two BS pairs $I = [\mathcal{O}_Y \to F]$ and $J = [\mathcal{O}_Y \to G]$ in $D^b(Y)$. The natural map $Hom(J, I) \to Hom(\mathcal{O}_Y, \mathcal{O}_Y) = \mathbb{C}$ is injective. In particular, morphisms in $Hom(I, I)$ are scalar multiplications.*

Proof. Consider the long exact sequence

$$\cdots \to \mathrm{Hom}(G, \mathcal{O}_Y) \to \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \to \mathrm{Hom}(J, \mathcal{O}_Y) \to \mathrm{Ext}^1(G, \mathcal{O}_Y) \to \cdots$$

obtained by applying $\mathrm{Hom}(-, \mathcal{O}_Y)$ to the triangle (1). The first and the fourth sheaves are zero because G has support of codimension at least 2 in Y, so

$$\mathbb{C} = \mathrm{Hom}(\mathcal{O}_Y, \mathcal{O}_Y) \to \mathrm{Hom}(J, \mathcal{O}_Y)$$

is an isomorphism. Further, consider the exact sequence

$$\cdots \to \mathrm{Ext}^{-1}(J, F) \to \mathrm{Hom}(J, I) \to \mathrm{Hom}(J, \mathcal{O}_Y) \to \cdots$$

obtained by applying $\mathrm{Hom}(J, -)$ to the triangle (1). It is enough to show that $\mathrm{Ext}^{-1}(J, F) = 0$. There is a long exact sequence

$$\cdots \to \mathrm{Hom}(K, F) \to \mathrm{Ext}^{-1}(J, F) \to \mathrm{Ext}^{-1}(\mathcal{I}_D, F) = 0$$

obtained by applying $\mathrm{Hom}(-, F)$ to the distinguished triangle (2) for the pair J

$$\mathcal{I}_D \to J \to K[-1] \xrightarrow{[1]} .$$

The first term is zero by the assumptions that $Q \in \mathcal{T}$ and $F \in \mathcal{F}$, so we obtain that $\mathrm{Ext}^{-1}(J, F) = 0$.

□

Remark 3.3. Two stable pairs are isomorphic if and only the corresponding two term complexes in $D^b(Y)$ are isomorphic. Indeed, starting from a pair $\mathcal{O}_Y \to F$, consider the complex

$$I := [\mathcal{O}_Y \to F].$$

In the above proof we have showed that for such a complex I, we have that $\mathrm{Hom}(I, \mathcal{O}_Y) = \mathbb{C}$. Consider a non-zero morphism $I \to \mathcal{O}_Y$; the sheaf F is the cone of this map and the pair is recovered as $\mathcal{O}_Y \to F$.

Proposition 3.4. *The locus of BS pairs inside $\mathfrak{LM}$ is open. In particular, the functor Φ is open.*

Proof. By Lemma 2.2, the statement follows once we show that the categories $\mathcal{T}, \mathcal{F} \subset \mathrm{Coh}_{\leqslant 1}(Y)$ are open. This was verified in [3, Section 8.2]. □

Next, we discuss that the functor Φ_{BS} is separated and complete. For this, let R be a DVR with fraction field K, residue field k, and uniformizer π.

Proposition 3.5. *Consider a BS family $I = [\mathcal{O}_{Y \times K} \to F]$ over K. There exists at most one flat BS family $\mathcal{I} = [\mathcal{O}_{Y \times R} \to \mathcal{F}]$ over R such that $\mathcal{I}_K \cong I$.*

Proof. Consider two flat BS families over R

$$\mathcal{I}^1 = [\mathcal{O}_{Y\times R} \to \mathcal{F}^1] \text{ and } \mathcal{I}^2 = [\mathcal{O}_{Y\times R} \to \mathcal{F}^2]$$

such that $\mathcal{I}^1_K \cong \mathcal{I}^2_K \cong I$. A relative version over R of Proposition 3.2 shows that the following map is injective

$$\mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^2) \hookrightarrow \mathcal{O}_R.$$

The isomorphism $\mathcal{I}^1_K \cong \mathcal{I}^2_K$ is a section of $\mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^2)$ over K, so there exists a global section of $\mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^2)$, that is, there exists a morphism $\mathcal{I}_1 \to \mathcal{I}_2$ which extends the isomorphism over K. In particular, this means that

$$\mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^2) \cong R.$$

Consider the morphisms $\mathcal{I}^1 \to \mathcal{I}^2$ and $\mathcal{I}^2 \to \mathcal{I}^1$ corresponding to $1 \in R = \mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^2)$ and $1 \in R = \mathcal{H}om_R(\mathcal{I}^2, \mathcal{I}^1)$, respectively. Similarly, we obtain a morphism $\mathcal{I}^2 \to \mathcal{I}^1$ corresponding to $1 \in R \cong \mathcal{H}om_R(\mathcal{I}^2, \mathcal{I}^1)$. Their composite $\varphi : \mathcal{I}^1 \to \mathcal{I}^1$ restricts to an isomorphism $\mathcal{I}^1_K \cong \mathcal{I}^1_K \cong I$. By the relative version of Proposition 3.2, there is a natural isomorphism $\mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^1) \cong R$. Consider the diagram

$$\begin{array}{ccc} \mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^2) \times \mathcal{H}om_R(\mathcal{I}^2, \mathcal{I}^1) & \longrightarrow & \mathcal{H}om_R(\mathcal{I}^1, \mathcal{I}^1) \\ \downarrow & & \downarrow \\ R \times R & \longrightarrow & R. \end{array}$$

The map φ corresponds to $1 \in \mathcal{O}_R$, so the composition

$$\varphi_k : \mathcal{I}^1_k \to \mathcal{I}^2_k \to \mathcal{I}^1_k$$

is an isomorphism. Both $\mathcal{I}^1$ and $\mathcal{I}^2$ have the same Hilbert polynomial, so they are isomorphic. □

Proposition 3.6. *Let $I = [\mathcal{O}_{Y\times K} \to F_K]$ be a BS pair. There exists a flat BS family $\mathcal{I} = [\mathcal{O}_{Y\times R} \to \mathcal{F}]$ over R such that $\mathcal{I}_K \cong I$.*

Proof. Let $\mathcal{H}$ be a flat extension of F_K over R. Then $\mathcal{H}om_R(\mathcal{O}_{Y\times R}, \mathcal{H})$ has a section over K, so it has a non-zero section over R

$$\mathcal{O}_{Y\times R} \xrightarrow{s} \mathcal{H}.$$

All subsheaves of $\mathcal{H}$ are flat over R.

Step 1. We first show that there exists a subsheaf $\mathcal{H}' \subset \mathcal{H}$ such that $\mathcal{H}'_k \in \mathcal{F}$. Assume that this not the case. In particular, this implies that $\mathcal{H}_k$ is not in $\mathcal{F}$. Let $P_0 \in \mathcal{F}$ and $Q_0 \in \mathcal{T}$ be such that

$$0 \to Q_0 \to \mathcal{H}_k \to P_0 \to 0.$$

The sheaf $\mathcal{H}_k$ is not in $\mathcal{F}$, so $Q_0 \neq 0$. Let $\mathcal{H}^1 \subset \mathcal{H}$ be the kernel of the map $\mathcal{H} \to \mathcal{H}/\pi = \mathcal{H}_k \to P_0$. We have a short exact sequence

$$0 \to \mathcal{H}^1 \to \mathcal{H} \to P_0 \to 0.$$

The restriction of the above sequence to the central fiber gives an exact sequence:

$$\mathrm{Tor}_1^R(\mathcal{H}, k) \to \mathrm{Tor}_1^R(P_0, k) \to \mathcal{H}_k^1 \to \mathcal{H}_k \to P_0 \to 0.$$

The first term is $\mathrm{Tor}_1^R(\mathcal{H}, k) = 0$ because $\mathcal{H}$ is flat over R. The second term is $\mathrm{Tor}_1^R(P_0, k) = P_0$, so the sequence becomes

$$0 \to P_0 \to \mathcal{H}_k^1 \to \mathcal{H}_k \to P_0 \to 0.$$

We thus have a short exact sequence

$$0 \to P_0 \to \mathcal{H}_k^1 \to Q_0 \to 0.$$

By our assumption, the sheaf $\mathcal{H}_k^1$ is not in $\mathcal{F}$. Thus, there exists a short exact sequence with $Q_1 \in \mathcal{T}$ and $P_1 \in \mathcal{F}$

$$0 \to Q_1 \to \mathcal{H}_k^1 \to P_1 \to 0.$$

Consider the pushforward of both these sequences to X:

$$0 \to f_*P_0 \to f_*\mathcal{H}_k^1 \to f_*Q_0 \;\text{ and }\; 0 \to f_*Q_1 \to f_*\mathcal{H}_{1,k} \to f_*P_1 \to 0.$$

Let $L = \ker(f_*Q_1 \to f_*\mathcal{H}_{1,k} \to f_*Q_0)$. By diagram chasing we see that $L \subset f_*P_0$, so we obtain a map $f^*L \to P_0$. The sheaf L is in $\mathcal{T}$ because $Rf_*f^*L = L$, which means that $L = 0$, and thus $f_*Q_1 \subset f_*Q_0$. We repeat the procedure above to define $\mathcal{H}^n \subset \mathcal{H}^{n-1}$ for every $n \geqslant 1$, and the above arguments implies that we have a descending sequence of torsion sheaves on

$$f_*Q_n \subset f_*Q_{n-1}.$$

There exists $n_0 \geqslant 0$ such that the above inclusions are equalities for $n \geqslant n_0$. Let $n \geqslant n_0$, and define

$$M = \ker(Q_{n+1} \to \mathcal{H}_k^{n+1} \to Q_n).$$

We have $f_*Q_{n+1} \cong f_*Q_n$ and $R^1f_*Q_{n+1} = R^1f_*Q_n = 0$. This implies that

$$Rf_*M = 0,$$

and thus that $M \in \mathcal{T}$. A diagram chasing implies that $M \subset P_n$, which is possible only if $M = 0$. Thus

$$Q_n \subset Q_{n-1}$$

for $n \geqslant n_0$. There exists $n_1 \geqslant 0$ such that the above inclusion is an equality for $n \geqslant n_1$. Without loss of generality, we can assume that $n_1 = 1$. This implies that the short exact sequences

$$0 \to Q_n \to \mathcal{H}_k^{n+1} \to P_n \to 0$$

split for $n \geqslant 1$. This means that $\mathcal{H}_k^{n+1} = Q_n \oplus P_n$, and thus that $P_n \cong P_{n-1}$ for $n \geqslant 0$.

Next, we construct by induction on $n \geqslant 1$ a natural surjection

$$\mathcal{H}/\pi^n \to \mathcal{H}/\mathcal{H}_n.$$

For $n = 1$, the map is the canonical surjection $\mathcal{H}_k = P_0 \oplus Q_0 \to P_0$. In general, it follows from the surjection:

$$\mathcal{H}^n/\pi = P_0 \oplus Q_0 \to \mathcal{H}^n/\mathcal{H}^{n+1} = P_0.$$

Further, the sheaf $\mathcal{H}/\mathcal{H}_n$ is flat over R/π^n: this follows by induction because $\mathcal{H}_{n-1}/\mathcal{H}_n \cong P_0$ for every n.

Let p the the Hilbert polynomial of P_0. The proper map

$$\pi : \mathrm{Quot}(\mathcal{H}, \chi) \to \mathrm{Spec}(R)$$

contains $\mathrm{Spec}(R/\pi^n)$ in its image for every $n \geqslant 1$ because the maps

$$\mathrm{Quot}(\mathcal{H}/\pi^n, \chi) \to \mathrm{Spec}(R/\pi^n)$$

are all surjective. This means that the map π is surjective, so there exists a quotient $\mathcal{H} \twoheadrightarrow P$, where P is flat over R and has Hilbert polynomial p. Let

$$Q = \ker(\mathcal{H} \to P)$$

be the kernel. We have that $\mu(Q_K) = \mu(Q_0)$. Thus Q_K is supported on exceptional fibers and $\chi(Q_K) = \chi(Q_0) \geqslant 0$ because $Q_0 \in \mathcal{Q}$. The generic fiber lies in a short exact sequence

$$0 \to Q_K \to \mathcal{H}_K = F_K \to P_K \to 0.$$

Consider the torsion pair sequence for Q_K

$$0 \to A \to Q_K \to B \to 0,$$

with $A \in \mathcal{T}$ and $B \in \mathcal{F}$. We thus have that $A \subset F_K$. Further, $F_K \in \mathcal{F}$, so $A = 0$, and thus $Q_K \in \mathcal{F}$. However, all sheaves supported on exceptional locus and in $\mathcal{F}$ have Euler characteristic $\chi < 0$, which contradicts $\chi(Q_K) \geqslant 0$. This means that our assumption in the beginning of the proof was false. Replace $\mathcal{H}$ with $\mathcal{H}'$; there exists a flat extension $\mathcal{H}$ of F_K over R such that $\mathcal{H}_k \in \mathcal{F}$.

Step 2. The section $\mathcal{O}_{Y\times K} \to F_K$ extends to a section

$$\mathcal{O}_{Y\times R} \xrightarrow{s} \mathcal{H}.$$

Let $\mathcal{K} = \mathrm{coker}(s)$. We next claim that there exists a subsheaf $\mathcal{H}' \subset \mathcal{H}$ such that the similarly defined cokernel $\mathcal{K}'$ fits in a short exact sequence $0 \to \mathcal{K}'_f \to \mathcal{K}' \to A \to 0$ with $\mathcal{K}'_f$ flat over R and $A \in \mathcal{T}$ supported on the central fiber.

Indeed, write $0 \to \mathcal{K}_f \to \mathcal{K} \to M \to 0$, where $\mathcal{K}^f \subset \mathcal{K}$ is the largest flat subsheaf of $\mathcal{K}$ over R and M is supported on the central fiber. Further, consider the short exact sequence

$$0 \to A \to M \to B \to 0$$

with $A \in \mathcal{T}$ and $B \in \mathcal{F}$. Define $\mathcal{H}' = \ker(\mathcal{H} \to \mathcal{H}_k \to B)$. The section $\mathcal{O}_{Y\times R} \to \mathcal{H}$ factors through $\mathcal{H}'$ with cokernel $\mathcal{K}'$, and we have a short exact sequence

$$0 \to \mathcal{K}' \to \mathcal{K} \to B \to 0,$$

which implies that $0 \to \mathcal{K}_f \to \mathcal{K}' \to A \to 0$. The subsheaf $\mathcal{H}'$ has the desired property. Replace $\mathcal{H}$ with $\mathcal{H}'$.

Step 3. We prove that the change in Step 2 keeps the sheaf $\mathcal{H}$ in $\mathcal{F}$. For this, suppose $\mathcal{H} \in \mathcal{F}$ and let $\mathcal{H}'$ be defined by

$$0 \to \mathcal{H}' \to \mathcal{H} \to B \to 0,$$

where B is supported on the central fiber in $\mathcal{F}$. We claim that $\mathcal{H}'_k \in \mathcal{F}$.

First, we have that $\mathcal{H}'_K \cong \mathcal{H}_K$. For the central fiber, consider the Tor sequence

$$0 \to B \to \mathcal{H}'_k \to \mathcal{H}_k \to B \to 0.$$

We obtain short exact sequences

$$0 \to A \to \mathcal{H}_k \to B \to 0 \text{ and } 0 \to B \to \mathcal{H}'_k \to P \to 0.$$

The sheaf A is a subsheaf of $\mathcal{H}_k$, so $A \in \mathcal{F}$. The extension of two sheaves in $\mathcal{F}$ is in $\mathcal{F}$, so $\mathcal{H}'_k \in \mathcal{F}$. This means that there exists a flat extension $\mathcal{H}$ of F_K over R such that $\mathcal{H}_k \in \mathcal{F}$ and the cokernel $\mathcal{K}$ has the property from Step 2.

Step 4. Next, we claim that there exists a subsheaf $\mathcal{H}' \subset \mathcal{H}$ such that the section

$$\mathcal{O}_{Y\times R} \xrightarrow{s} \mathcal{H}'$$

has cokernel $\mathcal{K} = \operatorname{coker}(s)$ in $\mathcal{Q}$. For a sheaf $C \in \mathrm{Coh}_{\leqslant 1}(Y)$, let $F(C) \in \mathcal{F}$ be the sheaf in $\mathcal{F}$ from the torsion pair short exact sequence. First, a diagram chasing shows that $F(\mathcal{K}^f) = F(\mathcal{K})$. Let $P_0 = F(\mathcal{K})$. Consider the short exact sequence with $Q_0 \in \mathcal{T}$

$$0 \to Q_0 \to \mathcal{K}_k \to P_0 \to 0.$$

Let $\mathcal{H}^1 = \ker(\mathcal{H} \to P_0)$. The section $\mathcal{O}_{Y\times R} \to \mathcal{H}$ factors through $\mathcal{O}_{Y\times R} \to \mathcal{H}^1$. After restricting this sequence to the central fiber and a Tor computation, we obtain the short exact sequence

$$0 \to P_0 \to \mathcal{H}^1_k \to A_0 \to 0.$$

Let $\mathcal{K}^1$ be the cokernel of $\mathcal{O}_Y \to \mathcal{H}^1_k$, then we get two short exact sequences, where the first one is using the torsion pair sequence for $\mathcal{K}^1_k$ where R_0 is a quotient of P_0

$$0 \to Q_1 \to \mathcal{K}^1_k \to P_1 \to 0 \text{ and } 0 \to R_0 \to \mathcal{K}^1_k \to Q_0 \to 0.$$

By the argument from the first part of the proof, we have that either $R_0 = 0$, and then the claim follows because $\operatorname{coker}(\mathcal{O}_Y \to \mathcal{H}^1) = Q^0 \in \mathcal{Q}$, or $f_*Q_1 \subset f_*Q_0$. Using the argument from Step 1, the sequence

$$Q_n \subset Q_{n-1}$$

stabilizes for n large enough. The sequence for P_n thus also becomes eventually constant; we assume this happens from $n = 0$.

Using an argument as in Step 1, the quotient $\mathcal{K}_f/\pi \to P_0$ on the central fiber can be lifted to a quotient

$$\mathcal{K}_f \to \mathcal{L},$$

where $\mathcal{L}$ is flat over R and thus $\mu(\mathcal{L}) = \mu(P_0)$. In particular, the map $(\mathcal{K}_f)_K \to \mathcal{L}_K$ is surjective. Consider the torsion pair sequence for $\mathcal{L}_K$

$$0 \to A \to \mathcal{L}_K \to B \to 0,$$

with $A \in \mathcal{Q}$ and $B \in \mathcal{F}$. We have a surjection $(\mathcal{K}_f)_K \to B$. Further, we have that $(\mathcal{K}_f)_K \in \mathcal{T}$. This means that $B = 0$ and thus that $\mathcal{L}_K \in \mathcal{T}$. There are no sheaves in $\mathcal{F}$ and $\mathcal{T}$ with the same slope. This contradiction explains that there is a subsheaf $\mathcal{H}' \subset \mathcal{H}$ such that $\operatorname{coker}(\mathcal{O}_{Y\times R} \to \mathcal{H}') \in \mathcal{Q}$.

Replace $\mathcal{H}'$ with $\mathcal{H}$. Using the argument in Step 3, we see that this change also has the property that $\mathcal{H}_k$ is in $\mathcal{F}$. The family $\mathcal{O}_{Y\times R} \xrightarrow{s} \mathcal{H}$ is thus a flat BS family extending I. ☐

Next, we explain that $\mathrm{BS}_n(Y,\beta)$ has a natural virtual fundamental class. Proposition 2.2 implies that the locus of BS pairs inside $\mathfrak{LM}$ is open. We now explain how the result follows from [13]. Let $\mathbb{I} \in D^b(Y \times \mathrm{BS}_n(Y,\beta))$ be the universal BS pair. Let $L_{Y\times \mathrm{BS}(Y)}$ be the Illusie cotangent complex of $Y \times \mathrm{BS}(Y)$, then

$$\mathbb{L}_{Y\times\mathrm{BS}(Y)} = \tau_{\geqslant -1} L_{Y\times\mathrm{BS}(Y)}.$$

Consider $\mathcal{H}om\,(\mathbb{I},\mathbb{I})_0$ the kernel of the trace map

$$\mathrm{Tr} : \mathcal{H}om\,(\mathbb{I},\mathbb{I}) \to \mathcal{O}_{Y\times\mathrm{BS}(Y)}.$$

The Atiyah class $\mathrm{At} \in \mathrm{Ext}^1(\mathbb{I}, \mathbb{I}\otimes \mathbb{L}_{Y\times\mathrm{BS}(Y)})$ induces a map, see [13, Sections 4.2, 4.5]:

$$R\pi_{2*}(R\mathcal{H}om(\mathbb{I},\mathbb{I})_0 \otimes \pi_1^*\omega_Y)[2] \to L_{\mathrm{BS}(Y)} \to \mathbb{L}_{\mathrm{BS}(Y)}. \tag{3}$$

Theorem 3.7. *The map in Equation (3) is a perfect obstruction theory. Thus the algebraic space $BS_n(Y,\beta)$ carries a virtual fundamental class:*

$$[BS_n(Y,\beta)]^{vir} \in A_d(BS_n(Y,\beta)), H_{2d}(BS_n(Y,\beta),\mathbb{Z}),$$

where $d = -\chi(RHom(I,I)_0)$ for I a BS pair in $BS_n(Y,\beta)$.

Proof. The map in Equation (3) is an obstruction theory by [13, Theorem 4.1 and Section 4.5]. It is further perfect by the comment at the end of [13, Section 4.3] and Proposition 3.2. The space $\mathrm{BS}_n(Y,\beta)$ thus carries a virtual fundamental class by [6]. ☐

3.3. **BS generating series.** Let $\mathbb{I} \in D^b(Y \times \mathrm{BS}_n(Y,\beta))$ be the universal BS pair. For an insertion $\gamma \in H^l(Y,\mathbb{Z})$ and an integer $k \geqslant 0$. Define

$$\mathrm{ch}_{2+k}(\gamma)(-) : H_*(\mathrm{BS}_n(Y,\beta),\mathbb{Q}) \to H_{*-2k+2-l}(\mathrm{BS}_n(Y,\beta),\mathbb{Q})$$

by the formula

$$\mathrm{ch}_{2+k}(\gamma)(-) = \pi_{2*}(ch_{2+k}(\mathbb{I})\pi_1^*(\gamma) \cap \pi_2^*(-)).$$

The BS invariants with insertions $\gamma_1, \cdots, \gamma_r$ and descendant levels $k_1, \cdots, k_r \geqslant 0$ are defined by

$$\langle \tau_{k_1}(\gamma_1), \cdots, \tau_{k_r}(\gamma_r)\rangle_{\beta,n} = \int_{[\mathrm{BS}_n(Y,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{2+k_1}(\gamma_1)\cdots \mathrm{ch}_{2+k_r}(\gamma_r).$$

The generating series for BS invariants with insertions and descendant levels as above and class β is given by the Laurent series in q:

$$\mathrm{BS}_{\beta,\gamma}(q) = \sum_{n\in\mathbb{Z}} \langle \tau_{k_1}(\gamma_1), \cdots, \tau_{k_r}(\gamma_r)\rangle_{\beta,n} q^n.$$

The total generating series of BS invariants with insertions and descendant levels is defined by the generating series in $\mathbb{C}[\Delta]_\Phi$:

$$\mathrm{BS}_\gamma(q) = \sum_{\beta\in N_1(Y)} \mathrm{BS}_{\beta,\gamma}(q)\, q^\beta.$$

4. Degeneration formula for BS invariants

In this section, we define relative BS invariants and prove a degeneration formula for BS invariants following the degeneration formula for DT invariants of Li-Wu [21].

4.1. **Relative BS pairs.** Let X be a projective threefold with rational, Gorenstein singularities with a resolution $f : Y \to X$ of relative dimension at most 1. Let $E \subset Y$ be the exceptional locus. Consider $S \subset Y$ a divisor which does not intersect E. For $k \geqslant 1$, consider the k-step degeneration of Y:

$$Y[k] = Y \cup_S \mathbb{P}_S(N \oplus \mathcal{O}) \cup_S \cdots \cup_S \mathbb{P}_S(N \oplus \mathcal{O}),$$

where the union has k copies of $\mathbb{P}_S(N \oplus \mathcal{O})$. There are natural projection maps $\pi : Y[k] \to Y$. Further, $Y[k]$ had $\mathbb{G}_m^k$ automorphisms covering the identity on $Y[0]$.

A relative BS pair is a two term complex

$$\mathcal{O}_{Y[k]} \xrightarrow{s} F,$$

where F is a sheaf on $Y[k]$ with $\pi_*[F] = \beta \in H_2(Y,\mathbb{Z})$ and $\chi(F) = n$, and such that:

(i) the restriction of $\mathcal{O}_{Y[k]} \to F$ to $Y - S$ is a BS pair,

(ii) the restriction of $\mathcal{O}_{Y[k]} \to F$ to $Y[k] - E$ is a PT relative pair, see Subsection 2.6 or [26] for definitions.

Consider the functor $\Phi : (\text{Schemes}/k)^{\text{op}} \to \text{Sets}$ with $\Phi(B)$ the set of equivalence classes of pairs $\mathcal{O}_{Y[k]\times B} \to \mathcal{F}$ for some $k \geqslant 0$ such that $\mathcal{O}_{Y[k]\times b} \to \mathcal{F}|_{Y\times b}$ is a relative BS pair for every $b \in B$. Two families $\mathcal{F}$ and $\mathcal{F}'$ are equivalent if there exists a line bundle $\mathcal{L}$ on B such that $\mathcal{F} \cong \mathcal{F}' \otimes \pi_2^*\mathcal{L}$.

Theorem 4.1. *The functor Φ is represented by a DM stack $BS_n(Y/S,\beta)$.*

Proof. We need to show that the functor Φ is bounded, open, separated, complete, and has trivial automorphisms. The functor Φ is a subfunctor of

$$\Phi \hookrightarrow \Phi_{\text{BS}(Y)} \times \Phi_{\text{PT}(Y-E/D)}.$$

It is immediate to see that this implies that Φ is bounded and open. Further, the analogue of Proposition 3.2 holds, and in particular Φ has trivial automorphisms. Separateness follows as in Proposition 3.5 using the analogue of Proposition 3.2.

To show that Φ is complete, let $\mathcal{O}_{Y[k]\times K} \to F_K$ be a relative BS pair over K. There exists an extension of the BS pair $\mathcal{O}_{Y\times K} \to F_{Y\times K}$ to a BS pair over R

$$\mathcal{O}_{Y\times R} \to \mathcal{F}^1.$$

Further, there exists an extension of the PT pair $\mathcal{O}_{(Y[k]-E)\times K} \to F_{(Y[k]-E)\times K}$ to a PT pair over R:

$$\mathcal{O}_{(Y[k]-E)\times R} \to \mathcal{F}^2.$$

To see this, extend the pair trivially over E and use properness of the functor $\Phi_{\text{PT}(Y/D)}$. Let $D^1 \subset D$ be the locus where $\mathcal{F}^1$ does not intersect D transversely, and let $C = E \cap D^1$. The codimension of C in Y is at least 2. We want to glue $\mathcal{F}^1|_{(Y-C)\times R}$ and $\mathcal{F}^2|_{(Y[k]-E)\times R}$. For this, we need to show that their restriction to $U := Y - C - E$ are isomorphic. We can use the argument in Proposition 3.2 to show that

$$\mathcal{H}om_R(\mathcal{F}^1, \mathcal{F}^2)|_U \cong \mathcal{H}om_R(\mathcal{F}^2, \mathcal{F}^1)|_U = R.$$

An argument similar to the proof of Proposition 3.5 shows that indeed:

$$\mathcal{F}^1|_U \cong \mathcal{F}^2|_U.$$

□

We next check that relative BS pairs are open in the derived category.

Proposition 4.2. *Let B_0 be a scheme over k, and $i_0 : B_0 \hookrightarrow B$ a nilpotent thickening. Consider*

$$I_0 = [\mathcal{O}_{Y[k]\times B_0} \to F_0]$$

a BS pair over B_0. Let $I \in D^b(Y[k] \times B)$ be a complex such that

$$Li_0^* I \cong I_0.$$

Then there exists a sheaf F on $Y[k] \times B$, flat over B, such that

$$I = [\mathcal{O}_{Y[k]\times B} \to F].$$

Proof. Proposition 3.4 implies that the restriction of I to Y of the complex is

$$I|_Y \cong [\mathcal{O}_Y \to \mathcal{F}^1],$$

where $\mathcal{F}^1$ is flat over B. Similarly, using openness of relative PT pairs in the derived category, the restriction of I to $Y[k]-E$ is

$$I|_{Y[k]-E} = [\mathcal{O}_{Y[k]-E} \to \mathcal{F}^2],$$

where $\mathcal{F}^2$ is flat over B. The restrictions to $Y-E-D$ are isomorphic, so the whole complex is a relative BS pair. □

Let $\mathbb{I} \in D^b(Y \times \mathrm{BS}_n(Y/S,\beta))$ be the universal relative BS pair. The Atiyah class $\mathrm{At} \in \mathrm{Ext}^1(\mathbb{I}, \mathbb{I} \otimes \mathbb{L}_{Y\times BS(Y/S)})$ induces a map, see [13, Sections 4.2, 4.5]:

$$R\pi_{2*}(R\mathcal{H}om(\mathbb{I},\mathbb{I})_0 \otimes \pi_1^*\omega_Y)[2] \to \mathbb{L}_{\mathrm{BS}(Y/S)}. \tag{4}$$

Using the results in [13], we obtain that:

Theorem 4.3. *The map in Equation (4) is a perfect obstruction theory. Thus the DM stack $BS_n(Y/S,\beta)$ carries a virtual fundamental class*

$$[BS_n(Y/S,\beta)]^{vir} \in A_d(BS_n(Y/S,\beta)), H_{2d}(BS_n(Y/S,\beta),\mathbb{Q}),$$

where $d = -\chi(RHom(I,I)_0)$ for I a BS pair in $BS_n(Y/S,\beta)$.

4.2. **Relative BS invariants.** Recall the definitions of cohomologically weighted partition, of the basis C_η for $H^{\cdot}(\mathrm{Hilb}(S,\beta),\mathbb{Q})$, and of the invariants $|\eta|, l(\eta), \xi(\eta)$ from Subsection 2.6. Consider the morphism

$$\varepsilon : BS_n(X/S,\beta) \to \mathrm{Hilb}(S,d)$$

which sends a sheaf F to its restriction to S, where $S.\beta = d$. The relative BS invariants for insertions $\gamma_1,\cdots,\gamma_r \in H^{\cdot}(Y,\mathbb{Z})$ and descendant levels $k_1,\cdots,k_r \geqslant 0$ are defined by the formula:

$$\langle \tau_{k_1}(\gamma_1),\cdots,\tau_{k_r}(\gamma_r)|\,\eta\rangle_{\beta,n} = \int_{[\mathrm{BS}_n(Y/S,\beta)]^{\mathrm{vir}}} \mathrm{ch}_{2+k_1}(\gamma_1)\cdots\mathrm{ch}_{2+k_r}(\gamma_r) \cap \varepsilon^*(C_\eta).$$

The generating series for the relative BS invariants with insertions and descendant levels as above, support β, and cohomologically weighted partition η is defined by:

$$\mathrm{BS}^{\mathrm{rel}}_{\beta,\eta,\gamma}(q) = \sum_{n\in\mathbb{Z}} \langle \tau_{k_1}(\gamma_1),\cdots,\tau_{k_r}(\gamma_r)|\,\eta\rangle_{\beta,n} q^n.$$

4.3. **The degeneration formula for BS invariants.** Let B be a smooth curve, and let $o \in B$. Consider families $\mathcal{Y}$ and $\mathcal{X}$ over B $\begin{array}{ccc} \mathcal{Y} & \xrightarrow{f} & \mathcal{X} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$ such that:

- the restriction $f_b : \mathcal{Y}_b \to \mathcal{X}_b$ of the map f over $b \in B$ is a resolution of singularities of a threefold $\mathcal{X}_b$ with rational, Gorenstein singularities, such that the relative dimension of f_b is 1, and

- the restriction over o of the map f is a morphism

$$f_o : Y_1 \cup_S Y_2 \to X_1 \cup_S X_2$$

such that Y_1 and Y_2 are smooth threefolds that intersect transversely in a smooth divisor S, $f|_{Y_1} : Y_1 \cong X_1$, X_1 and X_2 intersect transversely in S, and $f|_{Y_2} : Y_2 \to X_2$ is a resolution of singularities of the threefold X_2 with rational, Gorenstein singularities, such that the relative dimension of $f|_{Y_2}$ has dimension 1.

In particular, S and the exceptional locus of $f|_{Y_2} : Y_2 \to X_2$ do not intersect.

Theorem 4.4. *Consider a family $f : \mathcal{Y} \to \mathcal{X}$ as above. Let $\beta \in N_1(Y)$, and consider insertion classes $\gamma_1, \cdots, \gamma_r \in H^{\cdot}(Y, \mathbb{Z})$ and descendant levels $k_1, \cdots, k_r \geqslant 0$. The BS invariants for the generic fiber and the relative BS invariants for the special fiber are related as follows:*

$$BS_{\beta,\gamma}(q) = \sum PT^{rel}_{\beta_1,\eta,\gamma_A}(q) BS^{rel}_{\beta_2,\eta^\vee,\gamma_B}(q) \frac{(-1)^{|\eta|-l(\eta)}\xi(\eta)}{q^{|\eta|}}.$$

The sum on the right hand side is taken after all splittings $\beta_1 + \beta_2 = \beta$, all partitions $A \cup B = \{1, \cdots, k\}$ of the insertions, and after all cohomologically weighted partitions η.

The proof of the above theorem is similar to the proof of the degeneration formula for DT or PT invariants, see [21], [25]. We sketch the proof using the general strategy from these two papers.

Let $(\beta, n) \in N_{\leqslant 1}(Y)$. Let $\mathcal{B} = \mathcal{B}(\beta, n)$ be the Artin stack of (β, n) decorated semistable models of $\mathcal{Y}/B$ with universal family

$$\widetilde{\mathcal{Y}} \to \mathcal{B}.$$

There is a natural map $\mathcal{B} \to B$. Denote by $\mathcal{B}_o$ the fiber over o. We have that $\mathcal{B} - \mathcal{B}_o \cong B - o$, and the universal family restricted to $\mathcal{B} - \mathcal{B}_o$ is $\mathcal{Y} - \mathcal{Y}_o \to B - o$. We can write

$$\mathcal{B}_o = \bigcup_{k \geqslant 0} \mathcal{Y}_o[k],$$

where $\mathcal{Y}_o[k]$ is defined by:

$$\mathcal{Y}_o[k] = Y_1 \cup_S \mathbb{P}_S(N \otimes \mathcal{O}) \cup_S \cdots \cup_S S \times \mathbb{P}_S(N \otimes \mathcal{O}) \cup Y_2,$$

with k copies of $\mathbb{P}_S(N \otimes \mathcal{O})$, and where N is the normal bundle of S in Y_1.

Let $\mathcal{P} \to \mathcal{B}$ be the stack of relative BS pairs on the fibers of $\widetilde{\mathcal{Y}} \to \mathcal{B}$. One can show as in Proposition 4.1 that $\mathcal{P}$ is a proper DM stack with finite automorphism over B. Using the same argument as in Proposition 4.2 and the results of [13], there is a relative perfect obstruction theory of $\mathcal{P}/\mathcal{B}$ defined as follows. Let $\mathbb{I}$ be the universal complex over $\widetilde{\mathcal{Y}} \times_{\mathcal{B}} \mathcal{P}$:

$$E = R\pi_{2*}(R\mathcal{H}om(\mathbb{I}, \mathbb{I})_0 \otimes \pi_1^* \omega_{\widetilde{\mathcal{Y}}/\mathcal{B}})[2] \to \mathbb{L}_{\mathcal{P}/\mathcal{B}}.$$

We can obtain a perfect obstruction theory

$$\mathcal{E} \to \mathbb{L}_{\mathcal{P}}$$

from the above relative perfect obstruction theory as follows. Consider the natural map $E \to \mathbb{L}_{\mathcal{P}/\mathcal{B}} \to \mathbb{L}_{\mathcal{B}}[1]$, and define $\mathcal{E}[1]$ to be its cone. The perfect obstruction theory $\mathcal{E} \to \mathbb{L}_{\mathcal{P}}$ is obtained from the following diagram:

$$\begin{array}{ccccc} \mathcal{E} & \longrightarrow & E & \longrightarrow & \mathbb{L}_{\mathcal{B}}[1] \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \mathbb{L}_{\mathcal{P}} & \longrightarrow & \mathbb{L}_{\mathcal{P}/\mathcal{B}} & \longrightarrow & \mathbb{L}_{\mathcal{B}}[1]. \end{array}$$

Let $\mathcal{P}_o = \mathcal{P} \times_B o$. We then obtain a perfect obstruction theory

$$\mathcal{E}_o \to \mathbb{L}_{\mathcal{P}_o}$$

which fits in the diagram:

$$\begin{array}{ccccc} \mathcal{E}_o & \longrightarrow & E|_{\mathcal{P}_o} & \longrightarrow & \mathbb{L}_{\mathcal{B}_o}[1] \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \mathbb{L}_{\mathcal{P}_o} & \longrightarrow & \mathbb{L}_{\mathcal{P}_o/\mathcal{B}_0} & \longrightarrow & \mathbb{L}_{\mathcal{B}_o}[1]. \end{array}$$

Let L_o be line bundle over $\mathcal{B}$ corresponding to $\mathcal{B}_o \subset \mathcal{B}$. The perfect obstruction theories for $\mathcal{P}_o$ and $\mathcal{P}$ can be compared via the triangle, see [25, Diagram 54]:

$$\mathcal{E}|_{\mathcal{P}_o} \to \mathcal{E}_o \to L_o^{\vee}[1]. \tag{5}$$

Next, denote by ν the data of two pairs (β_1, n_1), (β_2, n_2), and a positive integer k such that $\beta_1 + \beta_2 = \beta$ and $n + k = n_1 + n_2$. Then $S.\beta_1 = S.\beta_2 = k$, see [12, Lemma 2.2]. Define

$$\mathcal{P}_\nu := \mathrm{PT}_{n_1}(Y_1/S, \beta_1) \times_{\mathrm{Hilb}(S,k)} \mathrm{BS}_{n_2}(Y_2/S, \beta_2).$$

Then $\mathcal{P}_0 = \bigcup_\nu \mathcal{P}_\nu$. Further, for every ν, there exists a divisor $\mathcal{B}_\nu \subset \mathcal{B}$ whose pull-back to $\mathcal{P}$ is $\mathcal{P}_\nu$. Let L_ν be the associated line bundle to $\mathcal{B}_\nu$. Then

$$\bigotimes_\nu L_\nu = L_0.$$

We can define a perfect obstruction theory $\mathcal{E}_\nu \to \mathbb{L}_{\mathcal{P}_\nu}$ as above which fits in a distinguished triangle:

$$\mathcal{E}|_{\mathcal{P}_\nu} \to \mathcal{E}_\nu \to L_\nu^{\vee}[1]. \tag{6}$$

Further, the perfect obstruction theories of $\mathcal{P}_\nu$ and its factors $\mathcal{P}_1 = \mathrm{PT}_{n_1}(Y_1/S, \beta_1)$ and $\mathcal{P}_2 = \mathrm{BS}_{n_2}(Y_2/S, \beta_2)$ are compared as follows, see [25, Diagram 61]:

$$\begin{array}{ccccc} \mathcal{E}_1 \oplus \mathcal{E}_2 & \longrightarrow & \mathcal{E}_\nu & \longrightarrow & \Omega_{\mathrm{Hilb}\,(S,k)}[1] \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{L}_{\mathcal{P}_1 \times \mathcal{P}_2} & \longrightarrow & \mathbb{L}_{\mathcal{P}_\nu} & \longrightarrow & \mathbb{L}_{\mathcal{P}_\nu/\mathcal{P}_1 \times \mathcal{P}_2}. \end{array} \tag{7}$$

The following results implies a cycle version of the degeneration formula:

Theorem 4.5. *(a) Let $b \in B - o$. Then $i_b^![\mathcal{P}]^{vir} = [BS_n(\mathcal{Y}_b, \beta)]^{vir}$.*

(b) The restriction over o has virtual fundamental class $i_o^![\mathcal{P}]^{vir} = [\mathcal{P}_o]^{vir}$.

(c) The fundamental class of the special fiber decomposes $[\mathcal{P}_o]^{vir} = \sum_\nu i_{\nu *}[\mathcal{P}_\nu]^{vir}$, where the sum is after all data ν consisting of two pairs (β_1, n_1), (β_2, n_2), and a positive integer k such that $\beta_1 + \beta_2 = \beta$ and $n + k = n_1 + n_2$.

(d) The fundamental class $[\mathcal{P}_\nu]^{vir}$ further decomposes in the contributions of the two factors as follows:

$$[\mathcal{P}_\nu]^{vir} = \Delta^!([PT_{n_1}(Y_1/S, \beta_2)]^{vir} \times [BS_{n_2}(Y_2/S, \beta_2)]^{vir}).$$

Here, Δ is the diagonal embedding $\Delta : Hilb(S, k) \to Hilb(S, k) \times Hilb(S, k)$.

Proof. (a) The restriction of the perfect obstruction theory of $\mathcal{P}$ to the fiber over b is the same as the perfect obstruction theory for $\mathrm{BS}_n(\mathcal{Y}_b, \beta)$.

(b) Use the triangle (5), the proof then follows as in [21].

(c) Use the triangles (6) and that $L_0 = \otimes_\nu L_\nu$. The proof then follows as in [21].

(d) The claim follows from the diagram (7), see the proof in the PT case given in [21]. □

The degeneration formula in Theorem 4.4 then follows from its cycle version in Theorem 4.5 as in the DT case [21, Section 6.3].

5. The BS/ PT correspondence

We now start the proof of Theorem 1.4. For this, fix $f : Y \to X$ a contraction of a curve $C \cong \mathbb{P}^1$ with normal bundle

$$N_{C/Y} = \mathcal{O}_C(-1)^{\oplus 2}.$$

We will use the notations

- $\mathbb{P} = \mathbb{P}_C(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O})$,
- $S = \mathbb{P}_C(\mathcal{O}(-1)^{\oplus 2})$,
- $N = \mathbb{P} - S = \mathrm{Tot}_C(\mathcal{O}(-1)^{\oplus 2})$.

Consider the two-dimensional torus $T \subset \mathbb{G}_m^3$ which acts on $\mathbb{P}$ and preserves the natural Calabi-Yau form on N. The torus T also acts naturally on $BS_n(\mathbb{P}, \beta)$, $BS_n(\mathbb{P}/S, \beta)$, and on the analogous PT moduli spaces.

Let $I = [\mathcal{O}_\mathbb{P} \to F]$ be a T-fixed BS or PT pair on $\mathbb{P}$ which intersects the divisor $S \subset \mathbb{P}$ transversely. The restriction of I to $Y - C$ is an ideal sheaf on a toric variety which intersects S transversely. By [23], the ideal sheaf $I|_{Y-C}$ corresponds to ideals $\pi_i \subset \mathbb{C}[x, y]$ generated by monomials for every leg $1 \leqslant i \leqslant 4$. We use the notation

$$[\pi] = \sum_{i=1}^4 [\pi_i] = \left(\sum_{i=1}^4 l(\pi_i) \right) h \in H_2(\mathbb{P}),$$

where $h \in H_2(\mathbb{P})$ is the class of a leg and $l(\pi_i) = \dim_\mathbb{C} \mathbb{C}[x, y]/\pi_i$. For such a partition profile $\pi = (\pi_i)_{1 \leqslant i \leqslant 4}$ and an integer $m \geqslant 0$, let

$$\mathrm{PT}_n(\pi, m) \subset \mathrm{PT}_n(\mathbb{P}, [\pi] + m[C]),\ \mathrm{PT}_n(\mathbb{P}/\mathbb{D}, [\pi] + m[C])$$

be the subspace of T-fixed PT pairs $\mathcal{O}_{\mathbb{P}} \to F$ with partition profile π. Define similarly $\mathrm{BS}_n(\pi, m) \subset \mathrm{BS}_n(\mathbb{P}, [\pi] + m[C])$, $\mathrm{BS}_n(\mathbb{P}/\mathbb{D}, [\pi] + m[C])$.

Proposition 5.1. *The subspaces $BS_n(\pi, m)$ and $PT_n(\pi, m)$ are proper algebraic spaces with a natural symmetric perfect obstruction theory constructed from the perfect obstruction theories for $BS_n(\mathbb{P}, \pi+m[C])$ and $PT_n(\mathbb{P}, \pi+m[C])$, respectively.*

Before we start the proof of the above proposition, we need the following preliminary result:

Lemma 5.2. *Let $\lambda_i : \mathbb{G}_m \to T$ for $i = 1, 2$ be characters such that:*

- *λ_1 fixes one point $s \in S$ and acts with positive weights on $N_{s/\mathbb{P}}$,*
- *λ_2 fixes a different point $t \in S$, acts with negative weights on $N_{t/S}$ and with positive weights on $N_{S/\mathbb{P}}|_t$.*

Let I_1 and I_2 be two T-equivariant BS or PT complexes intersecting S transversely, and let L be a T-equivariant line bundle on $\mathbb{P}$ such that $\langle \lambda_i, L\rangle \leqslant 0$ for $i = 1, 2$. The restriction map

$$Ext^k_{\mathbb{P}}(I_1, I_2 \otimes L)^T \to Ext^k_N(I_1, I_2 \otimes L)^T$$

is an isomorphism for any $k \geqslant 0$.

Proof. The characters and their attracting loci (λ_1, s), $(\lambda_2, S - s)$ are Kempf-Ness loci for the action of T on $\mathbb{P}$. The result follows as in [12, Theorem 3.16]. We explain it for one Kempf-Ness stratum V with associated character $\lambda : \mathbb{G}_m \to T$. Let $W \subset V$ be the λ-fixed locus. Denote by $\Gamma_V(-)$ the sections of a complex with support on V, and by $D^b(V)_{<0}$, $D^b(V)_{\geqslant 0}$ the subcategories of complexes F in $D^b_{\mathbb{G}_m}(V)$ such that λ acts with negative or non-negative weights on $F|_W$. It is enough to show that

$$R\Gamma_V(R\mathcal{H}om_{\mathbb{P}}(I_1, I_2 \otimes L))^{\mathbb{G}_m} = 0.$$

Let $i : V \hookrightarrow \mathbb{P}$, then $Ri^* R\mathcal{H}om_{\mathbb{P}}(I_1, I_2 \otimes L) = R\mathcal{H}om_V(j^* I_1, j^! I_2 \otimes L)$. It is thus enough to show that

$$R\Gamma_V(J)^{\mathbb{G}_m} = 0$$

for J a complex such that $Ri^! J \in D^b(V)_{<0}$. We have that

$$R\Gamma_V(J)^{\mathbb{G}_m} = \lim R\mathrm{Hom}_{\mathbb{P}}(\mathcal{O}_{\mathbb{P}}/\mathcal{I}^n_V, J)^{\mathbb{G}_m},$$

where $\mathcal{I}_V$ is the ideal sheaf of V. The sheaf $\mathcal{O}_{\mathbb{P}}/\mathcal{I}^n_V$ is generated by $i_*\mathrm{Sym}^k(N^\vee_{V/\mathbb{P}})$ for $k \geqslant 0$. The sheaves $\mathrm{Sym}^k(N^\vee_{V/\mathbb{P}})$ are in $D^b(V)_{\geqslant 0}$. We have that $Rj^! J \in D^b(V)_{<0}$, so we obtain the desired conclusion. □

Proof of Proposition 5.1. We need to show that the T-fixed perfect obstruction theories for $\mathrm{BS}_n(\pi, m)$ and $\mathrm{PT}_n(\pi, m)$ are symmetric. Let I be a BS of a PT pair. The perfect obstruction theory for $\mathrm{BS}_n(\mathbb{P}, \pi + m[C])$ or $\mathrm{PT}_n(\mathbb{P}, \pi + m[C])$ restricts over

I to $[E_I^{-1} \to E_I^0]$, with cohomology $h^{-1} = \mathrm{Ext}^1(I, I)$ and $h^0 = \mathrm{Ext}^2(I, I)$. The perfect obstruction theory on the T-fixed locus restricts over I to

$$[(E_I^{-1})^T \to (E_I^0)^T]$$

with cohomology $h^{-1} = \mathrm{Ext}^1_{\mathbb{P}}(I, I)^T$ and $h^0 = \mathrm{Ext}^2_{\mathbb{P}}(I, I)^T$. Let λ_1 and λ_2 be characters as in Lemma 5.2. The canonical divisor ω satisfies $\langle \lambda_i, \omega \rangle \leqslant 0$ for $i = 1, 2$. We then have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Ext}^1_{\mathbb{P}}(I_1, I_2)^T \times \mathrm{Ext}^2_{\mathbb{P}}(I_2, I_1 \otimes \omega)^T & \longrightarrow & \mathrm{Ext}^3_{\mathbb{P}}(I_1, I_1 \otimes \omega)^T \\
\downarrow \text{iso} & & \downarrow \text{iso} \\
\mathrm{Ext}^1_{N}(I_1, I_2)^T \times \mathrm{Ext}^2_{N}(I_2, I_1)^T & \longrightarrow & \mathrm{Ext}^3_{N}(I_1, I_1)^T \\
\text{iso} \uparrow & & \text{iso} \uparrow \\
\mathrm{Ext}^1_{\mathbb{P}}(I_1, I_2)^T \times \mathrm{Ext}^2_{\mathbb{P}}(I_2, I_1)^T & \longrightarrow & \mathrm{Ext}^3_{\mathbb{P}}(I_1, I_1)^T.
\end{array}$$

The top horizontal line is a non-degenerate pairing by Serre duality for $\mathbb{P}$, and thus the two other horizontal lines are non-degenerate pairings. A similar argument in the global case shows the existence of a non-degenerate pairing on E^T, as desired. □

Proposition 5.1 implies that the virtual fundamental classes for $\mathrm{BS}_n(\pi, m)$ and $\mathrm{PT}_n(\pi, m)$ have dimension zero. For a partition profile π, define the generating series

$$\mathrm{BS}_{\pi,m}(q) = \sum_{n \in \mathbb{Z}} [\mathrm{BS}_n(\pi, m)]^{\mathrm{vir}} q^n$$

and similarly define the series $\mathrm{PT}_{\pi,m}(q)$. They are both Laurent series in q. Define also the generating series

$$\mathrm{BS}_{\pi}(q) := \sum_{m \geqslant 0} \mathrm{BS}_{\pi,m}(q) q^{m[C]}$$

and similarly $\mathrm{PT}_{\pi}(q)$. They are both elements of $\mathbb{C}[\Delta]_{\Phi}$. In Section 6, we prove the following:

Theorem 5.3. *Let π be a partition profile. The BS and PT generating series defined above satisfy*

$$BS_{\pi}(q) = \frac{PT_{\pi}(q)}{PT_0(q)}.$$

In this section, we explain that:

Proposition 5.4. *Theorem 5.3 implies Theorem 1.4.*

Before we begin the proof of Proposition 5.4, we discuss some preliminary results. Denote by $\mathrm{BS}_n(\mathbb{P}/D, \pi, m)$ and $\mathrm{PT}_n(\mathbb{P}/D, \pi, m)$ the components of the T-fixed locus of $\mathrm{BS}_n(\mathbb{P}/D, \pi + m[C])$ and $\mathrm{PT}_n(\mathbb{P}/D, \pi + m[C])$ whose restrictions to the four legs in $\mathbb{P} - D$ have partition profile π. Both spaces have a perfect obstruction theory obtained by localization, see Subsection 2.3.

Proposition 5.5. *There exists a rational number ω such that*

$$[BS_n(\mathbb{P}/D,\pi,m)^T]^{vir} = [BS_n(\pi,m)]^{vir}\omega \text{ in } H_0(BS_n(\mathbb{P}/D,\pi,m)^T,\mathbb{Q})$$

$$[PT_n(\mathbb{P}/D,\pi,m)^T]^{vir} = [PT_n(\pi,m)]^{vir}\omega \text{ in } H_0(PT_n(\mathbb{P}/D,\pi,m)^T,\mathbb{Q}).$$

Proof. Consider the maps

$$\pi_{\mathrm{BS}} : \mathrm{BS}_n(\mathbb{P}/D,\pi,m) \to \mathrm{BS}_n(\pi,m) \text{ and } \pi_{\mathrm{PT}} : \mathrm{PT}_n(\mathbb{P}/D,\pi,m) \to \mathrm{PT}_n(\pi,m)$$

which send a relative pair to its restriction to $\mathbb{P}-D$. The fibers of these maps do not depend on the restriction of the pair to the curve C, and they are the same for PT and BS because their definitions are the same when restricted to the complement of C.

Consider the perfect obstruction theory $\mathcal{E}_{\mathrm{BS}} \to \mathbb{L}_{\mathrm{BS}(\mathbb{P}/S)}$ constructed in Subsection 4.3 which fits in a triangle:

$$\begin{array}{ccccc} \mathcal{E}_{\mathrm{BS}} & \longrightarrow & E_{\mathrm{BS}} & \longrightarrow & \mathbb{L}_{\mathcal{B}}[1] \\ \downarrow & & \downarrow & & \downarrow \mathrm{id} \\ \mathbb{L}_{\mathcal{P}} & \longrightarrow & \mathbb{L}_{\mathcal{P}/\mathcal{B}} & \longrightarrow & \mathbb{L}_{\mathcal{B}}[1]. \end{array}$$

Here, the relative perfect obstruction theory $E_{\mathrm{BS}} \to \mathbb{L}_{\mathrm{BS}(\mathbb{P}/S)/\mathcal{B}}$ is defined by

$$E_{\mathrm{BS}} = R\pi_{2*}(R\mathcal{H}om(\mathbb{I},\mathbb{I})_0 \otimes \pi_1^*\omega_{\widetilde{\mathbb{P}}/\mathcal{B}})[2] \to \mathbb{L}_{\mathrm{BS}(\mathbb{P}/S)/\mathcal{B}},$$

where $\mathbb{I}$ is the universal complex over $\mathrm{BS}(\mathbb{P}/S) \times_{\mathcal{B}} \widetilde{\mathbb{P}}$. The restriction of E_{BS} to N splits, so there exists a splitting

$$E_{\mathrm{BS}}^T \cong E_{\mathrm{BS}}^T|_N \boxtimes E'$$

for a complex E' with trivial restriction on N. We have that $E_{\mathrm{BS}}^T|_N$ is the perfect obstruction theory of $\mathrm{BS}_n(\pi,m)$. The analogous statement holds for $\mathcal{E}_{\mathrm{BS}}$ by the above triangle. The perfect obstruction theory for $\mathrm{PT}(\mathbb{P}/S)$ is similarly defined, and let E_{PT} and $\mathcal{E}_{\mathrm{PT}}$ be the analogous complexes in this case. There exists a splitting

$$E_{\mathrm{PT}}^T \cong E_{\mathrm{PT}}^T|_N \boxtimes E'.$$

We thus obtain the desired conclusion. □

Recall the definition of the generating series for relative invariants from Subsections 2.8 and 4.2. We claim that:

$$\mathrm{BS}_\eta^{\mathrm{rel}}(q) = \frac{\mathrm{PT}_\eta^{\mathrm{rel}}(q)}{\mathrm{PT}_0(q)} \tag{8}$$

for any cohomologically weighted partition η. Let $N_{\mathrm{relBS}}^{\mathrm{vir}}$ be the virtual normal bundle to the fixed locus

$$\mathrm{BS}_n(\mathbb{P}/D,\pi,m) \subset \mathrm{BS}_n(\mathbb{P}/D,[\pi]+m[C]),$$

and $N_{\mathrm{BS}}^{\mathrm{vir}}$ be the virtual normal bundle to the fixed locus

$$\mathrm{BS}_n(\pi,m) \subset \mathrm{BS}_n(\mathbb{P},[\pi]+m[C]).$$

Define similarly $N^{\mathrm{vir}}_{\mathrm{relPT}}$ and $N^{\mathrm{vir}}_{\mathrm{PT}}$. We denote by $N^{\mathrm{vir}}_{m,\mathrm{relBS}}$ the subcomplex of $N^{\mathrm{vir}}_{\mathrm{relBS}}$ direct sum of subcomplexes where T acts with non-zero weights. Define the generating series for the T-fixed BS invariants:

$$\mathrm{BS}^{\mathrm{rel}}_{\pi,m,\eta}(q) = \sum_{n\in\mathbb{Z}} \left(\int_{[\mathrm{BS}_n(\mathbb{P}/D,\pi,m)^T]^{\mathrm{vir}}} \frac{\varepsilon^*(C_\eta)}{e(N^{m,\mathrm{vir}}_{\mathrm{relBS}})} \right) q^n$$

and similarly $\mathrm{PT}^{\mathrm{rel}}_{\pi,m,\eta}(q)$ for PT invariants. They are both Laurent polynomials in $\mathrm{Frac}\, H^{\cdot}(BT)[q^{\pm 1}]$. Define also the generating series:

$$\mathrm{BS}^{\mathrm{rel}}_{\pi,\eta}(q) := \sum_{m\geqslant 0} \mathrm{BS}^{\mathrm{rel}}_{\pi,m,\eta}(q) q^{m[C]}$$

and similarly $\mathrm{PT}^{\mathrm{rel}}_{\pi,\eta}(q)$ for PT invariants. They are both in the Laurent ring $\mathrm{Frac}\, H^{\cdot}(BT)[\Delta]_\Phi$ defined as in Subsection 2.4. We claim that:

$$\mathrm{BS}^{\mathrm{rel}}_{\pi,\eta}(q) = \frac{\mathrm{PT}^{\mathrm{rel}}_{\pi,\eta}(q)}{\mathrm{PT}^{\mathrm{rel}}_{0}(q)} \tag{9}$$

for every cohomologically weighted partition η. Define the generating series

$$\mathrm{BS}_{\pi,m,\eta}(q) = \sum_{n\in\mathbb{Z}} \left(\int_{[\mathrm{BS}_n(\pi,m)]^{\mathrm{vir}}} \frac{\varepsilon^*(C_\eta)}{e(N^{m,\mathrm{vir}})} \right) q^n.$$

Here $N^{\mathrm{vir}}_{\mathrm{BS}}$ is the restriction of $N^{\mathrm{vir}}_{\mathrm{relBS}}$ to $\mathbb{P} - S$. The series $\mathrm{BS}_{\pi,\eta}(q)$ and the analogous for PT invariants are defined is a similar way. For an element τ in $H^{\cdot}(X) \otimes \mathrm{Frac}\, H^{\cdot}(BT)$, denote by τ_0 its term in $H^0(X) \otimes \mathrm{Frac}\, H^{\cdot}(BT)$.

Proposition 5.6. *We have that*

$$BS^{rel}_{\pi,\eta}(q) = BS^{rel}_{\pi,\eta}(q)\omega \text{ and } PT^{rel}_{\pi,\eta}(q) = PT^{rel}_{\pi,\eta}(q)\omega.$$

(b) We have equalities in $\mathrm{Frac}\, H^{\cdot}(BT)[\Delta]_\Phi$*:*

$$BS_{\pi,\eta}(q) = BS_\pi(q) \left(\frac{\varepsilon^*(C_\eta)}{e(N^{vir}_{BS})} \right)_0 \text{ and } PT_{\pi,\eta}(q) = PT_\pi(q) \left(\frac{\varepsilon^*(C_\eta)}{e(N^{vir}_{PT})} \right)_0.$$

Proof. (a) The normal bundle $N^{\mathrm{vir}}_{\mathrm{relBS}}$ splits as

$$N^{\mathrm{vir}}_{\mathrm{relBS}} = N^{\mathrm{vir}}_{\mathrm{BS}} \boxtimes \widetilde{N}$$

as in the proof of Proposition 5.5. The complex $\widetilde{N}$ is trivial when restricted to $\mathbb{P} - S$, and so we also have that

$$N^{m,\mathrm{vir}}_{\mathrm{relBS}} = N^{m,\mathrm{vir}}_{\mathrm{BS}} \boxtimes \widetilde{N}^m.$$

The same result holds for the analogous complexes in the PT case. Using Proposition 5.5, we can thus write the integrand as:

$$\langle [\mathrm{BS}_n(\mathbb{P}/D,\pi,m)^T]^{vir}, \frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{\mathrm{relBS}})} \rangle = \langle [\mathrm{BS}_n(\pi,m)]^{vir}, \frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{\mathrm{BS}})} \rangle \, \langle \omega, 1 \rangle$$

and similarly for PT invariants:

$$\langle [\mathrm{PT}_n(\mathbb{P}/D,\pi,m)^T]^{vir}, \frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{\mathrm{relPT}})}\rangle = \langle [\mathrm{PT}_n(\pi,m)]^{vir}, \frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{\mathrm{BS}})}\rangle\, \langle \omega, 1\rangle.$$

Part (b) is immediate. □

Proposition 5.7. *Recall the definition of C_η and of the map ε from Subsection 2.6. The following equality*

$$\left(\frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{BS})}\right)_0 = \left(\frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{PT})}\right)_0$$

holds in $H^0(BS_n(\pi,m))\otimes \mathit{Frac}\, H^{\cdot}(BT) = H^0(PT_n(\pi,m))\otimes \mathit{Frac}\, H^{\cdot}(BT)$.

Proof. **Step 1.** Recall from Equation 4 that the perfect obstruction theory for the BS moduli space is given by

$$R\pi_{2*}(R\mathcal{H}om(\mathbb{I},\mathbb{I})_0\otimes \pi_1^*\omega_Y)\to \mathbb{L}_{\mathrm{BS}_n(Y,[\pi]+m[C])},$$

where $\mathbb{I}\in D^b(Y\times \mathrm{BS}_n(Y,[\pi]+m[C]))$ is the universal sheaf. In order to compute the perfect obstruction theory for the fixed locus and the normal bundle N^{vir}_{BS}, we restrict the above complex to the T-fixed locus $\mathrm{BS}_n(\pi,m)\subset \mathrm{BS}_n(\mathbb{P},[\pi]+m[C])$.

Let I be such a BS pair. The restriction of I to $\mathbb{P}-C$ is the ideal sheaf of $D'\subset \mathbb{P}-C$. Let $D\subset\mathbb{P}$ be the closure of D', and let $\widetilde{\mathcal{I}}$ be the ideal sheaf of D. The map $\widetilde{\mathcal{I}}\to h^0(I)|_{\mathbb{P}-C}\to I|_{\mathbb{P}-C}$ can be extended to a map:

$$\mathcal{I} := \widetilde{\mathcal{I}}^N \to h^0(I)\to I$$

for some large $N\geqslant 0$. We thus have a distinguished triangle

$$\mathcal{I}\to I\to E\xrightarrow{[1]}. \tag{10}$$

There is also a global distinguished triangle

$$\pi_2^*\mathcal{I}\to\mathbb{I}\to\mathbb{E}\xrightarrow{[1]}.$$

The complex $\mathbb{E}$ is flat over $\mathrm{BS}_n(\pi,m)$ and its fibers are supported over the exceptional curve C. For families A and B over $\mathrm{BS}_n(\pi,m)$, denote by

$$N^{m,\mathrm{vir}}_{A,B} = \begin{cases} R\pi_{2*}(R\mathcal{H}om(\pi_2^*\mathcal{I},\pi_2^*\mathcal{I})_0\otimes\pi_1^*\omega_Y)^m & \text{if } A=B=\pi_2^*\mathcal{I},\\ R\pi_{2*}(R\mathcal{H}om(A,B)\otimes\pi_1^*\omega_Y)^m & \text{otherwise.}\end{cases}$$

We will drop π_2^* when placed in front of $\mathcal{I}$. We have the following equality in $H^{\cdot}(\mathrm{BS}_n(\pi,m))\otimes\mathrm{Frac}\,H^{\cdot}(BT_0)$:

$$e(N^{m,\mathrm{vir}}_{\mathrm{BS}}) = e(N^{m,\mathrm{vir}}_{\mathcal{I},\mathbb{E}})e(N^{m,\mathrm{vir}}_{\mathbb{E},\mathcal{I}})e(N^{m,\mathrm{vir}}_{\mathcal{I},\mathcal{I}})e(N^{m,\mathrm{vir}}_{\mathbb{E},\mathbb{E}}). \tag{11}$$

Step 2. The complexes $N^{\mathrm{vir}}_{\mathcal{I},\mathbb{E}}$, $N^{\mathrm{vir}}_{\mathbb{E},\mathcal{I}}$, and $N^{\mathrm{vir}}_{\mathbb{E},\mathbb{E}}$ are all self-dual. Indeed, it is enough to show that

$$\mathrm{Ext}^1(\mathcal{I},E)\cong\mathrm{Ext}^2(E,\mathcal{I})^\vee \text{ and } \mathrm{Ext}^1(E,E)\cong\mathrm{Ext}^2(E,E)^\vee$$

for E a sheaf supported on the curve C. The sheaves $\mathcal{E}xt^i(\mathcal{I}, E)$, $\mathcal{E}xt^i(E, \mathcal{I})$, and $\mathcal{E}xt^i(E, E)$ are all supported on E, so the result follows from Serre duality for $\mathbb{P} - S$. Serre duality interchanges the weight w and $-w$ subspaces for every $w \in \mathbb{Z}$, so the analogous result holds for $N^{m,\mathrm{vir}}$.

Denote by $\mathrm{ext}^i(-,-) = \dim \mathrm{Ext}^i(-,-)$. We thus have that:

(12)

$$N^{m,\mathrm{vir}}_{\mathcal{I},\mathbb{E}} = (-1)^{\mathrm{ext}^1(\mathcal{I},\mathbb{E})^m}, N^{m,\mathrm{vir}}_{\mathbb{E},\mathcal{I}} = (-1)^{\mathrm{ext}^1(\mathbb{E},\mathcal{I})^m}, \text{ and } N^{m,\mathrm{vir}}_{\mathbb{E},\mathbb{E}} = (-1)^{\mathrm{ext}^1(\mathbb{E},\mathbb{E})^m}.$$

By Serre duality, we have that:

$$\mathrm{ext}^1(\mathcal{I}, \mathbb{E})^m = \mathrm{ext}^1(\mathbb{E}, \mathcal{I})^m. \tag{13}$$

Equations (11),(12), and (13) imply that:

$$\frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{\mathrm{BS}})} = \frac{\varepsilon^*(C_\eta)}{e(N^{m,vir}_{\mathcal{I},\mathcal{I}})}(-1)^{\mathrm{ext}^1(\mathbb{E},\mathbb{E})^m}.$$

The analogous equality holds for PT.

Step 3. The claim of the proposition follows from the following:

$$\mathrm{ext}^1(E, E)^m \equiv \mathrm{ext}^1(D, D)^m \pmod 2, \tag{14}$$

where E and D are constructed as in the triangle (10) for a BS and PT pair with same π, m, and n. Let $\mathbb{G}_m \subset T$ be a generic torus, and consider the character of the representation $\mathrm{Ext}^1(E, E)^m$ in $K_0(B\mathbb{G}_m)$:

$$\psi_E(q) = \sum_{i \in \mathbb{Z}} \mathrm{ext}^1(E, E)^i q^i,$$

where $\mathrm{Ext}^{\cdot}(E, E)^i \subset \mathrm{Ext}^{\cdot}(E, E)$ is the subspace on which $\mathbb{G}_m$ acts with weight i and $\mathrm{ext}^1(E, E)^i = \dim \mathrm{Ext}^1(E, E)^i$. For an element $\chi \in K_0(B\mathbb{G}_m) \cong \mathbb{Z}[q^{\pm 1}]$, denote by χ^+ the terms with positive q-exponent. Using Serre duality, we have that

$$\mathrm{Ext}^1(E, E)^{-i} \cong \left(\mathrm{Ext}^2(E, E)^i\right)^\vee.$$

Define the following polynomial in q:

$$\chi_E(q) = \sum_{i>0} \left(\dim \mathrm{Ext}^1(E, E)^i - \dim \mathrm{Ext}^2(E, E)^i\right) q^i.$$

Observe that:

$$\psi_E(1) \equiv \chi_E(1) \pmod 2. \tag{15}$$

Further, for a sheaf A, we have that

$$\chi_A(q) = \chi(A, A)^+(q) - \chi(\mathcal{O}_\mathbb{P}, \mathcal{O}_\mathbb{P})^+(q)$$

in $K_0(B\mathbb{G}_m)$, where the terms on the right hand side are the virtual characters of the respective Euler characteristics.

Step 4. We claim that $\chi(A,A)$ defines a group homomorphism:

$$\chi(A,A) : K_0^{\mathbb{G}_m}(N) \to \mathbb{F}_2[q^{\pm 1}]. \tag{16}$$

To show this, consider an exact triangle $A \to B \to C \xrightarrow{[1]}$. We have that

$$\chi(B,B) = \chi(A,A) + \chi(A,C) + \chi(C,A) + \chi(C,C).$$

Using Serre duality, we observe that $\chi(A,C) + \chi(C,A) = 0$ in $\mathbb{F}_2[q^{\pm 1}]$. Thus, we have that $\chi(B,B) = \chi(A,A) + \chi(C,C)$ in $\mathbb{F}_2[q^{\pm 1}]$, so $\chi(A,A)$ is indeed a group homomophism.

In particular, $\chi(A,A)^+$ is also a group homomorphism. This implies that $\chi_E(q) = \chi_D(q)$ in $q\mathbb{F}_2[q]$ and thus the claim in Equation (15) is true. □

Proof of Proposition 5.4. **Step 1.** Theorem 5.3 implies Claim (9) using Proposition 5.6.

Step 2. Claim (9) implies Claim (8) using the localization theorem for the action of T on $\mathbb{P}$ which implies that:

$$\mathrm{BS}^{\mathrm{rel}}_{\beta,\eta}(q) = \sum_{\pi,m} \mathrm{BS}^{\mathrm{rel}}_{\pi,m,\eta}(q)$$

where the sum is taken after profile partitions π such that $[\pi] + m[C] = \beta \in H_2(\mathbb{P})$. The analogous result holds for PT invariants.

Step 3. We explain that Claim (8) implies Theorem 1.4. Let p be the singular point of X. Consider the family

$$\mathrm{Bl}_{C\times 0}(Y \times \mathbb{C}) \to \mathrm{Bl}_{p\times 0}(X \times \mathbb{C})$$

over $\mathbb{A}^1$. For $t \neq 0$, the fiber is the original contraction $f : Y \to X$. For $t = 0$, the fiber is

$$\mathrm{Bl}_C Y \cup \mathbb{P}_C(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}) \to \mathrm{Bl}_p X \cup C_p(X).$$

The map $\mathrm{Bl}_C Y \to \mathrm{Bl}_p X$ is an isomorphism, and the map

$$\mathbb{P}_C(\mathcal{O}(-1)^{\oplus 2} \oplus \mathcal{O}) \to C_p(X)$$

is the contraction of the zero fiber C.

Observe that all insertions are in $f^*H_2(X,\mathbb{Z})$, which means that when using the degeneration theorem, there will be no insertions in the contribution of $\mathbb{P}$. The degeneration theorem for PT invariants, see Subsection 2.6, says that

$$\mathrm{PT}_{\beta,\gamma}(q) = \sum \mathrm{PT}^{\mathrm{rel},1}_{\beta_1,\eta,\gamma}(q)\mathrm{PT}^{\mathrm{rel},2}_{\beta_2,\eta^\vee}(q)\frac{(-1)^{|\eta|-l(\eta)}\xi(\eta)}{q^{|\eta|}}, \tag{17}$$

where the sum is after all splittings of the curve class $\beta_1 + \beta_2 = \beta$ and after all cohomologically weighted partitions η. Similarly, the degeneration theorem for BS invariants, see Theorem 4.4, says that

$$\mathrm{BS}_{\beta,\gamma}(q) = \sum \mathrm{PT}^{\mathrm{rel},1}_{\beta_1,\eta,\gamma}(q)\mathrm{BS}^{\mathrm{rel},2}_{\beta_2,\eta^\vee}(q)\frac{(-1)^{|\eta|-l(\eta)}\xi(\eta)}{q^{|\eta|}}, \tag{18}$$

where once again the sum is after all splittings of the curve class $\beta_1 + \beta_2 = \beta$, and after all cohomologically weighted partitions η. The decompositions (17) and (18) together with Claim (8) imply Theorem 1.4.

□

6. The Hall algebra argument

In this section, we prove Theorem 5.3 using the motivic Hall algebra techniques developed by Joyce [14], [15], [16], and Joyce-Song [17], see also [7]. The proof follows closely Bryan-Steinberg's proof of the relation between BS and DT invariants [9]. Let $\mathfrak{M}$ be the moduli stack of T-equivariant sheaves in $\mathrm{Coh}_{\leqslant 1}(N)$ with a section. The argument will involve the motivic Hall algebra $K(\mathrm{St}/\mathfrak{M})$. The main ingredients are an integration map which is a Poisson algebra homomorphism, see Theorem 6.3, and Joyce's no-pole theorem for the category $\mathcal{A}$, see [7]. The result then follows as in [9] using identities in the Hall algebra.

6.1. Preliminaries.

6.1. Preliminaries. Recall the definition of the motivic Hall algebra and of the Grothendieck ring of varieties from Subsection 2.12. Let $\mathfrak{M}_{\mathbb{P}}$ be the moduli stack of sheaves in $\mathrm{Coh}_{\leqslant 1}(\mathbb{P})$ with a section, and let $\mathfrak{M}_{\mathbb{P}}^T \subset \mathfrak{M}_{\mathbb{P}}$ be the moduli of T-invariant sheaves on $\mathfrak{M}_{\mathbb{P}}$. As discussed in Section 5, the connected components of both $\mathfrak{M}$ and $\mathfrak{M}_{\mathbb{P}}^T$ are parametrized by (π, m, n), where π is a partition profile, and $m \geqslant 0$ and n are integers. For each triplet (π, m, n), we have that

$$\mathfrak{M}_{\pi,m,n} \subset \mathfrak{M}^T_{\mathbb{P},\pi,m,n}$$

is a closed substack given by the condition that $F|_S$ is a fixed ideal sheaf with profile partition π. In particular, we have that $\mathfrak{M}$ is an Artin stack locally of finite type with affine stabilizers.

Definition 6.1. Let $H_{\mathrm{reg}} \subset K(\mathrm{St}/\mathfrak{M})$ be the $K(\mathrm{Var}/k)_{\mathrm{loc}}$ module generated by the span of classes $[V \xrightarrow{f} \mathfrak{M}]$ for V a variety.

The following proposition is proved as in [7]:

Proposition 6.2. *(a) The convolution product preserves H_{reg}, and thus H_{reg} is a $K(Var/k)_{loc}$ algebra.*

(b) Define $H_{sc} = H_{reg}/(\mathbb{L}-1)H_{reg}$. Then H_{sc} is a commutative $K(Var/k)_{loc}$ algebra.

Recall from Subsection 2.2 that the invariants $[\mathrm{BS}_n(\pi, m)]^{\mathrm{vir}}$ and $[\mathrm{PT}_n(\pi, m)]^{\mathrm{vir}}$ can be computed using the Behrend function ν of $\mathfrak{M}$:

$$\int_{\mathrm{BS}_n(\pi,m)} \nu = [\mathrm{BS}_n(\pi, m)]^{\mathrm{vir}}, \tag{19}$$

$$\int_{\mathrm{PT}_n(\pi,m)} \nu = [\mathrm{PT}_n(\pi, m)]^{\mathrm{vir}}. \tag{20}$$

Let $\Delta \subset N_{\leqslant 1}$ be the monoid generated by $\mathrm{ch}\,(F)$, where F is a T-equivariant sheaf on N with support of dimension at most 1. For a class $(\beta, n) \in N_{\leqslant 1}$ and for a partition profile π, we introduce

$$\mathfrak{M}_{\beta,n} = \bigcup_{[\pi]+m[C]=\beta} \mathfrak{M}_{\pi,m,n} \text{ and } \mathfrak{M}_{\pi} = \bigcup_{\beta,n} \mathfrak{M}_{\pi,m,n}.$$

Recall the definition of the algebra $\mathbb{C}[\Delta]$ from Subsection 2.4. Consider the Poisson algebra $\mathbb{C}[\Delta]$ with trivial Poisson bracket. Define the integration map

$$\Phi : H_{\mathrm{sc}} \to \mathbb{C}[\Delta]$$

by the following formula, where V is a variety:

$$\Phi([V \xrightarrow{f} \mathfrak{M}_{\beta,n}]) = \chi(V, f^*\nu) q^{\beta+n}.$$

Theorem 6.3. *The integration map $\Phi : H_{sc} \to \mathbb{C}[\Delta]$ defined above is a Poisson algebra homomorphism.*

Proof. Theorem 6.4 and the discussion in Subsection 2.2 imply that for a complex $I = [\mathcal{O} \to F]$, where $F \in \mathrm{Coh}^T_{\leqslant 1}(N)$, the Behrend function can be computed by

$$\nu(I) = (-1)^{\dim \mathrm{Ext}^1(I,I)}(1 - \chi(M_f)),$$

where G, V, and $f : V/G \to \mathbb{C}$ are as in Theorem 6.4, and M_f is the Milnor fiber of f at any point in $\Phi^{-1}(I)$. We can thus use localization as in [17] to obtain the following identities for $I_1, I_2 \in \mathcal{A}$:

$$\nu(I_1 \oplus I_2) = (-1)^{\chi(I_1,I_2)} \nu(I_1)\, \nu(I_2)$$

$$\int_{\mathbb{P}\mathrm{Ext}^1(I_2,I_1)} \nu(F) d\chi - \int_{\mathbb{P}\mathrm{Ext}^1(I_1,I_2)} \nu(F) d\chi = (\mathrm{ext}^1(I_2, I_1) - \mathrm{ext}^1(I_1, I_2))\nu(I_1 \oplus I_2),$$

where in the second identity, the complex $0 \to I_1 \to F \to I_2 \to 0$ corresponds to an element in $\mathbb{P}\mathrm{Ext}^1(I_2, I_1)$, and similarly for the second integral. The verification that Ψ is a Poisson algebra homomorphism follows easily from these identities, see for example [17]. ☐

Theorem 6.4. *Consider a point $I \in \mathfrak{M}$, and let $G \subset Aut(I)$ be a maximal reductive group. The group G acts naturally on $Ext^1(I, I)$. There exist an open G-invariant set $I \in \mathcal{W} \subset \mathfrak{M}$, an open G-invariant subset $V \subset Ext^1(E, E)$, a holomorphic function $f : V/G \to \mathbb{C}$, and a natural smooth map*

$$\Phi : crit(f)/G \to \mathcal{W}$$

of relative dimension $dim\, Aut(I) - dim\, G$.

Proof. The above statement is known for moduli stacks of objects on proper varieties, see Joyce-Song [17], Toda [29]. We explain how to modify Toda's proof to obtain the conclusion in our case.

Step 1. We briefly review the theory of deformation of complexes, see [29]. Let Y be a smooth projective variety with an action of a torus T, and let I be a complex of holomorphic vector bundles on Y. Let $S \subset T$ be a maximal compact subgroup. Let $\mathcal{A}^{0,*}(I)$ be the Dolbeault double complex of I. Let

$$A^{0,*}(I) := \operatorname{Tot}\left(\Gamma(Y, \mathcal{A}^{0,*}(I))\right)$$

be the total complex of the double complex $\mathcal{A}^{0,*}(I)$. Consider the graded complex $\mathfrak{g}_I^{\cdot}$ with graded k component

$$\mathfrak{g}_I^k := \bigoplus_{p+q=k} \prod_i A^{0,q}(\mathcal{H}om(I^i, I^{i+p})).$$

The complex $\mathfrak{g}_I^{\cdot}$ has a differential $d_{\mathfrak{g}}$ obtained from the Doulbeault differential. The multiplication is defined by

$$A^{0,q}(\mathcal{H}om(I^i, I^{i+p})) \times A^{0,q'}(\mathcal{H}om(I^{i+p}, I^{i+p+p'})) \to A^{0,q+q'}(\mathcal{H}om(I^i, I^{i+p+p'})).$$

We thus obtain a dg algebra $(\mathfrak{g}_I^{\cdot}, d_{\mathfrak{g}}, \cdot)$.

If I is an S-equivariant complex, define the graded complex $\mathfrak{g}_I^{S,\cdot}$ with graded k component

$$\mathfrak{g}_I^{S,k} := \bigoplus_{p+q=k} \prod_i A^{0,q}(\mathcal{H}om(I^i, I^{i+p})^S).$$

The multiplication of $\mathfrak{g}_I^{\cdot}$ preserves $\mathfrak{g}_I^{S,\cdot}$, so we obtain a dg subalgebra

$$(\mathfrak{g}_I^{S,\cdot}, d_{\mathfrak{g}}, \cdot) \subset (\mathfrak{g}_I^{\cdot}, d_{\mathfrak{g}}, \cdot).$$

Consider the Maurer-Cartan map

$$\mathfrak{mc} : \mathfrak{g}_I^1 \to \mathfrak{g}_I^2$$

defined by $\alpha \to d_{\mathfrak{g}}(\alpha) + \alpha \cdot \alpha$. For a solution α of the Maurer-Cartan equation, the complex

$$(\mathcal{A}^{0,*}(I), d_{\mathcal{A}^{0,*}(I)} + \alpha)$$

is a complex deformation of I, and all complex deformations of I are obtained this way. There is also a notion of gauge equivalence, see loc. cit. Finally, if I is S-equivariant and $\alpha \in \mathfrak{g}^{S,1}$ is a solution of the Maurer-Cartan equation, then

$$(\mathcal{A}^{0,*}(I), d_{\mathcal{A}^{0,*}(I)} + \alpha)$$

is an S-equivariant complex deformation of the complex I, and all S-equivariant complex deformations of I are obtained this way.

Step 2. The cohomology of $(\mathfrak{g}_I^{\cdot}, d_{\mathfrak{g}})$ is $\operatorname{Ext}^{\cdot}(I, I)$, and thus the cohomology of $(\mathfrak{g}_I^{S,\cdot}, d_{\mathfrak{g}})$ is $\operatorname{Ext}^{\cdot}(I, I)^S$. Consider the minimal A_∞ algebras obtained as in Polishchuk [27], Toda [29], which are A_∞ algebra structures on the cohomology of the two dg algebra in Step 1:

$$\begin{array}{ccc}
(\mathrm{Ext}^{\cdot}(I,I),(m_k)_{k\geqslant 2}) & \xrightarrow{\text{qis}} & (\mathfrak{g}_I^{\cdot}, d_{\mathfrak{g}}, \cdot) \\
\uparrow & & \uparrow \\
(\mathrm{Ext}^{\cdot}(I,I)^S,(m_k)_{k\geqslant 2}) & \xrightarrow{\text{qis}} & (\mathfrak{g}_I^{S,\cdot}, d_{\mathfrak{g}}, \cdot).
\end{array}$$

The Dolbeault differential on $(\mathfrak{g}_I^{S,\cdot}, d_{\mathfrak{g}}, \cdot)$ is induced by the Dolbeault differential on $(\mathfrak{g}_I^{\cdot}, d_{\mathfrak{g}}, \cdot)$. Fix a Kähler metric on Y, Hermitian metrics on I^i. The maps m_k are constructed from the Dolbeault differentials $d_{\mathfrak{g}}$, the Kähler and Hermitian metrics, and using the Green operator for Y, see [29], so the multiplicative structures for $\mathrm{Ext}^{\cdot}(I,I)$ and $\mathrm{Ext}^{\cdot}(I,I)^S$ are compatible:

$$\begin{array}{ccc}
\mathrm{Ext}^{\cdot}(I,I) & \xrightarrow{m_k} & \mathrm{Ext}^{\cdot+2-k}(I,I) \\
\uparrow & & \uparrow \\
\mathrm{Ext}^{\cdot}(I,I)^S & \xrightarrow{m_k} & \mathrm{Ext}^{\cdot+2-k}(I,I)^S.
\end{array}$$

Step 3. Toda [29] has proved that there exists an open set $0 \in U \subset \mathrm{Ext}^1(I,I)$ such that the following series is absolutely convergent and thus defines a holomorphic function:

$$f := \sum_{k\geqslant 2} m_k(x,\cdots,x) : U \to \mathrm{Ext}^2(I,I).$$

Consider the morphism

$$\Phi : U \to \mathfrak{M}_{\mathbb{P}}$$

which sends x in U to the complex $E^{\cdot}$ corresponding to the element

$$f(x) \in \ker \mathfrak{mc} \subset \mathrm{Ext}^2(I,I).$$

Toda in loc. cit. proves that Φ is smooth of relative dimension $\dim \mathrm{Aut}\, E$.

Let $Y = \mathbb{P}$ with the action of the torus T, and let $I = [\mathcal{O} \to F]$ be a T-equivariant complex which intersects S transversely. Let $U^T = U \cap \mathrm{Ext}^1(I,I)^T$ and let $\mathfrak{M}_{\mathbb{P}}^T \subset \mathfrak{M}_{\mathbb{P}}$ be the substack corresponding to T-fixed pairs. For a T-equivariant complex I and $x \in \mathrm{Ext}^1(I,I)^T$, the deformation $\Phi(x)$ is also T-equivariant, and we thus obtain a diagram:

$$\begin{array}{ccc}
U & \xrightarrow{\Phi} & \mathfrak{M}_{\mathbb{P}} \\
\uparrow & & \uparrow \\
U^T & \xrightarrow{\Phi} & \mathfrak{M}_{\mathbb{P}}^T,
\end{array}$$

where the maps Φ are smooth of relative dimension $\dim \mathrm{Aut}\, E$. The connected component of $I \in \mathfrak{M}_{\mathbb{P}}^T$ has all the complexes intersect S transversely, so we get obtain a diagram:

$$\begin{array}{ccc} U & \xrightarrow{\Phi} & \mathfrak{M}_{\mathbb{P}} \\ \uparrow & & \uparrow \\ U^T & \xrightarrow{\Phi} & \mathfrak{M}, \end{array}$$

where the maps Φ are smooth of relative dimension $\dim \operatorname{Aut} E$.

Step 4. The formal series f and Φ are G-equivariant. Let $V = G \cdot U$ and $V^T = G \cdot U^T$. We obtain maps $\begin{array}{ccc} V & \xrightarrow{f} & \operatorname{Ext}^2(I,I) \\ \uparrow & & \uparrow \\ V^T & \xrightarrow{f} & \operatorname{Ext}^2(I,I)^T \end{array}$ and $\begin{array}{ccc} V & \xrightarrow{\Phi} & \mathfrak{M}_{\mathbb{P}} \\ \uparrow & & \uparrow \\ V^T & \xrightarrow{\Phi} & \mathfrak{M}. \end{array}$

Consider the open subset $\mathcal{W} = \Phi(V^T)$ of $\mathfrak{M}$.

Step 5. Using Lemma 5.2, define the pairing

$$\operatorname{Ext}^j_N(I_1, I_2)^T \times \operatorname{Ext}^{3-j}_N(I_2, I_1)^T \xrightarrow{\text{iso}} \operatorname{Ext}^j_{\mathbb{P}}(I_1, I_2)^T \times \operatorname{Ext}^{3-j}_{\mathbb{P}}(I_2, I_1 \otimes \omega)^T \to \operatorname{Ext}^3_{\mathbb{P}}(I_1, I_1 \otimes \omega)^T \cong \mathbb{C}.$$

By Polishchuk [27], the multiplications

$$m_k : \operatorname{Ext}^i_{\mathbb{P}}(I,I)^T \cong \operatorname{Ext}^i_N(I,I)^T \to \operatorname{Ext}^{i+2-k}_{\mathbb{P}}(I,I)^T \cong \operatorname{Ext}^{i_2-k}_N(I,I)^T$$

are cyclic, that is, for any complexes $I_1, \cdots, I_k$ as above, $n \geqslant 1$, a map $\psi : \{1, \cdots, n+1\} \to \{1, \cdots, k\}$ with $\psi(1) = \psi(n+1)$, and elements $a_i \in \operatorname{Ext}^i_N(I_{\psi(i)}, I_{\psi(i+1)})^T$ for $1 \leqslant i \leqslant n$, we have that

$$(m_{n-1}(a_1, \cdots, a_{n-1}), a_n) = (m_{n-1}(a_2, \cdots, a_n), a_1).$$

The argument given by Toda in [29] constructs a holomorphic function f such that

$$\operatorname{grad} f = \nu : \operatorname{Ext}^1_N(I,I)^T \to \operatorname{Ext}^2_N(I,I)^T \cong \left(\operatorname{Ext}^1_N(I,I)^T\right)^\vee.$$

□

6.2. **Identities in the Hall algebra.** The Hall algebras $K(\mathrm{St}/\mathfrak{M})$, H_{reg}, H_{sc}, and the Poisson algebra $\mathbb{C}[\Delta]$ are naturally Δ-graded. We consider the corresponding Laurent algebras $K(\mathrm{St}/\mathfrak{M})_\Phi$, $H_{\mathrm{reg},\Phi}$, $H_{\mathrm{sc},\Phi}$, and $\mathbb{C}[\Delta]_\Phi$. The integration map

$$\Psi : H_{\mathrm{sc}} \to \mathbb{C}[\Delta]$$

extends to a continuous integration map

$$\Psi : H_{\mathrm{sc},\Phi} \to \mathbb{C}[\Delta]_\Phi.$$

Definition 6.5. A morphism of stacks $f : \mathcal{W} \to \mathfrak{M}$ is called Φ-*finite* if for every $(\beta, n) \in N_{\leqslant 1}$, the substack $\mathcal{W}_{\beta,n} = f^{-1}(\mathfrak{M}_{\beta,n})$ is an Artin stack of finite type, and there exists a Laurent subset $S \subset \Delta$ such that $\mathcal{W}_{\beta,n}$ is non-empty unless $(\beta, n) \in S$.

A Φ-finite morphism $f : \mathcal{W} \to \mathfrak{M}$ defines an element of $K(\mathrm{St}/\mathfrak{M})_\Phi$ and of the other Hall algebras:

$$\sum_{(\beta,n)\in S} [\mathcal{W}_{\beta,n} \to \mathfrak{M}].$$

Recall the stability condition $\mu : \mathrm{Coh}_{\leqslant 1}(Z) \to S$ from Subsection 3.1. For each $s \in S$, there is a torsion pair $(\mathcal{T}_s, \mathcal{F}_s)$ of $\mathrm{Coh}_{\leqslant 1}(Z)$ constructed in Subsection 2.10. For $a = (\infty, 0)$, the pair $(\mathcal{T}_a, \mathcal{F}_a)$ is the torsion pair used in the definition of BS pairs; for $b = (\infty, \infty)$, the pair $(\mathcal{T}_b, \mathcal{F}_b)$ is the torsion pair used in the definition of PT pairs.

Definition 6.6. (a) For $s \in S$, define $\mathrm{PT}^s \subset \mathfrak{M}$ as the locus of complexes $\mathcal{O}_N \xrightarrow{s} F$ such that $F \in \mathcal{F}_s$ and $\mathrm{coker}(s) \in \mathcal{T}_s$.

(b) If $I \subset S$, define $1_{SS(I)}$ as the sublocus of $\mathfrak{M}$ represented by sheaves in $SS(I)$, and by $1^{\mathcal{O}}_{SS(I)}$ as the sublocus of $\mathfrak{M}$ represented by complexes $\mathcal{O} \to F$ with F in $SS(I)$.

(c) Define $\mathrm{DT} \subset \mathfrak{M}$ as the locus of complexes $\mathcal{O} \twoheadrightarrow F$, and define $\mathrm{DT}^s \subset \mathrm{DT}$ as the locus of surjections $\mathcal{O} \twoheadrightarrow F$ where $F \in \mathcal{T}_s$.

Proposition 6.7. *(a) Let $s \in \{a, b\}$. The following substacks define elements of the Hall algebra $K(St/\mathfrak{M})_\Phi$:*

$$PT^s \to \mathfrak{M},\ DT \to \mathfrak{M},\ DT^s \to \mathfrak{M}.$$

(b) Let $I \subset S$ be an interval bounded from below. The stack $1_{SS(I)} \to \mathfrak{M}$ defines an element of $K(St/\mathfrak{M})_\Phi$.

Proof. (a) The DT, PT, and BS moduli spaces over $\mathbb{P}$ are proper algebraic spaces, and the same is true for their T-equivariant components. The DT, PT, and BS moduli spaces in $\mathfrak{M}$ have components which appear in these T-equivariant moduli spaces, so the DT, PT, and BS spaces are proper. The spaces $\mathrm{DT}^a, \mathrm{DT}^b \subset \mathrm{DT}$ consist of surjections $\mathcal{O}_N \twoheadrightarrow F$ such that F is zero dimensional or F is supported on C, respectively, so they are closed in DT.

(b) The same argument used in [9, Lemma 68] works here as well. □

We will use subscripts indexed by partition profiles π, for example $[\mathrm{PT}^s_\pi \to \mathfrak{M}_\pi \hookrightarrow \mathfrak{M}]$, for the corresponding components of the DT, DT^s, and PT^s moduli spaces.

Proposition 6.8. *Consider the integration map $\Psi : H_{sc,\Phi} \to \mathbb{C}[\Delta]_\Phi$. Then:*

(a) $\Psi([PT^a_\pi \to \mathfrak{M}]) = BS_\pi(q)$,

(b) $\Psi([PT^b_\pi \to \mathfrak{M}]) = PT_\pi(q)$,

(c) $\Psi([PT^b_0 \to \mathfrak{M}]) = PT_0(q)$.

Proof. The identities are immediate from the definition of the integration map Ψ and from the Equation (19). □

Proposition 6.9. *The following identities are true in the Hall algebra $H_{sc,\Phi}$:*

(a) $1_{\mathcal{T}_s} * 1_{SS[\mu,s)} = 1_{SS(\geqslant\mu)}$.

(b) $1^{\mathcal{O}}_{\mathcal{T}_s} * 1^{\mathcal{O}}_{SS[\mu,s)} = 1^{\mathcal{O}}_{SS(\geqslant\mu)}$.

(c) $DT^s * 1_{\mathcal{T}_s} = 1^{\mathcal{O}}_{\mathcal{T}_s}$.

(d) $PT^s * 1_{SS[\mu,s)} - 1^{\mathcal{O}}_{SS[\mu,s)} \to 0$ *as* $\mu \to -\infty$.

(e) $DT * 1_{SS(\geqslant\mu)} - 1^{\mathcal{O}}_{SS(\geqslant\mu)} \to 0$ *as* $\mu \to -\infty$.

Proof. The analogous identities in the Calabi-Yau case were proved by Bryan-Steinberg, and the proofs in loc. cit. work in this case as well. ☐

Proposition 6.10. *(a) Let* $s \in \{a, b\}$*, then* $DT * 1_{\mathcal{T}_s} = DT^s * 1_{\mathcal{T}_s} * PT^s$.

(b) Let π *be a partition profile, then* $DT_\pi * 1_{\mathcal{T}_s} = DT^s * 1_{\mathcal{T}_s} * PT^s_\pi$.

Proof. (a) We follow the proof given in [9]. By Proposition 6.9, part (e), we have that

$$\lim_{\mu\to-\infty} \left(\mathrm{DT} * 1_{SS(\geqslant\mu)} - 1^{\mathcal{O}}_{SS(\geqslant\mu)}\right) \to 0.$$

Using Proposition 6.9 part (a) and (b), this can be rewritten as

$$\lim_{\mu\to-\infty} \left(\mathrm{DT} * 1_{SS(\geqslant s)} * 1_{SS[\mu,s)} - 1^{\mathcal{O}}_{SS(\geqslant s)} * 1^{\mathcal{O}}_{SS[\mu,s)}\right) \to 0.$$

Using Proposition 6.9, part (d), we also have that

$$\lim_{\mu\to-\infty} \left(\mathrm{PT}^s * 1_{SS(\mu,s)} - 1^{\mathcal{O}}_{SS[\mu,s)}\right) \to 0.$$

Multiply the above relation on the left by $1^{\mathcal{O}}_{SS(\geqslant s)}$, who by Proposition 6.9, part (c), equals $\mathrm{DT}^s * 1^{\mathcal{O}}_{SS(\geqslant s)}$ to obtain that:

$$\lim_{\mu\to-\infty} \left(\mathrm{DT}^s * 1^{\mathcal{O}}_{SS(\geqslant s)} * \mathrm{PT}^s * 1_{SS(\mu,s)} - 1^{\mathcal{O}}_{SS(\geqslant s)} * 1^{\mathcal{O}}_{SS[\mu,s)}\right) \to 0.$$

Combining this relation with the second relation, we obtain that

$$\lim_{\mu\to-\infty} \left(\mathrm{DT}^s * 1^{\mathcal{O}}_{SS(\geqslant s)} * \mathrm{PT}^s * 1_{SS(\mu,s)} - \mathrm{DT} * 1_{SS(\geqslant s)}\right) * 1^{\mathcal{O}}_{SS[\mu,s)} \to 0.$$

The element $1_{SS[\mu,s)}$ is invertible [7, Lemma 5.2], and thus we obtain the desired statement.

(b) Convolution with $1_{\mathcal{T}_s}$ and DT^s for $s \in \{a, b\}$ does not change the partition profile π, so the statement follows from the identity in (a) by taking the fiber of both sides over $\mathfrak{M}_\pi$. ☐

Proposition 6.11. *Let* $\mu \in S$ *be a slope. There exists* $\nu_\mu = (\mathbb{L} - 1)\varepsilon(\mu) \in H_{reg}$ *such that*

$$1_{SS(\mu)} = \exp(\varepsilon(\mu)) \in H_\Phi.$$

Further, $1_{SS(\mu)} \in H_\Phi$ *is invertible and the automorphism* $Ad_{1_{SS(\mu)}} : H_\Phi \to H_\Phi$ *preserves regular elements and the induced Poisson automorphism of* H_{sc} *is given by*

$$Ad_{1_{SS(\mu)}} = exp(\{\nu_\mu, -\}).$$

Proof. Joyce's no-pole theorem applies to any abelian category with a weak stability condition, so it applies in our case as well. The rest of the statements follow as in [9]. □

Proof of Theorem 3.4. Use Propositions 6.8 and 6.11 to obtain that

$$\Psi([\mathrm{DT}_\pi \to \mathfrak{M}]) = \Psi([\mathrm{DT}_{\mathrm{tors}} \to \mathfrak{M}])\mathrm{PT}_\pi(q).$$

Further, for the trivial partition profile, we obtain the equality:

$$\Psi([\mathrm{DT}_0 \to \mathfrak{M}]) = \Psi([\mathrm{DT}_{\mathrm{tors}} \to \mathfrak{M}])\mathrm{PT}_0(q).$$

Let $s = a$ in Proposition 6.10, and consider the components above $\mathfrak{M}_\pi$. Use Propositions 6.8 and 6.11 to obtain that

$$\Psi([\mathrm{DT}_\pi \to \mathfrak{M}]) = \Psi([\mathrm{DT}_0 \to \mathfrak{M}])\mathrm{BS}_\pi(q).$$

Putting together these equalities, we see that indeed

$$\mathrm{BS}_\pi(q) = \frac{\mathrm{PT}_\pi(q)}{\mathrm{PT}_0(q)}.$$

□

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