K-theoretic Hall algebras for quivers with potential

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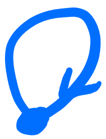
Definition of quivers.

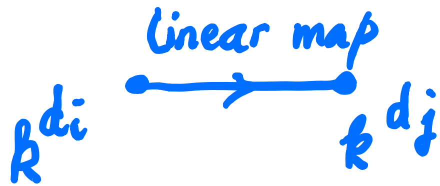
A quiver is a directed graph.
$Q = (V, E, s, t : E \to V)$
Examples: Jordan quiver



For k a field, the path algebra kQ is the k-vector space with basis paths in Q and multiplication given by concatenation of paths.
Example: $kJ = k[x]$.

Representations of quivers.

Let $d \in \mathbb{N}^V$ be a dimension vector. Representations of the path algebra can be described as follows:



The stack of representations of Q of dimension d is

$$\mathcal{X}(d) = \prod_{e \in E} \mathbb{A}^{d_{s(e)} d_{t(e)}} / \prod_{v \in V} GL(d_v).$$

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Classical Hall algebras

(Steinitz $\sim$ 1905, Hall, Ringel $\sim$ 1990)
Let Q be a quiver, k a finite field with q elements, $v = q^{1/2}$.

The Hall algebra $\mathbb{H}_Q$ is the $\mathbb{C}$-vector space with basis isomorphism classes of Q-reps over k and multiplication encoding extensions of such representations.

Consider the subalgebra $\mathbb{H}_Q^{\mathrm{sph}} \subset \mathbb{H}_Q$ generated by dimension vectors $(0, \cdots, 0, 1, 0, \cdots, 0)$.

Theorem (Ringel-Green)

Let Q be a quiver with no loops with corresponding Kac-Moody algebra $\mathfrak{g}$. Then

$$\mathbb{H}_Q^{sph} = U_v^{>}(\mathfrak{g}).$$

From classical Hall algebras to preprojective Hall algebras

The algebra $\mathbb{H}_Q^{\mathrm{sph}}$ can be categorified (Lusztig) by a category $\mathcal{C}(d)$ of constructible sheaves on the stacks $\mathcal{X}(d)$. Multiplication is defined by $m = p_* q^*$, where

$$\mathcal{X}(d) \times \mathcal{X}(e) \xleftarrow{q} \mathcal{X}(d, e) \xrightarrow{p} \mathcal{X}(d + e)$$

$$(A, B/A) \leftarrow (A^d \subset B^{d+e}) \to B.$$

Consider the singular support functor:

$$\mathcal{C}(d) \to D^b_{\mathbb{C}^*}\mathrm{Coh}(T^*\mathcal{X}(d)).$$

Grojnowski proposed the study of algebras defined using $D^b_{\mathbb{C}^*}\mathrm{Coh}(T^*\mathcal{X}(d))$.

Preprojective/ 2d Hall algebras

(Grojnowski 1995, Schiffmann-Vasserot 2009, Yang-Zhao 2014)
For a quiver Q, define

$$KHA(Q) = \bigoplus_{d \in \mathbb{N}^I} K_0^{\mathbb{C}^*}(T^*\mathcal{X}(d)).$$

Example: for J the Jordan quiver, $T^*\mathcal{X}(d)$ is the stack of sheaves on $\mathbb{A}^2$ with support dimension 0 and length d.

Conjecture

Let Q be an arbitrary quiver. Then

$$KHA(Q) = U_q^{>}(\widehat{\mathfrak{g}_Q}),$$

where $U_q(\widehat{\mathfrak{g}_Q})$ is the Okounkov-Smirnov quantum group.

Hall algebras for CY 3d categories.

(Kontsevich-Soibelman 2010)
Conjecturally, for any Calabi-Yau 3-category one can construct a Hall algebra. When the category is $D^b\mathrm{Coh}(X)$ for X a Calabi-Yau 3-fold, the number of generators is expected to be given by enumerative invariants of X.

Locally, these algebras are expected to be modelled by quivers with potential. In this situation, one can construct the Hall algebra.

This construction generalizes the preprojective/ 2d KHA for a quiver Q.

$$\text{classical HAs} \xrightarrow{\text{affinization}} \text{2d KHAs} \subset \text{3d KHAs}.$$

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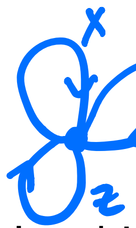
Quivers with potential.

A potential W is a linear combination of cycles in Q.
Example:

Q_3



$W_3 = xyz - xzy$

Define the Jacobi algebra

$$\mathrm{Jac}(Q, W) = \mathbb{C}Q / \left(\frac{\partial W}{\partial e}, e \in E \right).$$

Example: $\mathrm{Jac}(Q_3, W_3) = \mathbb{C}[x, y, z]$.

Let $\mathcal{J}(d)$ be the stack of reps of $\mathrm{Jac}(Q, W)$ of dimension d. We have that

$$\mathcal{J}(d) = \mathrm{crit}\,(\mathrm{Tr}\, W : \mathcal{X}(d) \to \mathbb{A}^1).$$

Example: $\mathcal{T}ors\,(\mathbb{A}^3, d) = \mathrm{crit}\,(\mathrm{Tr}\, W_3 : \mathfrak{gl}(d)^3 / GL(d) \to \mathbb{A}^1)$.

Definition of KHAs for quivers with potential.

Consider the $\mathbb{N}^V$-graded vector space

$$KHA(Q,W)=\bigoplus_{d\in\mathbb{N}^V} K_0(D_{sg}(\mathcal{X}(d)_0))=\bigoplus_{d\in\mathbb{N}^V} K_{\text{crit}}(\mathcal{J}(d)).$$

Here $D_{sg}(\mathcal{X}(d)_0)=D^b\text{Coh}\,(\mathcal{X}(d)_0)/\text{Perf}\,(\mathcal{X}(d)_0)$ is the category of singularities of $\mathcal{X}(d)_0=(\text{Tr}\,W)^{-1}(0)\subset\mathcal{X}(d)$.

Theorem (P.)

$KHA(Q,W)$ is an associative algebra with multiplication $m=p_*q^*$, where $\mathcal{X}(d)\times\mathcal{X}(e)\xleftarrow{q}\mathcal{X}(d,e)\xrightarrow{p}\mathcal{X}(d+e)$.

Relations between 2d and 3d KHAs.

For any quiver Q, there exists a pair $(\widetilde{Q}, \widetilde{W})$ such that

$$KHA^{2d}(Q) = KHA^{3d}(\widetilde{Q}, \widetilde{W}).$$

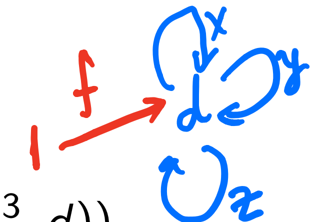
Example: For the Jordan quiver J, the pair is (Q_3, W_3) and the algebra is the positive part of $U_q\left(\widehat{\widehat{\mathfrak{gl}_1}}\right)$.

Representations of KHAs

Theorem (P.)

For any pair (Q, W), KHA has natural representations constructed using framings of Q.

Example: $\mathrm{KHA}(Q_3, W_3)$ acts on

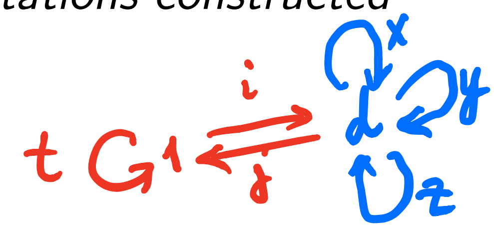


$$\bigoplus_{d \in \mathbb{N}^I} K_0(D_{sg}(\mathcal{X}(1,d)_0^{ss})) = \bigoplus_{d \in \mathbb{N}^I} K_{\mathrm{crit}}(\mathrm{Hilb}\,(\mathbb{A}^3, d)).$$

Theorem (P.)

For a pair $(\widetilde{Q}, \widetilde{W})$, KHA has natural representations constructed using Nakajima quiver varieties.

Example: $\mathrm{KHA}(Q_3, W_3)$ acts on



$$\bigoplus_{d \in \mathbb{N}} K_0(\mathrm{Hilb}\,(\mathbb{A}^2, d)).$$

$W_3 = xyz - xzy$

$W = W_3 + tji$

PBW theorem for KHAs.

Let Q be a symmetric quiver.

Theorem (P.)

There exists a filtration $F^\bullet$ on $KHA(Q,W)$ such that

$$gr\, KHA(Q,W) = dSym\,(F^1).$$

The first piece in the filtration is

$$F^1 = \bigoplus K_0(\mathbb{M}(d)) \subset \mathrm{KHA}(Q,W)$$

for some natural categories $\mathbb{M}(d) \subset D_{sg}(\mathcal{X}(d)_0)$.

Sketch of the proof of the PBW theorem.

The proof follows from semi-orthogonal decompositions in the zero potential case

$$D^b(\mathcal{X}(d)) = \langle p_* q^* \boxtimes \overline{\mathbb{M}}(d_i), \overline{\mathbb{M}}(d) \rangle,$$

where the summands on the left are generated by multiplication from non-trivial decompositions $d = d_1 + \cdots + d_k$.

The category $\overline{\mathbb{M}}(d)$ is the subcategory of $D^b(\mathcal{X}(d))$ generated by $\mathcal{O}_{\mathcal{X}(d)} \otimes V(\chi)$, where χ is a dominant weight of $G(d)$ such that

$$\chi + \rho \in \frac{1}{2}\text{sum}\,[0, \beta] \subset M_{\mathbb{R}}.$$

The above semi-orthogonal decomposition induces corresponding semi-orthogonal decompositions of $D_{sg}(\mathcal{X}(d)_0)$.

Thank you for your attention!