INTERSECTION $K$-THEORY

TUDOR PĂDURARIU

Abstract. For a proper map $f : X \to S$ between varieties over $\mathbb{C}$ with $X$ smooth, we introduce increasing filtrations $P_f^\leq \subseteq P_f^\geq$ on $gr K(X)$, the associated graded on $K$-theory with respect to the codimension filtration, both sent by the cycle map to the perverse filtration on cohomology $pH^*_f(X)$. The filtrations $P_f^\leq$ and $P_f^\geq$ are functorial with respect to proper pushforward; $P_f^\leq$ is functorial with respect to pullback.

We use the above filtrations to propose two definitions of intersection $K$-theory $grK^*(S)$ and $grIK^*(S)$. Both have cycle maps to intersection cohomology $IH^*(S)$. We conjecture a version of the decomposition theorem for semismall surjective maps and prove it in some particular cases.

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1. Introduction

For a complex variety $X$, intersection cohomology $IH^*(X)$ coincides with singular cohomology $H^*(X)$ when $X$ is smooth and has better properties than singular cohomology when $X$ is singular, for example it satisfies Poincaré duality and the Hard Lefschetz theorem. Many applications of intersection cohomology, for example in representation theory [18], [6, Section 4] are through the decomposition theorem of Beilinson–Bernstein–Deligne–Gabber [2].

A construction of intersection $K$-theory is expected to have applications in computations of $K$-theory via a $K$-theoretic version of the decomposition theory, and in representation theory, for example in the construction of representations of vertex algebras using (framed) Uhlenbeck spaces [4]. The Goresky–MacPherson construction of intersection cohomology [16] does not generalize in an obvious way to $K$-theory.

1.1. The perverse filtration and intersection cohomology. For $S$ a variety over $\mathbb{C}$, intersection cohomology $IH^*(S)$ is a subquotient of $H^*(X)$ for any resolution
of singularities \( \pi : X \to S \). More generally, let \( L \) be a local system on an open smooth subscheme \( U \) of \( S \) satisfying the following

**Assumption:** \( L = R^0f_*\mathbb{Q}_{f^{-1}(U)} \) for a generically finite map \( f : X \to S \) from \( X \) smooth such that \( f^{-1}(U) \to U \) is smooth.

The decomposition theorem implies that \( IH(S, L) \) is a (non-canonical) direct summand of \( H^*(X) \). Consider the perverse filtration

\[
pH^i_f(X) := H^*(S, p^\tau \leq i Rf_* IC_X) \subseteq H^*(S, Rf_* IC_X) = H^*(X).
\]

For \( V \hookrightarrow S \), denote by \( X_V = f^{-1}(V) \). Let \( A_V \) be the set of irreducible components of \( X_V \) and let \( c^a_V \) be the codimension on \( X^a_V \hookrightarrow X \). For any component \( X^a_V \), consider a resolution of singularities

\[
\begin{array}{c}
Y^a_V \\
\downarrow \pi^a_V \\
X^a_V \leftarrow \rightarrow \rightarrow X.
\end{array}
\]

Let \( g^a_V := f\pi^a_V : Y^a_V \to V \). Define

\[
pP^i_{f,V} := \bigoplus_{a \in A_V} \iota^a_V \pi^a_V \cdot \mathbb{P}^\leq i-c^a_V(Y^a_V) \subseteq pH^i_f,
\]

\[
pP^{\leq i}_f := \bigoplus_{V \subseteq S} pP^i_{f,V} \subseteq pH^i_f.
\]

The decomposition theorem implies that

\[
IH(S, L) = pP^{\leq 0}_f H^0(X) / pP^{\leq 0}_f H^0(X).
\]

1.2. **Perverse filtration in \( K \)-theory.** Inspired by the above characterization of intersection cohomology via the perverse filtration, we propose two \( K \)-theoretic perverse filtrations \( P^i_f \subset P^{\leq i}_f \) on \( \text{gr} K.(X) \) for a proper map \( f : X \to S \) of complex varieties with \( X \) smooth. Here, the associated graded \( \text{gr} K.(X) \) is with respect to the codimension of support filtration on \( K.(X) \) [14, Definition 3.7, Section 5.4].

The precise definition of the filtration \( P^i_f \text{gr} K.(X) \) is given in Subsection 3.3. Roughly, it is generated by (subspaces of) images

\[
\Gamma : \text{gr} K(T) \to \text{gr} K.(X)
\]

induced by correspondences \( \Gamma \) on \( X \times T \) of restricted dimension, see (3), for \( T \) a smooth variety with a generically finite map onto a subvariety of \( S \). These subspaces satisfy conditions when restricted to the subvarieties \( Y^a_V \) from Subsection 1.1.

The definition of filtration \( P^i_f \text{gr} K.(X) \) is given in Subsection 3.3. We further impose that \( \Gamma \) is a quasi-smooth scheme surjective over \( T \). This further restricts the possible dimension of the cycles \( \Gamma \), see Proposition 3.9 and allows for more computations.
Theorem 1.1. Let \( f : X \to S \) be a proper map with \( X \) smooth. Then the cycle map \( ch : grK_0(X)_\mathbb{Q} \to H^*(X) \) respects the perverse filtration

\[
P^\leq_1 grK_0(X)_\mathbb{Q} \subset P^\leq_1 grK_0(X)_\mathbb{Q} \xrightarrow{ch} pH^\leq_1(X).
\]

Perverse filtrations in \( K \)-theory have the following functorial properties. Let \( X \) and \( Y \) be smooth varieties with \( c = \dim X - \dim Y \). Consider proper maps

\[
Y \xrightarrow{h} X \xleftarrow{g} S.
\]

There are induced maps

\[
h_* : P^\leq_{i-c} grK.(Y) \to P^\leq_i grK.(X),
\]

\[
h_* : P^\leq_{i-c} grK.(Y) \to P^\leq_i grK.(X),
\]

\[
h^* : P^\leq_{i-c} grK.(X) \to P^\leq_i grK.(Y).
\]

If \( h \) is surjective, then there is also a map

\[
h^* : P^\leq_{i-c} grK.(X) \to P^\leq_i grK.(Y).
\]

Let \( S \) be a singular scheme, a local system \( L \), and a smooth variety \( X \) as in Subsection 1.1. We define \( \widetilde{P}^0 grK.(X) \) and \( \widetilde{P}^0 grK.(X) \) similarly to \( P^\leq_i H^*(X) \). Inspired by the discussion in cohomology from Subsection 1.1, define

\[
grIK.(S, L) := P^0 grK.(X)/ \left( P^0 grK.(X) \cap \ker f_* \right)
\]

\[
grIK.(S, L) := P^0 grK.(X)/ \left( P^0 grK.(X) \cap \ker f_* \right).
\]

Theorem 1.2. The definitions of \( grIK.(S, L) \) and \( grIK.(S, L) \) do not depend on the choice of the map \( f : X \to S \) with the properties mentioned above. Further, there are cycle maps

\[
ch : gr^j IK_0(S, L)_\mathbb{Q} \to IH^{2j}(S, L)
\]

\[
ch : gr^j IK_0(S, L)_\mathbb{Q} \to IH^{2j}(S, L).
\]

1.3. Properties of the perverse filtration intersection \( K \)-theory. The perverse filtration in \( K \)-theory and intersection \( K \)-theory have similar properties to their counterparts in cohomology.

For a map \( f : X \to S \), let \( s := \dim X \times_S X - \dim X \) be the defect of semismallness. In Theorem 3.11 we show that

\[
P^\leq_{s-1} grK_0(X) = 0,
\]

\[
P^\leq_s grK_0(X) = P^\leq_s grK_0(X) = grK_0(X).
\]
This implies that
\[
gr^iIK(S) = gr^iK(S) = gr^iK_0(S) \quad \text{for } S \text{ smooth},
\]
\[
gr^iIK_0(S) = gr^iK_0(X) \quad \text{if } S \text{ has a small resolution } f : X \to S.
\]

For more computations of perverse filtrations in $K$-theory and intersection $K$-theory, see Subsections 3.7 and 4.4.

In cohomology, there are natural maps
\[
H^i(S) \to IH^i(S) \to H^{BM}_{2d-i}(S)
\]
\[
IH^i(S) \otimes IH^j(S) \to H^{BM}_{2d-i-j}(S).
\]
The composition in the first line is the natural map $H^i(S) \to H^{BM}_{2d-i}(S)$. The second map is non-degenerate for cycles of complementary dimensions. In Subsection 4.3 we explain that there exist natural maps
\[
gr^iIK(S) \to gr^iG(S)
\]
\[
gr^iIK(S) \times gr^jIK(S) \to gr_{d-i-j}G(S)
\]
and their analogues for $IK$. The above filtration on $G$-theory is by dimension of supports, see [14, Section 5.4].

1.4. The decomposition theorem for semismall maps. As mentioned above, many applications of intersection cohomology are based on the decomposition theorem. When the map
\[
f : X \to S
\]
is semismall, the statement of the decomposition theorem is more explicit, which we now explain. Let \( \{S_a | a \in I\} \) be a stratification of $S$ such that $f_a : f^{-1}(S_a^0) \to S_a^0$ is a locally trivial fibration, where $S_a^0 = S_a - \bigcup_{b \in f} (S_a \cap S_b)$. Let $A \subset I$ be the set of relevant strata, that is, those strata such that for $x_a \in S_a^0$:
\[
\dim f^{-1}(x_a) = \frac{1}{2} (\dim S - \dim S_a).
\]
For $x_a \in S_a^0$, the monodromy group $\pi_1(S_a^0, x_a)$ acts on the set of irreducible components of $f^{-1}(x_a)$ of top dimension; let $L_a$ be the corresponding local system. Let $c_a$ be the codimension of $X_a = f^{-1}(S_a)$ in $X$. The decomposition theorem for the map $f : X \to S$ says that there exists a canonical decomposition [6, Theorem 4.2.7]:
\[
H^j(X) = \bigoplus_{a \in A} IH^{j-c_a}(S_a, L_a).
\]

We conjecture the analogous statement in $K$-theory.

**Conjecture 1.3.** Let $f : X \to S$ be a semismall map and consider $\{S_a | a \in I\}$ a stratification as above, and let $A \subset I$ be the set of relevant strata. There is a decomposition for any integer $j$:
\[
gr^jK_0(X)_Q = \bigoplus_{a \in A} gr^{j-c_a}IK(S_a, L_a)_Q.
\]
See Conjecture 5.1 for a more precise statement. In Theorem 5.4, we check the above conjecture for $K_0$ under the extra condition that for any $a \in A$, there are small maps $\pi_a : T_a \to S_a$ satisfying the Assumption in Subsection 1.1. The proof of the above result is based on a theorem of de Cataldo–Migliorini [5, Section 4]. In Subsection 4.4.4, we prove the statement for $K_0$ when $f : X \to S$ is a resolution of singularities of a surface.

1.5. Past and future work. When $X$ is smooth, $\text{gr}^r K_0(X)_\mathbb{Q} = CH^i(X)_\mathbb{Q}$. Thus $\text{gr}^r IK_0(S)_\mathbb{Q}$ is a candidate for intersection Chow groups of $S$. Corti–Hanamura already defined intersection Chow groups (or Chow motives) in [9], [10] inspired by the decomposition theorem. One proposed definition assumes conjectures of Grothendieck and Murre and proves a version of the decomposition theorem for Chow groups; the other approach defines a perverse-type filtration on Chow groups by induction on level $i$ of the filtration and via correspondences involving all varieties $W \to S$ with certain properties for the perverse filtration in cohomology. The advantage in our definition is that one can control the correspondences used to define $P_{f}^{\leq i}$ and $P_{f}^{\leq i}$ and allows for computations, see Subsection 4.4 and Theorem 5.3.

For varieties $S$ with a semismall resolution $f : X \to S$ with $L$ a local system satisfying the Assumption in Subsection 1.1 de Cataldo–Migliorini [5] proposed a definition of Chow motives $ICH(S,L)$ and proved a version of the decomposition theorem for semismall maps.

It is an important problem to find a definition of the perverse filtration on $K_*(X)$ which recovers the above definition when passing to $\text{gr}^r K_*(X)$. A natural such definition will also provide a definition of equivariant intersection $K$-theory with applications to geometric representation theory, for example in understanding the $K$-theoretic version of [4]. However, our approach uses functoriality of the perverse filtration in an essential way for which it is essential to pass to $\text{gr}^r K_*(X)$.

There are proposed definitions of intersection $K$-theory in particular cases. Cautis [7], Cautis–Kamnitzer [8] have an approach for categorification of intersection sheaves for certain subvarieties of the affine Grassmannian. Eberhardt defined intersection $K$-theoretic sheaves for varieties with certain stratifications [11]. In [19], we proposed a definition of intersection $K$-theoretic for good moduli spaces which has applications to the structure theory of Hall algebras of Kontsevich–Soibelman [20].


We plan to compare some of these intersection $K$-theoretic/ Chow groups in future work.

1.6. Acknowledgements. I thank the Institute of Advanced Studies for support during the preparation of the paper. This material is based upon work supported by the National Science Foundation under Grant No. DMS-1926686.
2. Preliminary material

2.1. Notations and conventions. All schemes considered in this paper are finite type quasi-projective over \( \mathbb{C} \). The definition of the filtration in Subsection 3.1 works over any field, but to define intersection \( K \)-theory we use resolution of singularities, and the construction works over any field of characteristic zero. A variety is an irreducible reduced scheme.

For \( S \) a scheme, let \( D^b \text{Coh}(S) \) be the derived category of bounded complexes of coherent sheaves and \( \text{Perf}(S) \) its subcategory of bounded complexes of locally free sheaves on \( S \). The functors used in the paper are derived; we sometimes drop \( R \) or \( L \) from notation, for example we write \( f^* \) instead of \( Rf^* \). When \( S \) is smooth, the two categories coincide. Define

\[
G·(S) = K.(D^b \text{Coh}(S))
\]

\[
K·(S) = K.(\text{Perf}(S)).
\]

For \( Y \) a subvariety of \( X \), let \( D^b \text{Coh}_Y(X) \) be the subcategory of \( D^b \text{Coh}(X) \) of complexes supported on \( Y \), and define

\[
G_Y·(X) := K \left( D^b \text{Coh}_Y(X) \right).
\]

When \( X \) is smooth, we also use the notation \( K_Y·(X) \) for the above. We will usually drop the subscript \( · \) from the notation.

Singular and intersection cohomology are used only with rational coefficients.

2.2. Filtrations in \( K \)-theory. A reference for the following is [13], especially Section 5 in loc. cit. Let \( F^n G.(S) \) be the filtration on \( G.(S) \) by sheaves with support of codimension \( \geq i \); it induces a filtration on \( K.(S) \). The associated graded will be denoted by \( \text{gr} G.(S), \text{gr} K.(S) \). A morphism \( f : X \to Y \) of smooth varieties induces maps:

\[
f^* : F^n K.(Y) \to F^n K.(X)
\]

\[
f^* : \text{gr}^i K.(Y) \to \text{gr}^i K.(X).
\]

Further, let \( F_{i\dim}^n G.(S) \) be the filtration on \( G.(S) \) by sheaves with support of dimension \( \leq i \); it induces a filtration on \( K.(S) \). The associated graded will be denoted by \( \text{gr} G.(S), \text{gr} K.(S) \). A proper morphism \( f : X \to Y \) of schemes induces maps:

\[
f_* : F_{i\dim}^n G.(X) \to F_{i\dim}^n G.(Y)
\]

\[
f_* : \text{gr} G.(X) \to \text{gr} G.(Y).
\]

There are similar filtrations and associated graded on \( G_Y(X) \) for \( Y \hookrightarrow X \) a subvariety. If \( X \) is smooth of dimension \( d \), then \( \text{gr}_i G_Y(X) = \text{gr}^{d-i} G_Y(X) \).

Proposition 2.1. Let \( S \to \text{Spec} \mathbb{C} \) be a variety of dimension \( d \). Then

\[
\left( a^*, \bigoplus_{T \supseteq S} t_T \right) : G_0(\text{Spec} \mathbb{C}) \oplus \bigoplus_{T \supseteq S} \text{gr}_0 G_0(T) \to \text{gr}_0 G_0(S),
\]
where the sum is taken over all proper subvarieties $T$ of $S$.

Proof. For $i < d$, the map
\[
\bigoplus_{T \subseteq S} \bigoplus_{T \subseteq S} \text{gr}_i G_0(T) \to \text{gr}_i G_0(S)
\]
is surjective by definition of the filtration $F^i_{\text{dim}}$. Finally, the following map is an isomorphism
\[
a^* : G_0(\text{Spec } \mathbb{C}) \xrightarrow{\sim} \text{gr}_d G_0(S).
\]
\[\square\]

Proposition 2.2. Let $S$ be a singular variety of dimension $d$, and let $f : X \to S$ be a resolution of singularities. The following map is surjective:
\[
f_* : \text{gr}_i G_0(X) \to \text{gr}_i G_0(S).
\]

Proof. We use induction on $d$. By Proposition 2.1, the following is an isomorphism
\[
f_* : \text{gr}_d G_0(X) \xrightarrow{\sim} \text{gr}_d G_0(S) \xrightarrow{\sim} G_0(\text{Spec } \mathbb{C}).
\]
For $V \subseteq S$ a subvariety, consider $g$ a resolution of singularities as follows:
\[
\begin{array}{ccc}
Y & \longrightarrow & X \\
\downarrow^g & & \downarrow^f \\
V & \longleftarrow & S.
\end{array}
\]
The surjectivity of $f_*$ for $i < d$ follows using Proposition 2.1 and the induction hypothesis. \[\square\]

2.3. The perverse filtration in cohomology. Let $S$ be a scheme over $\mathbb{C}$. Let $D^b_c(S)$ be the derived category of bounded complexes of constructible sheaves \[6, \text{Section 2}\]. Consider the perverse $t$-structure \[(P^{\leq i}, P^{\geq i})_{i \in \mathbb{Z}}\] on this category. There are functors:
\[
\begin{align*}
p_{\tau^\leq i} & : D^b_c(S) \to P^{\leq i}, \\
p_{\tau^\geq i} & : D^b_c(S) \to P^{\geq i}
\end{align*}
\]
such that for $F \in D^b_c(S)$ there is a distinguished triangle in $D^b_c(S)$:
\[
p_{\tau^\leq i} F \to F \to p_{\tau^{\geq i}+1} F \xrightarrow{[1]}.
\]
For a proper map $f : X \to S$ and $F \in D^b_c(X)$, the perverse filtration on $H^\cdot(X, F)$ is defined as the image of
\[
pH^{\leq i}_f(X, F) := H^\cdot(S, p_{\tau^\leq i} Rf_* F) \to H^\cdot(S, Rf_* F) = H^\cdot(X, F).
\]
For $F = IC_X$, the decomposition theorem implies that
\[
pIH^{\leq i}_f(X) \hookrightarrow IH^\cdot(X).
\]
Let $f : X \to S$ be a generically finite morphism from $X$ smooth, let $U$ be a smooth open subset of $X$ such that $f^{-1}(U) \to U$ is smooth, and let $L = R^0 f_* \mathbb{Q}_{f^{-1}(U)}.$
For \( V \hookrightarrow S \), denote by \( X_V := f^{-1}(V) \). Let \( A_V \) be the set of irreducible components of \( X_V \). Let \( c_V^a \) be the codimension on \( X_V \hookrightarrow X \). Further, consider a resolution of singularities \( \pi_V^a \) such that:

\[
Y_V^a \xrightarrow{\pi_V^a} X_V^a \xrightarrow{c_V^a} X.
\]

Let \( g_V^a := f\pi_V^a : Y_V^a \to V \). Then

\[
p_{\tau \leq 0} Rf_*IC_X = \ker \left( Rf_*IC_X \to \bigoplus_{V \subseteq S, a \in A_V} (p_{\tau > c_V^a} Rg_V^a_*IC_{Y_V^a}) [c_V^a] \right).
\]

Define the subspace

\[
p_{\tau \leq 0} Rf_*IC_X = \image \left( \bigoplus_{V \subseteq S, a \in A_V} (p_{\tau \leq -c_V^a} Rg_V^a_*IC_{Y_V^a}) [-c_V^a] \to p_{\tau \leq 0} Rf_*IC_X \right).
\]

By a computation of Corti–Hanamura \[10\], Proposition 1.5, Theorem 2.4], we have that:

1. \( IC_S(L) = p_{\tau \leq 0} Rf_*IC_X / p_{\tau \leq 0} Rf_*IC_X \).

Further, consider a more general morphism \( f : X \to S \) with \( X \) smooth. Let \( V \subseteq S \) be a subvariety. For \( i \in \mathbb{Z} \), denote by \( p^H_i(Rf_*IC_X)_V \) the direct sum of simple summands of \( p^H_i(Rf_*IC_X) \) with support equal to \( V \). A computation of Corti–Hanamura \[10\], Proposition 1.5] shows that:

2. \( p^H_i(Rf_*IC_X)_V \hookrightarrow \bigoplus_{a \in A_V} p^H_i^{+c_V^a}(Rg_V^a_*IC_{Y_V^a}) \).

3. The perverse filtration in \( K \)-theory

3.1. Definition of the filtration \( P^{\leq i}_{\tau} \). Let \( f : X \to S \) be a proper map between varieties. We define an increasing filtration

\[ P^{\leq i}_{\tau} \text{gr}^G.(X) \subset \text{gr}^G.(X). \]

It induces a filtration on \( \text{gr}^K.(X) \). We use the notations from Subsection 2.3. Let \( Y \hookrightarrow X \) be a subvariety and let \( T \xrightarrow{\pi} S \) be a map generically finite onto its image from \( T \) smooth. Consider the diagram:

\[
\begin{array}{ccc}
T \times X & \xrightarrow{p} & X \\
\downarrow{q} & & \downarrow{f} \\
T & \xrightarrow{\pi} & S
\end{array}
\]

For a correspondence \( \Gamma \in \text{gr}_{\dim X - s} G_{T \times _SY}^0(T \times X) \), define

\[
\Phi_{\Gamma} := p_*(\Gamma \otimes q^!(\cdot)) : \text{gr}^K_i(T) \to \text{gr}^{-s} G_{Y, i}(X).
\]
We usually drop the shift by $s$ in the superscript of $\text{gr}G_Y(X)$. We also drop the subscript on relative $K$-theory. We define the subspace of $\text{gr}G_Y(X)$:

$$P_{f,T}^{i \leq i} := \text{span}_T \left( \Phi_T : \text{gr} \, K(T) \to \text{gr}G_Y(X) \right)$$

$$P_f^{i \leq i} := \text{span} \left( P_{f,T}^{i \leq i} \text{ for all maps } \pi \text{ as above} \right),$$

where the dimension of the correspondence satisfies

$$(3) \quad \left\lfloor \frac{i + \dim X - \dim T}{2} \right\rfloor \geq s.$$

We also define a quotient of $\text{gr}G_Y(X)$:

$$P_f^{i \leq i} \text{gr}G_Y(X) \hookrightarrow \text{gr}G_Y(X) \twoheadrightarrow P_f^{i > i} \text{gr}G_Y(X).$$

### 3.2. Functoriality of the filtration $P^{i \leq i}$

**Proposition 3.1.** Let $X$ and $Y$ be smooth varieties with $c = \dim X - \dim Y$. Consider proper maps

$$Y \xrightarrow{h} X \xleftarrow{g} T \xrightarrow{f} S.$$

There are induced maps

$$h^* : P_f^{i \leq i - c} \text{gr} \, K(X) \to P_g^{i \leq i} \text{gr} \, K(Y).$$

**Proof.** Let $T \to S$ be a generically finite map onto its image with $T$ smooth. It suffices to show that

$$h^* : P_f^{i \leq i - c} \text{gr} \, K(X) \to P_g^{i \leq i} \text{gr} \, K(Y).$$

Consider the diagram:

$$\xymatrix{ Y \ar[r]^h & X \ar[d]^{g} & \ar[l]_f S \ar[d]^{h} \ar[dr]^{q_Y} & \ar[d]^{q_X} \ar[r] & T \ar[r]^\theta & X \times T \ar[l]_{q_Y} }$$

Let $\Theta \in \text{gr}_{\dim X - s} G_{T \times S, 0}(T \times X)$ be a correspondence such that

$$i \geq 2s - \dim X + \dim T.$$

For $j \in \mathbb{Z}$, we have that:

$$\text{gr}^j K(T) \xrightarrow{\Phi_{\Theta}} \text{gr}^{j - s} K(X)$$

$$\xrightarrow{\Phi_{h^* \Theta}} \text{gr}^{j - s} K(Y).$$
To see this, we compute:

\[ h^* \Phi(F) = h^* p_{X*}(\Theta \otimes q_X^* F) = p_{Y*} \tilde{h}^* (\Theta \otimes q_Y^* F) = p_{Y*} (\tilde{h}^* \Theta \otimes q_Y^* F) = \Phi_{\tilde{h}*\Theta}(F). \]

The correspondence \( \tilde{h}^* \Theta \in \text{gr}_{\dim Y-s} G_{T \times S} Y(T \times Y) \) satisfies

\[ i + c \geq 2s - \dim Y + \dim T, \]

and this implies the desired conclusion. \( \square \)

**Proposition 3.2.** Let \( X \) and \( Y \) be varieties with proper maps

\[ Y \xrightarrow{h} X \]

\[ S \xrightarrow{g} \]

Let \( c = \dim X - \dim Y \). There are induced maps

\[ h_* : P^{\leq i-c} g_* G(Y) \rightarrow P^{\leq i} f_* G(X). \]

**Proof.** Let \( T \rightarrow S \) be a generically finite map onto its image from \( T \) smooth. We first explain that

\[ h_* : P^{\leq i-c} g_* G(Y) \rightarrow P^{\leq i} f_* G(X). \]

We use the notation from the proof of Theorem 3.1. Consider a correspondence \( \Gamma \in \text{gr}_{\dim Y-s} G_{T \times S} Y(T \times Y) \) such that

\[ i \geq 2s - \dim Y + \dim T. \]

For \( j \in \mathbb{Z} \), we have that:

\[ \text{gr}_{\dim Y-j} G(T) \xrightarrow{\Phi_*} \text{gr}_{\dim Y-j+s} G(Y) \]

\[ \xrightarrow{\Phi_{\tilde{h}_*\Gamma}} \]

\[ \text{gr}_{\dim Y-j+s} G(X). \]

To see this, we compute:

\[ h_* p_{Y*}(\Gamma \otimes q_Y^* F) = p_{X*} \tilde{h}_*(\Gamma \otimes \tilde{h}^* q_X^* F) = p_{X*} (\tilde{h}_* \Gamma \otimes q_X^* F). \]

The correspondence

\[ \tilde{h}_* \Gamma \in \text{gr}_{\dim Y-s} G_{T \times S} X(T \times X) = \text{gr}_{\dim X-(c+s)} G_{T \times s} X(T \times X) \]

satisfies

\[ i + c \geq 2(s + c) - \dim X + \dim T, \]

and thus the conclusion follows. \( \square \)

We continue with some further properties of the filtration \( P^{\leq i-c} \). The following is immediate:

**Proposition 3.3.** Let \( f : X \rightarrow S \) be a proper map. Let \( U \) be an open subset of \( S \), \( X_U := f^{-1}(U) \), \( \iota : X_U \rightarrow X \), and \( f_U : X_U \rightarrow U \). Then

\[ \iota^* : P^{\leq i} f_* G(X) \rightarrow P^{\leq i} f_U_* G(X_U). \]
Proposition 3.4. Let \( f : X \to S \) be a proper map from \( X \) smooth and consider \( e \in \text{gr}^i K_0(X) \). Then
\[
e \cdot P_f^{\leq i} \text{gr}^i K(X) \subset P_f^{\leq i+2j} \text{gr}^{i+j} K(X).
\]

Proof. Let \( T \to S \) be a generically finite map onto its image with \( T \) smooth and let \( \Theta \in \text{gr}_a G_{T \times S, 0}(T \times X) \). Let \( p : T \times X \to X \) be the natural projection. Then
\[
p^*(e) \cdot \Theta \in \text{gr}_a G_{T \times S, 0}(T \times X).
\]
For \( x \in \text{gr} K(T) \), we have that
\[
e \cdot \Phi_\Theta(x) = \Phi_{p^*(e) \cdot \Theta}(x),
\]
and the conclusion thus follows. \( \square \)

Proposition 3.5. Let \( X \) and \( Y \) be smooth varieties with proper maps
\[
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow{g} & & \downarrow{f} \\
S & & \\
\end{array}
\]
such that \( h \) is surjective. Let \( c = \dim X - \dim Y \). Then
\[
h_* \left( P_f^{\leq i} \text{gr} K(Y)_\mathbb{Q} \right) = P_f^{\leq i+c} \text{gr} K(X)_\mathbb{Q} \\
h_* \text{gr} K(X)_\mathbb{Q} \cap P_f^{\leq i+c} \text{gr} K(Y)_\mathbb{Q} = h_* P_f^{\leq i} \text{gr} K(X)_\mathbb{Q}.
\]
If there exists \( X' \to Y \) such that the induced map \( X' \to X \) is birational, then the above isomorphisms hold integrally.

Proof. The statement and its proof are similar to [10, Proposition 3.11].

Let \( i : X' \to Y \) be a map such that \( hi : X' \to X \) is generically finite and surjective. Then, by Proposition 3.2
\[
P_f^{\leq i} \text{gr} K(X') \xrightarrow{i_*} P_f^{\leq i} \text{gr} K(Y) \xrightarrow{h_*} P_f^{\leq i+c} \text{gr} K(X).
\]
The map \( h_* i_* : \text{gr} K(X') \to \text{gr} K(X) \) is multiplication by the degree of the map \( hi \), so it is an isomorphism rationally; it is an isomorphism integrally if \( X' \to X \) has degree 1. The pullback statement is similar. \( \square \)

3.3. The filtration \( P^\leq \). Let \( f : X \to S \) be a proper map from \( X \) smooth. Let \( V \subset S \) be a subvariety, and let \( A_V \) the set of irreducible components of \( f^{-1}(V) \). For an irreducible component \( X_V^0 \), consider a resolution of singularities \( \pi_V^0 \) as follows:
\[
\begin{array}{c}
\overline{X}_V^a & \xrightarrow{\pi_V^a} & X_V^a & \xrightarrow{\Delta^a} X \\
\downarrow{f_V^a} & & \downarrow{f} \\
V & \xrightarrow{f} & S,
\end{array}
\]
Let $c^a_V$ be the codimension of $X^a_V$ in $X$. Denote by $\pi^a_V = \iota^a_V \tau^a_V$. Consider a subvariety $Y \hookrightarrow X$. Define

$$P_f^{\leq i} \overline{\text{gr}} G_Y(X) := \bigcap_{V \subset S} \bigcap_{a \in A_V} \ker \left( \iota^a_V^* : P_f^{\leq i} \overline{\text{gr}} G_Y(X) \to P^r_{\bar{f}_V} \overline{\text{gr}} K(\bar{X}_V) \right).$$

The definition is independent of the resolutions $\pi^a_V$ chosen. For two different resolutions $\bar{X}^a_V$, $\bar{X}^{a'}_V$, there exists $W$ such that

$$W \bar{X}^a_V \bar{X}^{a'}_V$$

where the maps $\pi$ and $\pi'$ are successive blow-ups along smooth subvarieties of $\bar{X}^a_V$ and $\bar{X}^{a'}_V$, respectively. Let $\tau^a_V : \bar{X}^a_V \to X$ as above. Then $\tau^a_V \pi = \tau^a_V \pi'$. By Proposition 3.5

$$\ker \left( \tau^a_V^* : P_f^{\leq i} \overline{\text{gr}} G_Y(X) \to P^r_{\bar{f}_V} \overline{\text{gr}} K(\bar{X}_V) \right) =$$

$$\ker \left( \pi^* \tau^a_V^* : P_f^{\leq i} \overline{\text{gr}} G_Y(X) \to P^r_{\bar{f}_V} \overline{\text{gr}} K(W) \right) =$$

$$\ker \left( \tau^{a'}_{V'}^* : P_f^{\leq i} \overline{\text{gr}} G_Y(X) \to P^r_{\bar{f}_V} \overline{\text{gr}} K(\bar{X}'_V) \right).$$

**Theorem 3.6.** Let $X$ and $Y$ be smooth varieties with $c = \dim X - \dim Y$. Consider proper maps

$$Y \xrightarrow{h} X \xleftarrow{f} S.$$

There are induced maps

$$h^* : P_f^{\leq i - c} \overline{\text{gr}} K.(X) \to P_f^{\leq i} \overline{\text{gr}} K.(Y)$$

$$h_* : P_f^{\leq i - c} \overline{\text{gr}} K.(Y) \to P_f^{\leq i} \overline{\text{gr}} K.(X).$$

**Proof.** The functoriality of $h^*$ follows from Proposition 3.1 and induction on dimension of $S$.

We discuss the statement for $h_*$. We use induction on the dimension of $S$. The case of $S = \text{Spec}(C)$ is clear as $P_f^{\leq i} = P_f^{\leq i}$. We use the notation from the beginning of Subsection 3.3. Let $V$ be a subvariety of $S$. Let $X^a_V$ be an irreducible component of $f^{-1}(V)$ with a resolution of singularities $X^a_V \to X^a_V$. Let $B$ be the
set of irreducible component of $Y_V$ over $X_V^a$. For $b \in B$, consider a resolution of singularities $\widetilde{Y}_V^b \to Y_V^b$ such that

$$\bigcup_{b \in B} \widetilde{Y}_V^b \xrightarrow{\bigoplus_B h_b^V} \widetilde{X}_V^a \quad \xrightarrow{\bigoplus_B \tau_b^V} \tau_V^\ast.$$ 

Consider the cartesian diagram

$$\begin{array}{ccc}
Y_V^{\text{der}} & \xrightarrow{\tilde{h}} & \widetilde{X}_V^a \\
\downarrow{\tau} & & \downarrow{\tau_V^\ast} \\
Y & \xrightarrow{h} & X.
\end{array}$$

The scheme $Y_V^{\text{der}}$ is quasi-smooth, see [17] for a definition, and $\text{reldim} \tilde{h} = \text{reldim} h$.

For $b \in B$, there is a map $p_b : \widetilde{Y}_V^b \to Y_V^{\text{der}}$. Let $d_b = \dim \widetilde{Y}_V^b - \dim Y_V^{\text{der}}$ and define

$$e_b = \text{det} \left( L_{\tau_V^b} / h_b^V L_{\tau_V^b} \right) \in \text{gr}^{d_b} K_0 \left( \widetilde{Y}_V^b \right),$$

where by $L_{\tau}$ we denote the cotangent complex of the map $\tau$.

By a version of the excess intersection formula, the following diagram commutes:

$$\begin{array}{ccc}
\text{gr} K._V(Y) & \xrightarrow{h_{\ast}} & \text{gr} K._V(X) \\
\downarrow{\bigoplus_B \tau_V^b} & & \downarrow{\tau_V^\ast} \\
\bigoplus_B \text{gr} K.(Y_V^b) & \xrightarrow{\bigoplus_B e_b} & \bigoplus_B \text{gr} K.(\widetilde{Y}_V^b) \xrightarrow{\bigoplus_B h_b^V} K.(\widetilde{X}_V^a).
\end{array}$$

We ignore shifts in the gradings above. Consider the diagram

$$\begin{array}{ccc}
\bigcup_{b \in B} \widetilde{Y}_V^b & \xrightarrow{\bigoplus_B h_b^V} & \widetilde{X}_V^a \\
\downarrow{\bigcup_B p_b} & & \downarrow{\tau_V^\ast} \\
\bigoplus_B \tau_V^b & \xrightarrow{\bigoplus_B \tau_V^b} & Y \xrightarrow{h} X.
\end{array}$$

Then

$$\sum_{b \in B} h_b^V (e_b \cdot \tau_V^b) = \sum_{b \in B} \tilde{h}_b p_b (e_b \cdot p_b^\ast \tau_b^\ast) = \tilde{h}_b \left( \left( \sum_{b \in B} p_b e_b \right) \cdot \tau_b^\ast \right).$$
It suffices to show that
\[
\sum_{b \in B} p_b e_b = 1 \in \text{gr}^0 K_0 \left( Y_V^{\text{der}} \right).
\]

The underlying scheme $Y_V^{\text{cl}}$ has irreducible component indexed by $B$ birational to $\tilde{Y}_V^b$. There exist open sets $W = \bigsqcup_{b \in B} W^b \subset Y_V^{\text{der}}, U^b \subset \tilde{Y}_V^b$ whose complements have codimension $\geq 1$ and such that
\[
W^{b, \text{cl}} = U^b.
\]

After possibly shrinking the open sets, we can assume that for any $b \in B$:
\[
U^b = W^b \times Y_V^{\text{der}} \tilde{Y}_V^b
\]
\[
\mathcal{O}_{U^b} = \mathcal{O}_{U^b} \left[ \bigwedge \mathcal{E}[1]; d \right],
\]

where $\mathcal{E}$ is a vector bundle on $U^b$ of dimension $d_b$ and the differential $\mathcal{E} \to \mathcal{O}_{U^b}$ is zero. Let $i_b : W^b \to W^{b, \text{cl}} = U^b$ and let $\varepsilon_b := i_b^*(1) \in \text{gr}^{d_b} K_0 (U^b)$ be the Euler class of $\mathcal{E}$. Then $p_b (\varepsilon_b) = 1 \in \text{gr}^0 K_0 (W^b)$ and the restriction map sends
\[
\text{res} : \text{gr}^{d_b} K_0 \left( Y_V^b \right) \to \text{gr}^{d_b} K_0 \left( U^b \right)
\]
\[
e_b \mapsto \varepsilon_b.
\]

Back to proving (5), we have that $\text{gr}^0 K_0 \left( Y_V^{\text{der}} \right) \cong \bigoplus_{b \in B} \text{gr}^0 K_0 (W^b)$. Consider the diagram
\[
\begin{array}{ccc}
\text{gr}^{d_b} K_0 \left( Y_V^b \right) & \xrightarrow{\text{res}} & \text{gr}^{d_b} K_0 \left( U^b \right) \\
\downarrow & & \downarrow \\
\text{gr}^0 K_0 \left( Y_V^{\text{der}} \right) & \xrightarrow{\text{res}} & \text{gr}^0 K_0 (W^b),
\end{array}
\]

where the horizontal maps are restriction to open sets maps. Then
\[
\text{res} p_b (e_b) = p_b (\varepsilon_b) = 1 \text{ in } \text{gr}^0 K_0 (W^b).
\]

The diagram thus commutes. The conclusion now follows from Propositions 3.2 and 3.3.

3.4. Towards the filtration $P^e_j$. We continue with the notation from Subsection 3.1. Let $X$ be a smooth variety with a proper map $f : X \to S$. Let $T \xrightarrow{p} S$ a generically finite map onto its image from $T$ smooth.

We say that $\Gamma$ is a $T$-quasi-smooth scheme if $\Gamma$ is a derived scheme with maps
\[
\begin{array}{ccc}
X' & \xrightarrow{\iota} & X \\
\downarrow & & \downarrow \\
\Gamma & \xrightarrow{p} & X \\
\downarrow & & \downarrow \\
T & \longrightarrow & S
\end{array}
\]
such that $\iota$ is a closed immersion in a smooth variety $X'$ (i.e. the cotangent complex $L_\iota$ is a vector bundle on $\Gamma$), $t$ is smooth, and $q^{\text{cl}}$ is surjective. The conditions of the maps $\iota$ and $t$ imply that $\Gamma$ is quasi-smooth, see [17] for a definition. Let

$$\text{gr}^q K_{T \times S}^q(\Gamma) \subset \text{gr}^q K_{T \times S}^q(T \times X)$$

be the subspace generated by classes $[\Gamma]$ for $T$-quasi-smooth schemes.

**Proposition 3.7.** Let $h$ be a proper map:

$$
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow{g} \quad \quad \downarrow{f} & & \quad \quad \downarrow{f} \\
S & & X'
\end{array}
$$

There are induced maps

$$h_* : \text{gr}^q K_{T \times S}^q(T \times Y) \to \text{gr}^q K_{T \times S}^q(T \times X).$$

If $h$ is surjective, then there are induced maps

$$h^* : \text{gr}^q K_{T \times S}^q(T \times X) \to \text{gr}^q K_{T \times S}^q(T \times Y).$$

**Proof.** We discuss the statement about pullback. Consider the diagram:

$$
\begin{array}{ccc}
\Theta & \hookrightarrow & Y' \\
\downarrow{r} & & \downarrow{h} \\
\Gamma & \hookrightarrow & X' \\
\downarrow{q} & & \downarrow{t_X} \\
T & \longrightarrow & S,
\end{array}
$$

where $\Gamma$ is a quasi-smooth scheme with $q^{\text{cl}}$ is surjective, $t_X$ is smooth, and the upper squares are cartesian. Then the map $\Theta \hookrightarrow Y$ is a closed immersion and $t_Y$ is smooth. The map $h$ is surjective, so $r^{\text{cl}} : \Theta^{\text{cl}} \to \Gamma^{\text{cl}}$ is surjective, and thus $(qr)^{\text{cl}} : \Theta^{\text{cl}} \to T$ is surjective as well.

We next discuss the statement about pushforward. Consider

$$
\begin{array}{ccc}
Y' & \xrightarrow{t_Y} & Y \\
\downarrow{h'} \quad \quad \downarrow{h} & & \quad \quad \downarrow{f} \\
\Gamma & \xrightarrow{t_X} & X \\
\downarrow{q} & & \downarrow{f} \\
T & \longrightarrow & S,
\end{array}
$$

such that $\iota$ is a closed immersion, $t$ is smooth, and $q^{\text{cl}}$ is surjective. The map $Y' \to X$ is a proper map of smooth quasi-projective varieties, so we can choose $X'$ with maps

$$
Y' \xrightarrow{t'} X' \xrightarrow{t'} X.
$$
such that $i'$ is a closed immersion and $t'$ is smooth. Then

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{h_{p}} & \Gamma' \\
\downarrow^{q} & & \downarrow^{q'} \\
T & \xrightarrow{f} & S
\end{array}
$$

such that $i'_{h_{p}}$ is a closed immersion, $t'$ is smooth, and $q^c$ is surjective. \qed

Consider a diagram

$$
\begin{array}{ccc}
\Gamma & \xrightarrow{i} & \Gamma' \\
\downarrow^{q} & & \downarrow^{q'} \\
T & \xrightarrow{f} & S
\end{array}
$$

as above, with $t$ a smooth map and with $i$ a closed immersion. Let

$$T \times_{S} X = Z_1 \cup Z_2,$$

where $Z_1$ is the union of irreducible components of $T \times_{S} X$ dominant over $T$ and $Z_2$ is the union of the other irreducible components. Denote by $Z_1' := Z_1 - (Z_1 \cap Z_2)$. Similarly define $Z_1'$ and $Z_2'$ for $T \times_{S} X'$. Let $b = \text{reldim } q$ and $a = b + \dim T = \dim \Gamma$.

**Proposition 3.8.** The class $[\Gamma] \in \text{gr}_{a} K_{T \times_{S} X}(T \times X')$ is not supported on $Z_2'$.

**Proof.** Let $\ell$ be an $ft$-ample divisor; it also induces a $g$-ample class. Denote by $\text{pr}_{1} : T \times X' \to T$. Then

$$\text{pr}_{1*} \left( [\Gamma] \cdot \ell^b \right) = d[T] \in \text{gr}_{\text{dim } T} K(T)$$

for $d$ a non-zero integer. Let $\eta$ be the generic point of $T$; by abuse of notation, we denote by $\eta$ its image in $S$. It suffices to show the analogous result when restricting to $\eta$, and $d$ is the intersection number $\ell^b \cdot \Gamma_{\eta}$ in $X'_{\eta}$.

Further, let $x \in \text{gr}_{a} K_{Z_1'}(T \times X')$. We have that

$$\text{pr}_{1*} (x \cdot \ell^b) = 0 \in \text{gr}_{\text{dim } T} K(T)$$

because the support on $x \cdot \ell^b$ is not dominant over $T$. The conclusion thus follows. \qed

**Proposition 3.9.** Let $T \xrightarrow{\pi} X$ be a generically finite map from $T$ smooth with image $V$. Let $a > \dim X_{V}$. Then $\text{gr}_{a} K_{T \times_{S} X}(T \times X) = 0$. Further, $\text{gr}_{\text{dim } X_{V}} K_{T \times_{S} X}(T \times X)$ is generated by irreducible components of $T \times_{S} X$ dominant over $T$ of dimension $X_{V}$. 

Proof. Suppose we are in the setting of (6) and let \( s : X \to X' \) be a section of \( t \).
Assume that
\[
t_*, t_*[\Gamma] = p_*[\Gamma] \neq 0 \in \text{gr}_{a} K^q_{T \times S Y}(T \times X).
\]
Then there exists a non-zero \( x \in \text{gr}_{a} K^q_{T \times S Y}(T \times X) \) such that
\[
p_*[\Gamma] = s_* (x) \in \text{gr}_{a} K^q_{T \times S Y}(T \times X').
\]
Consider the diagram
\[
\begin{array}{ccc}
\text{gr}_{a} K^q_{T \times S Y}(T \times X) & \xrightarrow{\text{res}} & \text{gr}_{a} K^q_{Z_1^0}(T \times X - Z_2) \\
\uparrow s_* & & \uparrow s_* \\
\text{gr}_{a} K^q_{T \times S Y}(T \times X') & \xrightarrow{\text{res}} & \text{gr}_{a} K^q_{Z_1^0}(T \times X - Z_2).
\end{array}
\]
By Proposition 3.8 we have that \( \text{res}(x) \neq 0 \in \text{gr}_{a} K^q_{Z_1^0}(T \times X - Z_2) \). We have that \( \dim Z_1^0 = \dim X_V \), and the conclusion follows from here.

3.5. The perverse filtration \( P^\leq_i \). We now define a smaller filtration \( P^{<i}_f \subset P^\leq_i \).
We use the notation from Subsection 3.1.

Let \( X \) be a smooth variety with a proper map \( f : X \to S \) and let \( T \rightarrow S \) be a generically finite map onto its image from \( T \) smooth. Consider a subvariety \( Y \hookrightarrow X \).
Define the subspaces of \( \text{gr} G_Y(X) \):
\[
P^{<i}_{f,T} := \text{span}_{\Gamma} \left( \Phi_{\Gamma} : \text{gr} K(T) \to \text{gr} G_Y(X) \right)
\]
\[
P^{<i}_{f,V} := \text{span} \left( P^{<i}_{f,T} \text{ for all maps } \pi \text{ as above } V \right),
\]
where \( \Gamma \in \text{gr}_{a \dim X - s} K^q_{T \times S Y,0}(T \times X) \) and
\[
\left\lfloor \frac{i + \dim X - \dim T}{2} \right\rfloor \geq s.
\]
Using the notation from Subsection 3.3 define
\[
P^{<i}_f \text{gr} G_Y(X) := \bigcap_{V \subset S} \bigcap_{a \in A_V} \ker \left( \tau^g_{V} : P^{<i}_f \text{gr} G_Y(X) \to P^{<i}_{g} \text{gr} K_h(X_{V_a}) \right).
\]
The definition is independent of the resolutions \( \overline{X_a} \) chosen, see Subsection 3.3.

Theorem 3.10. Let \( X \) and \( Y \) be smooth varieties with \( c = \dim X - \dim Y \). Consider proper maps
\[
\begin{array}{ccc}
Y & \xrightarrow{h} & X \\
\downarrow g & & \downarrow f \\
S & \xrightarrow{k} & f
\end{array}
\]
There are induced maps
\[
h_* : P^{c}_{g} \text{gr} K(Y) \to P^{c}_{f} \text{gr} K(X)
\]
\[
h_* : P^{c}_{g} \text{gr} K(Y) \to P^{c}_{f} \text{gr} K(X).
\]
If \( h \) is surjective, then there are induced maps

\[
h^* : P_{f}^{\leq i - c} \text{gr} K(X) \to P_{g}^{\leq i} \text{gr} K(Y)
\]

\[
h^* : P_{f}^{\leq i - c} \text{gr} K(X) \to P_{g}^{\leq i} \text{gr} K(Y).
\]

**Proof.** The functoriality follow as in Propositions 3.1, 3.2, and Theorem 3.6, using Proposition 3.7. \( \square \)

### 3.6. Properties of the perverse filtration.

Consider a proper map \( f : X \to S \) with \( X \) smooth. Define the defect of semismallness of \( f \) by

\[
s := s(f) = \dim X \times S - \dim X.
\]

Further, define \( s' = \max (\dim X + \dim S - 4, \dim X) \). It is known [6, Section 1.6] that the perverse filtration in cohomology satisfies

\[
pH_{f}^{\leq -s-1}(X) = 0 \quad \text{and} \quad pH_{f}^{\leq s}(X) = H(X).
\]

We prove an analogous result in \( K \)-theory:

**Theorem 3.11.** For \( f \) as above,

\[
P_{f}^{\leq -s'-1} \text{gr} K(X) = P_{f}^{\leq -s-1} \text{gr} K(X) = 0
\]

\[
P_{f}^{\leq s} \text{gr} K_0(X) = P_{f}^{\leq s} \text{gr} K_0(X) = \text{gr} K_0(X).
\]

**Proposition 3.12.** Let \( f : X \to S \) be a surjective map from \( X \) smooth and consider a subvariety \( Z \hookrightarrow X \) of codimension \( \geq 2 \). Then there exists a subvariety \( \iota : Y \hookrightarrow X \) of codimension 1 such that \( Z \subset Y \) and \( f \circ \iota : Y \to S \) is surjective.

**Proof.** It suffices to pass to an open subset of \( Z \), and we can thus assume that \( Z \hookrightarrow X \) is given by a regular closed immersion with functions \( f_1, \ldots, f_r \) with \( r \geq 2 \). Pick \( f \in (f_1, \ldots, f_r) \) such that \( Z(f) \) is surjective onto \( S \). \( \square \)

**Proposition 3.13.** Let \( f : X \to S \) be a proper surjective map from \( X \) smooth of relative dimension \( d \). Then

\[
P_{f}^{\leq d} \text{gr} K_0(X) = \text{gr} K_0(X).
\]

**Proof.** We use induction on \( d \). Assume that \( f \) is generically finite. Consider the correspondence \( \Delta \cong X \hookrightarrow X \times_S X:\)

\[
\Delta \overset{\sim}{\longrightarrow} X \\
\downarrow \sim \quad \downarrow f \\
X \overset{f}{\longrightarrow} S.
\]

This implies that \( P_{f}^{\leq 0} \text{gr} K(X) = \text{gr} K(X) \).

Consider a general \( f \). Let \( \iota : Z \hookrightarrow X \) be a subvariety of codimension \( \geq 2 \). By Proposition 3.12 there exists \( Y \to X \) of codimension 1 such that \( Z \subset Y \) and \( Y \to S \) is surjective. Let \( Y' \to Y \) be a resolution of singularities and denote the resulting map by \( g : Y' \to S \). By induction,

\[
P_{g}^{\leq d-1} \text{gr} K_0(Y') = \text{gr} K_0(Y').
\]
By Proposition 2.2

\[ \text{image } (\iota_* : \text{gr}\ G_0(Z) \to \text{gr}\ K_0(X)) \subset \text{image } (g_* : \text{gr}\ K_0(Y') \to \text{gr}\ K_0(X)). \]

Finally, assume that \( Z \hookrightarrow Y \) has codimension 1. By Proposition 2.1 it suffices to show that

\[ \text{image } (\text{gr}\dim Z G_0(Z) \to \text{gr}\dim Z K_0(X)) \subset \text{image } \Phi \Gamma \subset P_{f \leq d} \]

because \( \text{gr}\ i G_0(Z) \) for \( i < \dim Z \) is generated by varieties of smaller dimension than \( Z \). If \( Z \to S \) is surjective, then it has relative dimension \( d - 1 \) and we can treat it as above. If \( Z \to S \) is not surjective, let \( W \subset S \) be its image. Choose a resolution of singularities \( T \to W \) and a smooth variety \( \Gamma \) with surjective maps \( p \) and \( q \):

\[
\begin{array}{ccc}
\Gamma & \xrightarrow{p} & Z \\
& \downarrow q & \downarrow f \\
T & \xrightarrow{} & W \\
& & \xrightarrow{} S.
\end{array}
\]

Then \( [\Gamma] \in \text{gr}\dim X - b G_{T \times S} X (T \times X) \) and its image \( \Phi \Gamma \) is in \( P_{f \leq d} \text{gr}\ K_0(X) \). Then

\[ \text{image } (\text{gr}\dim Z G_0(Z) \to \text{gr}\dim Z K_0(X)) \subset \text{image } \Phi \Gamma \subset P_{f \leq d} \text{gr}\ K_0(X). \]

The conclusion now follows from Proposition 2.1.

\[ \square \]

**Proof of Theorem 3.11.** We first show that \( P_{f \leq -s} \text{gr}\ K(X) = 0 \). Consider a map \( \pi : T \to X \) generically finite onto its image \( V \subset S \) with \( T \) smooth and consider a correspondence

\[ \Gamma \in \text{gr}\dim X - b G_{T \times S} X (T \times S). \]

Then \( \dim X - b \leq \dim T \times S X \leq \max (\dim X, \dim X + \dim T - 2) \), and so

\[ b \geq \min (0, -\dim T + 2). \]

By the bound (3), it suffices to show that

\[ \left\lfloor \frac{-s' - 1 + \dim X - \dim T}{2} \right\rfloor < \min (0, -\dim T + 2) \]

\[ \max (\dim X - \dim T - 1, \dim X + \dim T - 5) < s', \]

which is true because \( 0 \leq \dim T \leq \dim S \).

We next explain that \( P_{f \leq -s} \text{gr}\ K(X) = 0 \). We keep the notation from the previous paragraph. Let \( [\Gamma] \in \text{gr}\dim X - b K_{f \times S} (T \times S) \). By Proposition 3.9 we have that

\[ b \geq \dim X - \dim X_V. \]

It suffices to show that

\[ \left\lfloor \frac{-s - 1 + \dim X - \dim T}{2} \right\rfloor < \dim X - \dim X_V \]

\[ 2 \dim X_V - \dim V \leq s - \dim X = \dim X \times_S X, \]

which is true because \( 2 \dim X_V - \dim V \leq \dim X_V \times_S X \leq \dim X \times_S X. \)
We next show that \( P^e f_* \text{gr} K_0(X) = \text{gr} K_0(X) \). We can assume that \( f \) is surjective of relative dimension \( d \). Use the notation from Subsection 3.3. We have that

\[
P^e f_* \text{gr} K_0(X) := \bigcap_{V \in S} \bigcap_{a \in A_V} \ker \left( r^a_V : P^e f_* \text{gr} K_0(X) \to P^e f_* \text{gr} K_0(X_{V}) \right).
\]

We claim that

\[
\text{reldim} (X_V^0 \to V) = \text{reldim} (X_V^0 \to V) \leq s + c_V^0.
\]

Indeed,

\[
\dim X_V^0 - \dim V \leq (\dim X \times S X - \dim X) + (\dim X - \dim X_V^0) = 2 \dim X_V^0 - \dim V \leq \dim X \times_S X, \quad \text{which is true. By Proposition 3.13 this implies that } P^e f_* \text{gr} K_0(X_{V}) = 0.
\]

Furthermore, \( s \geq d \), so Proposition 3.13 implies that \( P^e f_* \text{gr} K_0(X) = \text{gr} K_0(X) \), and thus \( P^e f_* \text{gr} K_0(X) = \text{gr} K_0(X) \). This also implies that \( P^e f_* \text{gr} K_0(X) = \text{gr} K_0(X) \).

□

3.7. Examples of perverse filtration in K-theory.

3.7.1. Let \( X \) be a smooth variety of dimension \( d \), and let \( f : X \to \text{Spec} \mathbb{C} \). Then

\[
P^e f_* \text{gr}^j K_0(X) = \begin{cases} \text{gr}^j K_0(X) & \text{if } j \leq \left\lfloor \frac{i + d}{2} \right\rfloor, \\ 0 & \text{otherwise.} \end{cases}
\]

3.7.2. Let \( X \) be a smooth variety and let \( E \) be a vector bundle on \( X \) of rank \( d + 1 \). Let \( Y := \mathbb{P}_X(E) \). Denote by \( h := c_1(\mathcal{O}_Y(1)) \in \text{gr}^2 K_0(Y) \). Consider the projection map \( f : Y \to X \). We have that \( s(f) = d \). For \( i \leq d \), there exists an isomorphism

\[
\bigoplus_{0 \leq j \leq \left\lfloor \frac{i + d}{2} \right\rfloor} \text{gr}^{a - 2j} K(X) \cong P^e f_* \text{gr}^a K(Y) \quad (x_0, \ldots, x_{\left\lfloor \frac{i + d}{2} \right\rfloor}) \mapsto \sum_{j \leq \left\lfloor \frac{i + d}{2} \right\rfloor} h^j f^*(x_j).
\]

The condition for \( P^e f^i \) is checked using projective bundles over varieties of smaller dimension, and we obtain that

\[
\bigoplus_{0 \leq j \leq \left\lfloor \frac{i + d}{2} \right\rfloor} \text{gr}^{a - 2j} K(X) \cong P^e f_* \text{gr}^a K(Y).
\]

3.7.3. Let \( X \) be a smooth variety and let \( Z \) be a smooth subvariety of codimension \( d + 1 \). Consider the blow-up diagram for \( Y = \text{Bl}_Z X \):

\[
\begin{array}{ccc}
E & \xrightarrow{\iota} & Y \\
\downarrow & & \downarrow \\
Z & \xleftarrow{j} & X.
\end{array}
\]
Let \( h := c_1(O_E(1)) \in \text{gr}^2 K_0(E) \). We have that \( s(f) = d - 1 \). For \( i \leq d - 1 \), there is an isomorphism:

\[
\text{gr}^a K(X)^{\varepsilon} \oplus \bigoplus_{0 \leq j \leq \left\lfloor \frac{i + d}{2} \right\rfloor - 1} \text{gr}^{a-2-2j} K(Z) \cong P_{f_j}^{\leq i} \text{gr}^a K(Y)
\]

\[
(x, z_0, \cdots, z_{\left\lfloor \frac{i + d}{2} \right\rfloor - 1}) \mapsto f^*(x) + \sum_{j \in \left( \frac{i + d}{2} \right) - 1} \iota_*(h^j q^*(z_j)).
\]

Here \( \varepsilon \) is 0 if \( i < 0 \) and is 1 otherwise. This follows from the computation in Subsection 3.7.2 and Proposition 4.4.

One can check that in the above examples, we have that \( P_{f_j}^{\leq i} = P_{f_j}^{\leq i} \).

### 3.8. Compatibility with the perverse filtration in cohomology.

Consider a proper map \( f : X \to S \) with \( X \) smooth. Define filtrations \( P^{\leq i}_f, P^{\leq i}_f \) on \( H^*(X), H^*(X)_{\text{alg}} \) as in Subsections 3.1 and 3.5. We have that

\[
\text{image} \left( \text{ch} : P^{\leq i}_f \text{gr} K_0(X)_Q \to P^{\leq i}_f \text{gr} H^*(X) \right) = P^{\leq i}_f \text{gr} H^*(X)_{\text{alg}}.
\]

We use the notation \( p^H_{\leq i} (X)_{\text{full}} \) for the cohomology of summands of \( p^H_{\leq i} f_* IC_X \) with support \( S \).

**Proposition 3.14.** There exist natural inclusions

\[
P^{\leq i}_f H^*(X) \subset P^{\leq i}_f H^*(X) \subset p^H_{\leq i} (X)
\]

\[
P^{\leq i}_f H^*(X)_{\text{alg}} \subset P^{\leq i}_f H^*(X)_{\text{alg}} \subset p^H_{\leq i} (X)_{\text{alg}}.
\]

Thus the cycle map restricts to

\[
\text{ch} : P^{\leq i}_f \text{gr} K_0(X)_Q \to p^H_{\leq i} (X)_{\text{alg}}
\]

\[
\text{ch} : P^{\leq i}_f \text{gr} K_0(X)_Q \to p^H_{\leq i} (X)_{\text{alg}}.
\]

**Proof.** Let \( \pi : T \to S \) be a generically finite map with \( T \) smooth. Consider a correspondence

\[
\Gamma \in \text{gr}_{\text{dim } X - s} K_{T \times S, 0}(T \times X)
\]

such that

\[
\left\lfloor \frac{i + \text{dim } X - \text{dim } T}{2} \right\rfloor \geq s.
\]

The correspondence \( \Gamma \) induces a map of constructible sheaves on \( S \):

\[
R\pi_* Q_T[-2s] \xrightarrow{q_T} Rf_* Q_X.
\]

\[
Rp_* IC_T[\text{dim } X - \text{dim } T - 2s] \xrightarrow{q_T} Rf_* IC_X.
\]

If \( \pi \) is not surjective, \( R\pi_* IC_T \) has summands with support \( W \subsetneq S \). If \( \pi \) is surjective, the complex \( R\pi_* IC_T \) has summands \( IC_S(L) \) of full support and of perverse
Thus $\mathcal{P}^{S} f H (X)$ contains cohomology of sheaves $\mathcal{I} \mathcal{C}_{S} ([L] j)$ with $j \leq i$ which appear as summands of $R f * \mathcal{I} \mathcal{C} X$ and of other sheaves with support $W \subsetneq S$. Thus

$$ P_{f}^{P \leq i} H (X) \rightarrow p H_{f}^{P \leq i} (X)_{\text{full}}. $$

Using the notation in Subsection 3.3 we have that

$$ P_{f}^{S \leq i} H (X) := \bigcap_{V \subseteq S} \bigcap_{a \in A_{V}} \ker \left( \tau_{V}^{a} : P_{f}^{S \leq i} H (X) \rightarrow P_{f}^{S \leq i + c_{V}^{a}} H (X_{V})_{\text{alg}} \right). $$

In particular,

$$ P_{f}^{S \leq i} H (X) \subset \bigcap_{V \subseteq S} \bigcap_{a \in A_{V}} \ker \left( \tau_{V}^{a} : P_{f}^{S \leq i} H (X) \rightarrow P_{f}^{S \leq i + c_{V}^{a}} (X_{V})_{\text{full}} \right). $$

Using (2), we obtain that $P_{f}^{S \leq i} H (X) \subset p H_{f}^{S \leq i} (X)$.

□

**Remark.** We expect equalities $P_{f}^{S \leq i} H (X)_{\text{alg}} = P_{f}^{S \leq i} H (X)_{\text{alg}} = p H_{f}^{S \leq i} (X)_{\text{alg}}$ in the above proposition.

4. Intersection $K$-theory

4.1. Definition of intersection $K$-theory. Let $S$ be a variety and let $L$ be a local system on an open smooth subset $U$ of $S$ such that there exists a generically finite proper map $f : X \rightarrow S$ such that $X$ is smooth, $f^{-1} (U) \rightarrow U$ is smooth, and $L = R^{0} f * \mathbb{Q}_{f^{-1} (U)}$. Recall the notation of Subsection 3.3 Define

$$ \tilde{P}_{f}^{S \leq i} \text{gr} K (X) := \text{image} \left( \bigoplus_{V \subseteq S} \bigoplus_{a \in A_{V}} P_{f}^{S \leq i} \text{gr} K_{X_{V}} (X) \rightarrow P_{f}^{S \leq i} \text{gr} K (X) \right). $$

$$ \tilde{P}_{f}^{S \leq i} \text{gr} K (X) := \text{image} \left( \bigoplus_{V \subseteq S} \bigoplus_{a \in A_{V}} P_{f}^{S \leq i} \text{gr} K_{X_{V}} (X) \rightarrow P_{f}^{S \leq i} \text{gr} K (X) \right). $$

Define

$$ \text{gr} I K (S, L) := P_{f}^{S \leq 0} \text{gr} K (X) / \left( \tilde{P}_{f}^{S \leq 0} \text{gr} K (X) \cap \ker f_{*} \right) $$

$$ \text{gr} I K (S, L) := P_{f}^{S \leq 0} \text{gr} K (X) / \left( \tilde{P}_{f}^{S \leq 0} \text{gr} K (X) \cap \ker f_{*} \right). $$

**Theorem 4.1.** The definitions of $\text{gr} I K (S, L)$ and $\text{gr} I K (S, L)$ do not depend on the choice of the map $f : X \rightarrow S$ with the properties mentioned above.
We start with some preliminary results. Let $f : X \to S$ be a proper map with $X$ smooth. Let $Z$ be a smooth subvariety of $X$ with normal bundle $N, Y = \text{Bl}_Z X$, and $E = \mathbb{P}_Z (N)$ the exceptional divisor

$$
\begin{array}{c}
E & \xleftarrow{\iota} & Y \\
\downarrow{\pi} & & \downarrow{\pi} \\
Z & \xrightarrow{j} & X.
\end{array}
$$

Consider the proper maps

$$
\begin{array}{c}
E & \xleftarrow{\iota} & Y & \xrightarrow{\pi} & X \\
\phantom{E} & \downarrow{\nu} & \phantom{Y} & \downarrow{\pi} & \phantom{X} \\
S' & \xrightarrow{j'} & Y & \xrightarrow{\pi'} & X
\end{array}
$$

Let $X' \hookrightarrow X$ be a closed subset, and denote its preimages in $Y, Z, E$ by $Y', Z', E'$ respectively. Denote by $\text{gr} K_{Y'}(Y)^0 = \ker (\pi_* : \text{gr} K_{Y'}(Y) \to \text{gr} K_{X'}(X))$.

**Proposition 4.2.** Let $T \to S$ be a map with $T$ smooth which is generically finite onto its image. Then

$$
\begin{align*}
\text{gr} K_{T \times_Z Y'}(T \times Y) &= \pi_* \text{gr} K_{T \times_Z X'}(T \times X) \oplus \text{gr} K_{T \times_Z E'}(T \times Y)^0 \\
\text{gr} K_{T \times_Z Y'}^q(T \times Y) &= \pi_* \text{gr} K_{T \times_Z X'}^q(T \times X) \oplus \text{gr} K_{T \times_Z E'}^q(T \times Y)^0.
\end{align*}
$$

**Proof.** Let $c + 1$ be the codimension of $Z$ in $X$. Denote by $O(1)$ the canonical line bundle on $E$ and let $\hbar = c_1(O(1)) \in \text{gr}^2 K_0(E)$. There is a semi-orthogonal decomposition [3] Theorem 4.2:

$$D^b(Y) = \left\langle \pi^* D^b(X), \tau_* \left( p^* D^b(Z) \otimes O(-1) \right), \cdots, \tau_* \left( p^* D^b(Z) \otimes O(-c) \right) \right\rangle,$$

which implies that

$$\text{gr}^j K(Y) = \pi_* \text{gr}^j K(X) \oplus \bigoplus_{0 \leq k \leq c - 1} \tau_* \left( h^k \cdot p^* \text{gr}^{j-2-2k} K(Z) \right).$$

Using the analogous decomposition for $Y - Y' = \text{Bl}_{Z - Z'}(X - X')$ and the localization sequence in K-theory [21] V.2.6.2, we obtain that

$$\text{gr}^j K_{Y'}(Y) = \pi_* \text{gr}^j K_{X'}(X) \oplus \bigoplus_{0 \leq k \leq c - 1} \tau_* \left( h^k \cdot p^* \text{gr}^{j-2-2k} K_{Z'}(Z) \right).$$

In particular, we have that

$$\text{gr}^j K_{T \times_{S} Y'}(T \times Y) = \pi_* \text{gr}^j K_{T \times_{S} X'}(T \times X) \oplus \bigoplus_{0 \leq k \leq c - 1} \tau_* \left( h^k \cdot p^* \text{gr}^{j-2-2k} K_{T \times_{S} Z'}(T \times Z) \right)$$

and thus that

$$\text{gr} K_{T \times_{S} Y'}(T \times Y) = \pi_* \text{gr} K_{T \times_{S} X'}(T \times X) \oplus \text{gr} K_{T \times_{S} E'}(T \times Y)^0.$$
By Proposition 3.7 we also have that
\[ \text{gr} \cdot K^q_{T \times T'}(T \times Y) = \pi^* \text{gr} \cdot K^q_{T \times x T'}(T \times X) \oplus \text{gr} \cdot K^q_{T \times x E'}(T \times Y)^0. \]

□

An immediate corollary of Proposition 4.2 is:

**Corollary 4.3.** We continue with the notation from Proposition 4.2. There are decompositions
\[ P^\leq_g \text{gr} \cdot K^q_X(Y) = \pi^* P^\leq_g \text{gr} \cdot K^q_X(X) \oplus P^\leq_g \text{gr} \cdot K^q_E(Y) \]
\[ P^\leq_g \text{gr} \cdot K^q_X(Y) = \pi^* P^\leq_g \text{gr} \cdot K^q_X(X) \oplus P^\leq_g \text{gr} \cdot K^q_E(Y). \]

We next prove:

**Proposition 4.4.** We continue with the notation from Proposition 4.2. There are decompositions
\[ P^\leq_g \text{gr} \cdot K^q_X(Y) = \pi^* P^\leq_g \text{gr} \cdot K^q_X(X) \oplus P^\leq_g \text{gr} \cdot K^q_E(Y) \]
\[ P^\leq_g \text{gr} \cdot K^q_X(Y) = \pi^* P^\leq_g \text{gr} \cdot K^q_X(X) \oplus P^\leq_g \text{gr} \cdot K^q_E(Y). \]

**Proof.** We use the notation from Subsection 3.3. For \( V \subseteq S \), let \( A_V \) be the set of irreducible components of \( f^{-1}(V) \). Let \( X^a_V \) be such a component.

If \( X^a_V \subset Z \), then there is only one irreducible component \( Y^a_V = \mathbb{P}_{X^a_V}(N) \) of \( g^{-1}(V) \) above it.

If \( X^a_V \) is not in \( Z \), then there is one component \( Y^a_V \) of \( g^{-1}(V) \) birational to \( X^a_V \). The other components are \( \mathbb{P}_{W^b_V}(N) \), where \( W^b_V \) is an irreducible component of \( X^a_V \cap Z \). Denote by \( B_a \) the set of such components. For \( a \in A \) and \( b \in B_a \), consider resolutions of singularities such that

\[
\begin{array}{cccccc}
\tilde{Y}^a_V & \rightarrow & Y^a_V & \rightarrow & Y \\
\downarrow & & \downarrow & & \downarrow \\
\tilde{X}^a_V & \rightarrow & X^a_V & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
X^a_V \cap Z & & X^a_V & \rightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
W^b_V & \rightarrow & W^b_V.
\end{array}
\]

Denote by \( \tau \) the maps as in Subsection 3.3 for example \( \tau^a_V : \tilde{X}^a_V \rightarrow X \), and by \( \mu \) the map

\[ \tau^b_V : W^b_V \rightarrow X^a_V \rightarrow X. \]
We consider the proper maps
\[ f^a_V : X^a_V \to X^a_V \to V \]
\[ g^a_V : Y^a_V \to Y^a_V \to V \]
\[ f^b_V : W^b_V \to W^b_V \to V \]
\[ g^b_V : \mathbb{P}^{-N}(N) \to \mathbb{P}^{-N}(N) \to V. \]

Denote by
\[ c^a_V = \text{codim} (X^a_V \text{ in } X) = \text{codim} (Y^a_V \text{ in } Y) \]
\[ c^b_V = \text{codim} (W^b_V \text{ in } X) \]
\[ c^b_V = \text{codim} (\mathbb{P}^{-N}(N) \text{ in } Y) \]
the codimensions as in Subsection 3.3. By Proposition 3.5 we have that
\[ \ker \left( \tau^a_V : \pi^* P^i_{f^a} \text{gr} K(X) \to P^{r^a+i^a}_{g^a_V} \text{gr} K. \left( Y^a_V \right) \right) \cong \]
\[ \ker \left( \tau^a_V : P^i_{f^a} \text{gr} K(X) \to P^{r^a+i^a}_{f^a_V} \text{gr} K. \left( X^a_V \right) \right). \]

By Proposition 3.5 and Proposition 3.1 for the map \( \mu \) in [7], we have that
\[ \ker \left( \tau^b_V : \pi^* P^i_{f^b} \text{gr} K(X) \to P^{r^b+i^b}_{g^b_V} \text{gr} K. \left( \mathbb{P}^{-N}(N) \right) \right) \cong \]
\[ \ker \left( \tau^b_V : P^i_{f^b} \text{gr} K(X) \to P^{r^b+i^b}_{f^b_V} \text{gr} K. \left( W^b_V \right) \right) \supset \]
\[ \ker \left( \tau^b_V : P^i_{f^b} \text{gr} K(X) \to P^{r^b+i^b}_{f^b_V} \text{gr} K. \left( \mathbb{P}^{-N}(N) \right) \right). \]

Let \( B_V \) be the set of irreducible components of \( g^{-1}(V) \). For \( d \in B_V \), denote by \( g^d_V : \tilde{Y}^d_V \to V \) and let \( c^d_V := \text{codim} (Y^d_V \text{ in } Y) \). We have that \( B_V = A \cup \bigcup_{a \in A} B_a \). The statements in [8] and [9] imply that
\[ \pi_* : \bigcap_{V \subseteq S \in B_V} \bigcap_{d \in B_V} \ker \left( \tau^d_V : \pi^* P^i_{f^d} \text{gr} K(X) \to P^{r^d+i^d}_{g^d_V} \text{gr} K. \left( \tilde{Y}^d_V \right) \right) \cong \]
\[ \bigcap_{V \subseteq S \in A_V} \bigcap_{a \in A} \ker \left( \tau^a_V : P^i_{f^a} \text{gr} K(X) \to P^{r^a+i^a}_{f^a_V} \text{gr} K. \left( \mathbb{P}^{-N}(N) \right) \right) \cong \]
\[ \bigcap_{V \subseteq S \in A_V} \bigcap_{a \in A} \ker \left( \tau^a_V : P^i_{f^a} \text{gr} K(X) \to P^{r^a+i^a}_{f^a_V} \text{gr} K. \left( \mathbb{P}^{-N}(N) \right) \right). \]

Using Corollary 4.3 we obtain that
\[ P^i_{g^q} \text{gr} K_{Y^d}(Y) = \pi^* P^i_{f^d} \text{gr} K_{X^d}(X) \oplus P^i_{g^q} \text{gr} K_{E^d}(Y)^0. \]

The analogous statement for \( P^{<i} \) follows similarly. \( \Box \)
Proof of Theorem 4.1. Any two such varieties \( f : X \to S \) and \( f' : X' \to S \) are birational, so by [1] there is a smooth variety \( W \) such that

\[
\begin{array}{ccc}
W & \xrightarrow{\pi} & X \\
\pi' & & \pi \\
\downarrow & & \downarrow \\
S & \xleftarrow{f} & X'
\end{array}
\]

and the maps \( \pi \) and \( \pi' \) are successive blow-ups along smooth subvarieties of \( X \) and \( X' \), respectively. It thus suffices to show that

\[
P^\leq 0 \text{gr} K(X) / \left( \tilde{P}^\leq 0 \text{gr} K(X) \cap \ker f_* \right) \cong P^\leq 0 \text{gr} K(Y) / \left( \tilde{P}^\leq 0 \text{gr} K(Y) \cap \ker g_* \right),
\]

where \( \pi : Y \to X \) is the blow up along smooth subvariety \( Z \hookrightarrow X \) and

\[
\begin{array}{ccc}
Y & \xrightarrow{\pi} & X \\
\downarrow & & \downarrow \\
S & \xleftarrow{g} & X'
\end{array}
\]

By Proposition 4.4, we have that

\[
P^\leq i \text{gr} K(Y) = \pi^* P^\leq i \text{gr} K(X) \oplus P^\leq i \text{gr} K E(Y)^0
\]

\[
\tilde{P}^\leq i \text{gr} K(Y) = \pi^* \tilde{P}^\leq i \text{gr} K(X) \oplus P^\leq i \text{gr} K E(Y)^0.
\]

Taking the quotients we thus obtain the isomorphism (10). The analogous statement for \( IK \) is similar. \( \square \)

4.2. Cycle map for intersection \( K \)-theory. Let \( S \) be a variety and consider a local system \( L \) on an open smooth subset \( U \) of \( S \) such that there exists a map \( f : X \to S \) as in Subsection 4.1.

**Proposition 4.5.** The cycle map \( ch : gr^j K_0(S) \to H^{2j}(X) \) induces cycle maps independent of the map \( f : X \to S \) as in Subsection 4.1:

\[
ch : gr^j IK_0(S, L) \to IH^{2j}(S, L)
\]

\[
ch : gr^j IK_0(S, L) \to IH^{2j}(S, L).
\]

**Proof.** Define \( P^\leq j X^j (X) \) as in Subsection 3.1 and denote by

\[
\tilde{P}^\leq 0 H^j (X) := \text{image} \left( \bigoplus_{V \subset S} \bigoplus_{u \in \mathcal{A}_r} P^\leq j X^j (X) \to H^j (X) \right) \cap P^\leq 0 H^j (X).
\]

Denote by \( P^\leq 0 H^j (X) \subset P^\leq 0 H^j (X) \) the sum of summands of \( P^\tau R^j IC_X \) with support strictly smaller than \( S \). By Proposition 3.14 the cycle map respects the perverse
filtrations in $K$-theory and cohomology

$$\text{ch} : P^0_f \text{gr}^j K_0(X) \rightarrow P^0_f H^{2j}(X) \hookrightarrow P^0_f H_f^{2j}(X).$$

Taking the quotient and using (1), we obtain a map

4.3. **Further properties of intersection $K$-theory.** Intersection cohomology satisfies the following properties, the second one explaining its name [9, Motivation]:

- The natural map $H^i(S) \rightarrow H_{2d-i}^{BM}(S)$ factors through $H^i(S) \rightarrow IH^i(S) \rightarrow H_{2d-i}^{BM}(S)$.

- There is a natural intersection map

$$IH^i(S) \otimes IH^j(S) \rightarrow H_{2d-i-j}^{BM}(S)$$

which is non-degenerate for $i + j = 2d$.

We prove analogous, but weaker versions of the above properties in $K$-theory.

**Proposition 4.6.** (a) There are natural maps

$$\text{gr}^j K.(S) \rightarrow \text{gr}_{d-i} G.(S)$$
$$\text{gr}^j K.(S) \rightarrow \text{gr}_{d-i} G.(S).$$

(b) There are natural intersection maps

$$\text{gr}^j K.(S) \otimes \text{gr}^j K.(S) \rightarrow \text{gr}_{d-i-j} G.(S)$$
$$\text{gr}^j K.(S) \otimes \text{gr}^j K.(S) \rightarrow \text{gr}_{d-i-j} G.(S).$$

**Proof.** Let $f : X \rightarrow S$ be a resolution of singularities. We discuss the claims for $IK_{\text{S}}$, the ones for $IK_{\text{K}}$ are similar. We construct maps as above using $f$; they are independent by $f$ by an argument as in Theorem 4.1

(a) There is a natural map $\text{gr}^i K.(X) = \text{gr}_{d-i} G.(X) \xrightarrow{f^*} \text{gr}_{d-i} G.(S)$, and we thus obtain a map

$$\text{gr}^i K.(S) = P^0_f \text{gr}^i K.(X)/\left(\tilde{P}^0_f \text{gr}^i K.(X) \cap \ker f^*\right) \rightarrow \text{gr}_{d-i} G.(S).$$

(b) Consider the composite map

$$P^0_f \text{gr}^i K.(X) \otimes P^0_f \text{gr}^j K.(X) \rightarrow \text{gr}^{i+j} K.(X \times X) \xrightarrow{\Delta_*} \text{gr}^{i+j} K.(X) \xrightarrow{f^*} \text{gr}_{d-i-j} G.(S).$$

The subspaces

$$\left(\tilde{P}^0_f \text{gr}^i K.(X) \cap \ker f^*\right) \otimes P^0_f \text{gr}^i K.(X)$$
$$P^0_f \text{gr}^i K.(X) \otimes \left(\tilde{P}^0_f \text{gr}^i K.(X) \cap \ker f^*\right)$$
are in the kernel of $f_*\Delta^* = \Delta^* (f_* \boxtimes f_*)$. We thus obtain the desired map. □

4.4. Computations of intersection $K$-theory.

4.4.1. If $S$ is smooth, then $\text{gr} IK_*(S) = \text{gr} IK_*(S) = \text{gr} K_*(S)$.

4.4.2. Let $f : X \to S$ be a small resolution of singularities. Then

$$\text{gr} IK_0(S) = \text{gr} K_0(X).$$

Let $T \xrightarrow{\pi} S$ be a generically surjective finite map from $T$ smooth. By Proposition 3.9, $\text{gr} \dim X K^T_q(X \times T)$ is generated by the irreducible components of $T \times_S X$ dominant over $S$. This means that the cycles in $\text{gr}_a K^T_q(X \times T)$ supported on the exceptional locus have $a < \dim X$, and thus they have perverse degree $\geq 1$, see (3).

Next, say that $T \xrightarrow{\pi} S$ has image $V \subseteq S$. Let $[\Gamma] \in \text{gr} \dim X \text{gr}\ K^T_q(X \times T)$. By Proposition 3.9, $a \leq \dim X - \dim X_V$. Its perverse degree $i$ satisfies

$$\left\lfloor \frac{i + \dim X - \dim V}{2} \right\rfloor \geq \dim X - \dim X_V,$$

and thus that

$$i \geq \dim X + \dim V - 2 \dim X_V \geq 1.$$

This means that $\tilde{P}^{\leq 0}_f \text{gr} K_*(X) = 0$. By Theorem 3.11, $\text{P}^{\leq 0}_f \text{gr} K_0(X) = \text{gr} K_0(X)$, and thus $\text{gr} IK_0(S) = \text{gr} K_0(X)$.

4.4.3. Let $S$ be a surface. Consider a resolution of singularities $f : X \to S$. Let $B$ be the set of singular points of $S$. For each $p$ in $B$, let $A_p = \{ C^a_p \}$ be the set of irreducible components of $X_p := f^{-1}(p)$. For each such curve, consider the diagram

$$C^a_p \xrightarrow{g^a_p} X \xrightarrow{h^a_p} p \xrightarrow{f} S.$$

Consider the maps

$$m^a_p := g^a_p h^a_* : K_*(p) \to \text{gr}^1 K_*(X)$$

$$\Delta^a_p := h^a_* g^a_* : \text{gr}^1 K_*(X) \to K_*(p).$$

We claim that

$$\tilde{P}^{\leq 0}_f \text{gr} K_*(X) = \text{image} \left( \bigoplus_{p \in B, a \in A_p} m^a_p : K_*(p) \to \text{gr}^1 K_*(X) \right).$$

The correspondences which contribute to $\tilde{P}^{\leq 0}_f$ are in $\text{gr}_2 K^T_q(X \times T)$ for $\pi : T \to S$ a generically finite map onto its image $V \subseteq S$ with $T$ smooth. By
Proposition 3.9

\[ \left\lfloor \frac{2 - \dim V}{2} \right\rfloor \geq s \geq \dim X - \dim X_V. \]

So the map \( T \to S \) is the inclusion of a point \( p \to S \) for \( p \in B \) and \( \Gamma \) is in \( \text{gr}_1 G_{X_p}(X) \). Further, for \( p, q \in B, a \in A_p, b \in A_q \):

\[ \Delta_q^b m_p^a = \delta_{pq} \delta_{ab} \text{id}. \]

This means that:

\[ \bigoplus_{p \in B} \bigoplus_{a \in A_p} m_p^a : \bigoplus_{p \in B} K_{(p)}^{[A_p]} \cong \text{P}^0_f \text{gr}^1 K(X). \]

The map \( f \) is semismall, so by Theorem 3.11 we obtain a form of the decomposition theorem for the map \( f \):

\[ \text{gr}^1 K_0(X) \cong \text{gr}^1 K_0(S) \oplus \bigoplus_{p \in B} K_0(p)^{[A_p]}. \]

See Section 5 for further discussions of the decomposition theorem for semismall maps.

4.4.4. Let \( Y \) be a smooth projective variety of dimension \( d \) and let \( \mathcal{L} \) be a line bundle on \( Y \). Consider the cone \( X := \text{Tot}_{Y} \mathcal{L} \xrightarrow{f} S \).

Let \( o \) be the vertex of the cone \( X \). There is only one fiber with nonzero dimension

\[ Y \leftarrow^i X \]

\[ \downarrow^g \]

\[ \downarrow^f \]

\[ o \leftarrow S. \]

Using the correspondence \( X \cong \Delta \xrightarrow{\iota} X \times_S X \), we see that

\[ P_{\leq 0}^f \text{gr}^1 K(X) = \text{gr}^1 K(X). \]

For \( V \subsetneq S \), the irreducible components of \( f^{-1}(V) \) are \( f_V : W \to V \) birational to \( V \) and, if \( V \) contains \( o \), the fiber \( Y \). As above, we have that \( P_{\leq 0}^f \text{gr}^1 G(W) = \text{gr} G(W) \), so the conditions in defining \( P_{\leq 1}^f \) are automatically satisfied for these irreducible components. We thus have that

\[ P_{\leq 0}^f \text{gr}^1 K(X) = \ker \left( \iota^* : \text{gr}^1 K(X) \to P_{> 1}^g \text{gr}^j K(Y) \right). \]

By the computation in Subsection 3.7.1

\[ P_{> 1}^g \text{gr}^j K(Y) = \begin{cases} 
\text{gr}^j K(Y) & \text{if } j > \left\lfloor \frac{d+1}{2} \right\rfloor, \\
0 & \text{otherwise}.
\end{cases} \]
The map \( \iota^* : \text{gr}^j K(X) \to \text{gr}^j K(Y) \) is an isomorphism, so we have that

\[
P_f^{\leq 0} \text{gr}^j K(X) = \begin{cases} 
\text{gr}^j K(Y) & \text{if } j \leq \lfloor \frac{d+1}{2} \rfloor, \\
0 & \text{otherwise}.
\end{cases}
\]

Further, \( \tilde{P}_f^{\leq 0} \text{gr}^j K(X) \) is generated by the cycles over \( X_o \cong Y \) of codimension between 0 and \( \lfloor \frac{d-1}{2} \rfloor \). The map \( \iota^* : \text{gr}^i K(Y) \to \text{gr}^{i+2} K(X) \cong \text{gr}^{i+2} K(Y) \)

is multiplication by the class \( \hbar := c_1 (\mathcal{L}|_Y) \in \text{gr}^2 K_0(Y) \). As a vector space, we thus have that

\[
\text{gr}^j I K_*(S) = \begin{cases} 
\text{gr}^j K(Y)/\hbar \text{gr}^{j-2} K(Y) & \text{if } j \leq \lfloor \frac{d+1}{2} \rfloor, \\
0 & \text{otherwise}.
\end{cases}
\]

The answer in cohomology is similar, see [6, Example 2.2.1].

5. The decomposition theorem for semismall maps

We will be using the notation from Subsection 1.4. For \( a, b \in A \), we write \( b \prec a \) if \( S_b \not\subseteq S_a \). Denote by \( \iota_{ba} : X_b \hookrightarrow X_a \). For \( a \in A \), define

\[
\tilde{P}_f^{\leq 0} \text{gr}^j K_a(X) = \text{image} \left( \bigoplus_{b \prec a} \iota_{ba*} : P_f^{\leq 0} \text{gr}^j K_a(X) \to P_f^{\leq 0} \text{gr}^j K_a(X) \right).
\]

First, we state a more precise version of Conjecture 1.3.

**Conjecture 5.1.** Let \( f : X \to S \) be a semismall map and consider \( \{S_a | a \in I\} \) a stratification as in Subsection 1.4, denote by \( A \subset I \) the set of relevant strata. For \( a \in A \), consider generically finite maps \( \pi_a : T_a \to S_a \) with \( T_a \) is smooth such that \( \pi_a^{-1}(S_a^o) \to S_a^o \) is smooth and \( R^1 f_* Q_{S_a^o} = L_a \). For each \( a \), there exists a rational map \( X_a \dashrightarrow T_a \), and let \( \Gamma_a \) be the closure of its graph

\[
\begin{array}{ccc}
\Gamma_a & \longrightarrow & X_a \\
\downarrow & & \downarrow \iota_a \\
T_a & \underset{\pi_a}{\longrightarrow} & S_a \\
\end{array}
\]

The correspondence \( \Gamma_a \) induces an isomorphism

\[
(11) \quad \iota_{a*} \Phi_{\Gamma_a} : P_f^{\leq 0} \text{gr}^{j-c_a} K_*(T_a)_{\mathbb{Q}} / \mathcal{P}^{\leq 0}_{\pi_a} \text{gr}^{j-c_a} K_*(T_a)_{\mathbb{Q}} \cong \mathcal{P}^{\leq 0}_{f*} \text{gr}^j K_*(X_a)_{\mathbb{Q}} / \mathcal{P}^{\leq 0}_{f*} \text{gr}^j K_*(X_a)_{\mathbb{Q}}
\]

and a decomposition

\[
\bigoplus_{a \in A} \text{gr}^{j-c_a} I K_*(S_a, L_a)_{\mathbb{Q}} \cong \text{gr}^j K_*(X)_{\mathbb{Q}}
\]

\[(x_a)_{a \in A} \mapsto \sum_{a \in A} \iota_{a*} \Phi_{\Gamma_a}(x_a).\]
In relation to (11), we propose the following:

**Conjecture 5.2.** Let $f : X \to S$ be a surjective map of relative dimension $d$ with $X$ is smooth. Let $U$ be a smooth open subset of $S$ such that $f^{-1}(U) \to U$ is smooth. For $y \in U$, $\pi_1(U, y)$ acts on the irreducible components of $f^{-1}(y)$ of top dimension; let $L$ be the associated local system. If $L$ satisfies the assumption on local systems in Subsection 4.1 then there is an isomorphism

$$
\mathcal{P}^{\leq -d} f^{\text{gr}} K(X)_{\mathbb{Q}} \cong \mathcal{P}^{\leq -d} f^{\text{gr}} K(X)_{\mathbb{Q}} \cong f^{\text{gr}} \mathcal{I} K(S, L)_{\mathbb{Q}}.
$$

The analogous statement in cohomology follows from the decomposition theorem.

In this section, we prove the following:

**Theorem 5.3.** We use the notation of Conjecture 5.1. Assume that the maps $\pi_a : T_a \to S_a$ are small. Then Conjecture 5.1 holds for $K_0$.

We first note a preliminary result.

**Proposition 5.4.** Consider varieties $S$ and $X$, and a smooth variety $Y$ with surjective maps $f : X \to S$ of relative dimension $d$ and $g : Y \to S$ of relative dimension 0. Assume there exists an open subset $U$ of $S$ and a map $h$ such that:

$$
g^{-1}(U) \quad \xrightarrow{h} \quad f^{-1}(U) \quad \xleftarrow{g} \quad U \quad \xrightarrow{f} \quad g^{-1}(U)
$$

Denote also by $h$ the rational map $h : X \dashrightarrow Y$. Consider a resolution of singularities $\pi : X' \to X$ such that there exists a regular map $h'$ as follows:

\[
\begin{array}{ccc}
X' & \xrightarrow{\pi} & X \\
\downarrow{h'} & & \downarrow{h} \\
\quad & \quad & \quad \\
Y & \xrightarrow{g} & S.
\end{array}
\]

Let $\Gamma$ be the closure of the graph of $h$ in $Y \times X$ and let $\Gamma'$ be the graph of $h'$ in $Y \times X'$. Then the following diagram commutes:

$$
\begin{array}{ccc}
gr^r K(Y) & \xrightarrow{\Phi_{\Gamma'}} & gr^r K(X') \\
\downarrow{\Phi_{\Gamma}} & & \downarrow{\pi_*} \\
& gr^r G(X). &
\end{array}
$$
**Proof.** Consider the maps:

\[
\begin{array}{ccc}
Y \times X' & \xrightarrow{\pi'} & X' \\
\downarrow & & \downarrow \\
Y \times X & \xrightarrow{\pi} & X \\
\downarrow q & & \downarrow f \\
Y & \xrightarrow{g} & S.
\end{array}
\]

Let \( x \in \text{gr} K(Y) \). We want to show that:

\[
\pi'_* p'_*(\Gamma' \otimes \pi'^* q^*(x)) = p_*(\Gamma \otimes q^*(x)).
\]

It suffices to show that

\[
(12) \quad \pi'_* \Gamma' = \Gamma \quad \text{in gr} G(X \times Y).
\]

Both \( \Gamma \) and \( \Gamma' \) have dimension equal to the dimension of \( X \). The map \( \pi' : \Gamma' \to \Gamma \) is birational, so the cone of

\[
\mathcal{O}_\Gamma \to \pi'_* \mathcal{O}_{\Gamma'}
\]

is supported on a proper set of \( \Gamma \), which implies the claim of \( 12 \). \( \square \)

**Proof of Theorem 5.3** Let \( a \in A \) and consider the diagram:

\[
\begin{array}{ccc}
Y_a & \xrightarrow{h_a} & X_a \\
\downarrow \pi_a & & \downarrow f_a \\
T_a & \xrightarrow{\pi_a} & S_a
\end{array}
\]

where the map \( \tau_a \) is a resolution of singularities. Let \( \Gamma_a \) be the closure of the natural rational map \( X_a \to T_a \). By Proposition 5.4 and Theorem 3.10, the map \( \Phi_{\Gamma_a} \) factors as:

\[
\Phi_{\Gamma_a} : \text{gr}^j K_0(T_a) \xrightarrow{h_a^*} \text{gr}^j K_0(Y_a) \xrightarrow{\tau_a^*} \text{gr}^j G_0(X_a) \to \text{gr}^j+ca K_{X_a}(X).
\]

By Theorem 3.11, the map \( \Phi_{\Gamma_a} \) factors as:

\[
\Phi_{\Gamma_a} : \text{gr}^j K_0(T_a) = \bigoplus_{a \in A} \text{gr}^j K_0(T_a) \xrightarrow{h_a^*} \bigoplus_{a \in A} \text{gr}^j K_0(Y_a) \xrightarrow{\tau_a^*} \bigoplus_{a \in A} \text{gr}^j K_0(X_a) \to \bigoplus_{a \in A} \text{gr}^j+ca K_{X_a,0}(X) = \text{gr}^j K_0(X).
\]

We thus obtain a map of vector spaces

\[
(13) \quad \bigoplus_{a \in A} \Phi_{\Gamma_a} : \bigoplus_{a \in A} \text{gr}^j-ca K_0(T_a) \to \bigoplus_{a \in A} \text{gr}^j+ca K_{X_a,0}(X) \to \text{gr}^j K_0(X).
\]
A theorem of de Cataldo–Migliorini [5, Theorem 4.0.4] says that there is an isomorphism:

\[ \bigoplus_{a \in A} \Phi_{T_a} : \bigoplus_{a \in A} \text{gr}^{j-\text{ca}} K_0(T_a)_Q \xrightarrow{\sim} \text{gr}^j K_0(X)_Q. \]

Combining with (13), we see that in this case

\[ \Phi_{T_a} : \text{gr}^{j-\text{ca}} K_0(T_a)_Q \xrightarrow{\sim} \iota_{a*} \left( \mathcal{P}_{\mathcal{I}_f} \text{gr}^j K_{X_a,0}(X)_Q / \tilde{\mathcal{P}}_{\mathcal{I}_f} \text{gr}^j K_{X_a,0}(X)_Q \right). \]

This implies the statement of Theorem 5.3.

\[ \square \]

\textbf{REFERENCES}


School of Mathematics, Institute of Advanced Studies, 1 Einstein Drive, Princeton, NJ 08540.

Email address: tpad@ias.edu