

DEFORMED DIMENSIONAL REDUCTION

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ABSTRACT. Since its first use by Bryan-Behrend-Szendrői in the computation of motivic DT invariants of $\mathbb{A}_{\mathbb{C}}^3$, dimensional reduction has proved to be an important tool in motivic and cohomological DT theory. Inspired by a conjecture of Cazzaniga-Morrison-Pym-Szendrői about motivic DT invariants and work of Orlov about equivalences of categories of singularities, we generalize the dimensional reduction theorem in motivic, cohomological, and K-theoretic DT theory, and use it to prove versions of the Cazzaniga-Morrison-Pym-Szendrői conjecture in all of these settings.

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1. INTRODUCTION

1.1. Dimensional reduction, old and new. The notion of dimensional reduction, introduced in [1], has proven to be extremely useful in Donaldson–Thomas theory and beyond. We start with a motivic version of the dimensional reduction theorem, which is a slight variant of the one proved in [1]. Let X be an algebraic variety, and let \mathbb{G}_m act on $\tilde{X} = X \times \mathbb{A}_{\mathbb{C}}^m$ by $z \cdot (x, t) = (x, zt)$. Assume that $g: \tilde{X} \rightarrow \mathbb{C}$ is a regular degree one function, so that we may write

$$g = \sum_{1 \leq j \leq m} g_j t_j$$

where g_j are functions pulled back from X and t_1, \dots, t_m are coordinates on $\mathbb{A}_{\mathbb{C}}^m$. Define $\tilde{Z} \subset \tilde{X}$ to be the reduced vanishing locus of the functions g_1, \dots, g_m . The theorem states that

$$(1) \quad \int [\Phi_g] = \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} [\tilde{Z}] \in \hat{K}_0(\text{Var} / \text{pt})$$

where $\int[\Phi_g]$ is the absolute motivic vanishing cycle defined by Denef and Loeser [14]. Their definition lies in a ring of $\hat{\mu}$ -equivariant motives, whereas the identity (1) takes place in a naive Grothendieck ring of motives with no monodromy; part of the statement of the theorem is that the monodromy on the left hand side of (1) is trivial.

There is a cohomological version of the theorem as well. Let

$$\phi_g^{\text{mon}}: \mathcal{D}(\text{MHM}(\tilde{X})) \rightarrow \mathcal{D}(\text{MMHM}(\tilde{X}))$$

be the vanishing cycles functor. Note the extra ‘‘M’’ appearing in the target category — this stands for monodromy, and is again accounted for by the monodromy automorphism on the vanishing cycles. By construction, for any mixed Hodge module \mathcal{F} , $\phi_g^{\text{mon}}\mathcal{F}$ is supported on $\tilde{X}_0 := g^{-1}(0)$, and moreover there is a natural transformation $\phi_g\mathcal{F} \rightarrow \mathcal{F}|_{\tilde{X}_0}$. Since $\tilde{Z} \subset \tilde{X}_0$ we can restrict further to obtain the natural transformation

$$(2) \quad \phi_g^{\text{mon}}\mathcal{F} \rightarrow \mathcal{F}|_{\tilde{Z}}.$$

Denote by $\tilde{\iota}: \tilde{Z} \hookrightarrow \tilde{X}$ the inclusion. We can alternatively obtain (2) by applying ϕ_g^{mon} to the natural transformation

$$(3) \quad \text{id} \rightarrow \tilde{\iota}_*\tilde{\iota}^*,$$

since the vanishing cycle functor commutes with proper maps, is the identity functor for the zero map, and $g \circ \tilde{\iota} = 0$. The cohomological dimensional reduction theorem states that the natural map

$$(4) \quad \pi_!\phi_g^{\text{mon}}(\text{id} \rightarrow \tilde{\iota}_*\tilde{\iota}^*)\pi^*$$

is an isomorphism for $\mathcal{G} \in \text{MHM}(X)$. Just as in the motivic version of the theorem, the target has trivial monodromy, since $\phi_g^{\text{mon}}\tilde{\iota}_*\tilde{\iota}^* \cong \tilde{\iota}_*\tilde{\iota}^*$. Let $i: X' \hookrightarrow X$ be the inclusion of a subvariety, and let $\tau: X' \rightarrow \text{pt}$ be the structure map. Then a consequence of the theorem is the statement that

$$\tau_!i^*\pi_!\phi_g^{\text{mon}}(\text{id} \rightarrow \tilde{\iota}_*\tilde{\iota}^*)\pi^*\mathbb{Q}_X$$

is an isomorphism, i.e there is an isomorphism of mixed Hodge structures:

$$\text{H}_c(\tilde{X}', \phi_g^{\text{mon}}\mathbb{Q}_{\tilde{X}}) \cong \text{H}_c(\tilde{Z} \cap \tilde{X}', \mathbb{Q})$$

and this is the special case that is used most often.

The starting point of the paper is the question of whether we can generalize in the following way. Assume instead that we have

$$g = g_0 + \sum_{1 \leq j \leq m} g_j \cdot t_j$$

where g_0, \dots, g_m are again pulled back from functions on X . Write

$$(5) \quad g^{\text{red}} := g_0|_{\tilde{Z}}.$$

The natural transformation (4) is still defined, and we obtain from it a natural transformation

$$(6) \quad \pi_!\phi_g^{\text{mon}}\pi^* \rightarrow \pi_!\tilde{\iota}_*\phi_{g^{\text{red}}}^{\text{mon}}\tilde{\iota}^*\pi^*,$$

bearing in mind that $g|_{\tilde{Z}} = g^{\text{red}}$.

Question 1.1. *Is (6) an isomorphism?*

By applying the natural transformation to \mathbb{Q}_X and taking total compactly supported hypercohomology we obtain as before a homomorphism

$$\text{H}_c(\tilde{X}, \phi_g^{\text{mon}}\mathbb{Q}_{\tilde{X}}) \rightarrow \text{H}_c(\tilde{Z}, \phi_{g^{\text{red}}}^{\text{mon}}\mathbb{Q}_{\tilde{Z}})$$

and we may ask if it is an isomorphism. Obviously a positive answer to the first question implies a positive answer to the second. There are several situations in which the answer to Question 1.1 is yes:

- (A) If $g_0 = 0$ then $\phi_{g^{\text{red}}} \cong \text{id}$ and (6) becomes naturally isomorphic to (4) which is an isomorphism by the usual dimensional reduction theorem.

- (B) Let $X = T \times \mathbb{A}_{\mathbb{C}}^p$, with g_1, \dots, g_m pulled back from T , and $g_0 = \sum_{1 \leq j \leq p} h_j t_j'$, with t_j' coordinates on $\mathbb{A}_{\mathbb{C}}^p$ and h_j pulled back from T , and assume furthermore that $\mathcal{G} \cong \pi_T^* \mathcal{G}'$ for some $\mathcal{G}' \in \mathcal{D}(\text{MHM}(T))$. Then by two applications of the dimensional reduction theorem, (6) is an isomorphism when applied to \mathcal{G} .
- (C) Let $X = X_1 \times X_2$ with g_1, \dots, g_m pulled back from X_1 and g_0 pulled back from X_2 . Then by the Thom–Sebastiani isomorphism, (6) is an isomorphism.
- (D) Let $X = \mathbb{A}_{\mathbb{C}}^1 = \text{Spec}(\mathbb{C}[x])$, and set $n = 1$. Let $g_1(x) = x^a$ and $g_0(x) = x^b$ with $b > a$. Then $g = x^a(t + x^{b-a})$, and has an isolated singularity at the origin, so that $\phi_g^{\text{mon}} \mathbb{Q}_{\tilde{X}} = \text{H}_c(\phi_g^{\text{mon}}, \mathbb{Q}) \otimes \mathbb{Q}_{\{0\}}$. Then $Z = Z(x)$, and $g^{\text{red}} = 0$, so that $\text{H}_c(\tilde{Z}, \phi_{g^{\text{red}}}^{\text{mon}} \mathbb{Q}) = \text{H}_c(\mathbb{A}_{\mathbb{C}}^1, \mathbb{Q})$ which is isomorphic to $\text{H}_c(\phi_{x^a(t+x^{b-a})}^{\text{mon}}, \mathbb{Q})$ by direct calculation.

As encouraging as these observations are, it turns out that it is not hard to cook up examples for which the answer to Question 1.1 is no. For instance, modify example (D) from above, so that now a and b satisfy $a \geq b$. Then again $Z = Z(x)$, and $g^{\text{red}} = 0$, so that the right hand side of (6) is given by $\text{H}_c(\mathbb{A}_{\mathbb{C}}^1, \mathbb{Q})$. On the other hand, we now can write

$$g = x^b(tx^{a-b} + 1)$$

for which 0 is not a critical value, so that the left hand side of (6) is zero, when applied to the constant sheaf $\mathbb{Q}_{\tilde{X}}$. Considering the well behaved and badly behaved variants of (D) above, we see that the dimensional reduction morphism is an isomorphism if and only if there is a positive weighting of x and y making g a quasihomogeneous function. This brings us to our main theorem, which will be proved in Section 7:

Theorem 1.2. *Let $\tilde{X} = X \times \mathbb{A}_{\mathbb{C}}^n \xrightarrow{g} \mathbb{C}$ be a \mathbb{G}_m -equivariant function, where \mathbb{G}_m acts trivially on X and on $\mathbb{A}_{\mathbb{C}}^n$ with positive weights. Assume furthermore that there is a decomposition $\mathbb{A}_{\mathbb{C}}^n = \mathbb{A}_{\mathbb{C}}^m \times \mathbb{A}_{\mathbb{C}}^{n-m}$ and we can write*

$$g = g_0 + \sum_{1 \leq j \leq m} g_j t_j$$

with the functions g_0, \dots, g_m pulled back from $X \times \mathbb{A}_{\mathbb{C}}^{n-m}$. Define g^{red} as in (5), and let $\pi: \tilde{X} \rightarrow X$ be the natural projection. Then the dimensional reduction morphism

$$\pi_! \phi_g^{\text{mon}} \pi^* \rightarrow \pi_! \tilde{t}_* \phi_{g^{\text{red}}}^{\text{mon}} \tilde{t}^* \pi^*$$

is an isomorphism. In particular, for $X' \subset X$ a subvariety, there is a natural isomorphism

$$(7) \quad \text{H}_c(\tilde{X}', \phi_g^{\text{mon}} \mathbb{Q}_{\tilde{X}}) \cong \text{H}_c(\tilde{Z} \cap \tilde{X}', \phi_{g^{\text{red}}}^{\text{mon}} \mathbb{Q}_{\tilde{Z}}).$$

1.2. Motivation from Donaldson–Thomas theory. One of the motivations for searching for this generalization of the dimensional reduction isomorphism was a conjecture of Cazzaniga, Morrison, Pym and Szendrői, regarding the motivic Donaldson–Thomas invariants of the quiver Q with three loops A, B, C , and with the potential $W_3 = A[B, C] + C^3$. The definition of these invariants is recalled in Section 3. They conjectured that for all $n \in \mathbb{Z}_{\geq 1}$ there is an equality

$$\Omega_{Q, W_3, n} = 1 - [\mu_3].$$

We can consider the vanishing cycles for the potential $W_d = A[B, C] + C^d$ via deformed dimensional reduction of the potential $W = A[B, C]$, the Donaldson–Thomas theory of which is very well understood: Behrend, Bryan and Szendrői proved that the motivic DT invariants for (Q, W) are given by $\mathbb{L}^{3/2}$ for all n . Using

the motivic dimensional reduction theorem, we verify the Cazzaniga-Morrison-Pym-Szendrői conjecture:

Theorem 1.3. *The motivic DT invariants of (Q, W_d) are:*

$$\Omega_{Q, W_d, n} = 1 - [\mu_d].$$

Later in [6] the cohomological Donaldson–Thomas invariants of (Q, W) were calculated, along with their relative versions. The cohomological version of DT-theory is recalled in Section 5 below: via cohomological dimensional reduction and the cohomological integrality theorem of [13] there are mixed Hodge modules \mathcal{F}_n on the coarse moduli space \mathcal{X}_n of n dimensional representations of the two-loop quiver (with loops labelled B and C) such that weight polynomial of the compactly supported hypercohomology of \mathcal{F}_n recovers the DT invariants calculated in [1]. Using purity, it was shown in [6] that these sheaves are given by a half Tate twist of $\Delta_{n,*}\mathbb{Q}$ where

$$\begin{aligned} \Delta_n: \mathbb{A}_{\mathbb{C}}^2 &\rightarrow P_n \\ (t_1, t_2) &\mapsto (t_1 \text{Id}_{n \times n}, t_2 \text{Id}_{n \times n}). \end{aligned}$$

A striking feature of the cohomological version of the Cazzaniga-Morrison-Pym-Szendrői conjecture is that it is obtained by applying the vanishing cycle functor $\Phi_{\text{Tr}(W_3)}^{\text{mon}}$ to this mixed Hodge module. We prove this result in Section 6 using the motivic computation and purity of the Hodge modules. We provide an alternative proof in Section 7 using Theorem 1.2.

Theorem 1.4. *Let \mathcal{M}_n be the coarse space of representations of dimension n of Q . There is an isomorphism in MMHM(\mathcal{M}_n)*

$$\mathcal{BPS}_{W_d} \cong \Delta_{n,*}\mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2} \otimes \mathbb{H}(\mathbb{A}_{\mathbb{C}}^1, \Phi_{x^d}^{\text{mon}}\mathbb{Q}) \otimes \mathfrak{L}^{-3/2}$$

as well as an isomorphism in MMHS

$$\text{BPS}_{W_d, n} \cong \mathbb{H}(\mathbb{A}_{\mathbb{C}}^1, \Phi_{x^d}^{\text{mon}}\mathbb{Q}) \otimes \mathfrak{L}^{1/2}.$$

1.3. Categorical and K-theoretic DT theory. For a regular function on a smooth variety $f: X \rightarrow \mathbb{A}_{\mathbb{C}}^1$, the vanishing cohomology $H^*(X_0, \varphi_f \mathbb{Q})$ is categorified by the category of singularities $D_{sg}(X_0)$, which is the quotient:

$$D_{sg}(X_0) = D^b(X_0)/\text{Perf}(X_0),$$

where $D^b(X_0)$ is the derived category of bounded complexes of sheaves on X_0 and $\text{Perf}(X_0) \subset D^b(X_0)$ is the full subcategory of perfect complexes. Efimov [16] shows that the periodic cyclic homology of $D_{sg}(X_0)$ recovers the vanishing cohomology:

$$HP_*(D_{sg}(X_0)) = \bigoplus_{i \in \mathbb{Z}} H^{+2i}(X_0, \phi_f \mathbb{Q}_X).$$

It is thus natural to ask whether Theorem 1.2 admits categorical or K-theoretic versions. Orlov [27] proved the analogue of Theorem 1.2 when $m = 1$ and \tilde{Z} is smooth:

$$D_{sg}(\tilde{X}_0) \cong D_{sg}(Z_0),$$

where Z_0 is the zero locus of g^{red} on Z . Later, Hirano proved the analogue of Theorem 1.2 for any $m \geq 1$ when \tilde{Z} is smooth [21].

Isik [22] proved a categorical version of the dimension reduction theorem (4), that is, the case $g_0 = 0$ of Theorem 1.2, without an assumption on \tilde{Z} . Denote by \mathcal{K} the Koszul complex associated to g_1, \dots, g_m . Isik's theorem say that there is an equivalence:

$$D_{sg}^{gr}(\tilde{X}_0) \cong D^b(\mathcal{K}),$$

where D_{sg}^{gr} is a graded version of D_{sg} . As a corollary of Isik's theorem, we obtain the following weak version of Theorem 1.2 in K-theory:

Lemma 1.5. *Let X be a variety with regular functions $g_0, \dots, g_n : X \rightarrow \mathbb{A}_{\mathbb{C}}^1$ and define the regular function*

$$g = g_0 + t_1 g_1 + \dots + t_m g_m : \tilde{X} = X \times \mathbb{A}_{\mathbb{C}}^m \rightarrow \mathbb{A}_{\mathbb{C}}^1.$$

Recall that $Z \subset X$ is the zero locus of $g_1 = \dots = g_m = 0$ and $g^{red} : Z \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is the restriction of g_0 to Z . Consider the maps $\pi : \tilde{Z}_0 = Z_0 \times \mathbb{A}_{\mathbb{C}}^m \rightarrow Z_0$ and $\iota : \tilde{Z}_0 \rightarrow \tilde{X}_0$. Then there is a natural surjection:

$$\iota_* \pi^* : K_0(D_{sg}(Z'_0/Z)) \rightarrow K_0(D_{sg}(\tilde{X}_0)).$$

Here Z'_0 is a Koszul complex with underlying scheme Z_0 and $D_{sg}(Z'_0/Z_0)$ is the relative category of singularities, see [17].

We expect the above result to be improved to an equivalence of categories

$$D_{sg}(Z'_0/Z) \cong D_{sg}(\tilde{X}_0).$$

We use Lemma 1.5 to prove an analogue of Theorems 1.3 and 1.4 in K-theory. The BPS invariants can be categorified using the categories $\mathbb{M}(n)$ defined in [29], see Subsection 8.1 for more details.

Theorem 1.6. *The K-theoretic BPS invariants of (Q, W_d) and dimension vector $n \geq 1$ are*

$$K_0(\mathbb{M}(n)) = 0.$$

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2. MOTIVIC DEFORMED DIMENSIONAL REDUCTION

2.1. Background and definitions. Let G be an algebraic group. For Y a G -variety we denote by $\hat{K}_0^G(\text{Var}/Y)$ the free Abelian group generated by symbols

$$(8) \quad [X \xrightarrow{f} Y]$$

where f is a morphism of G -equivariant varieties, and X is reduced. We impose two types of relations

- (1) The cut and paste relations:

$$[X \xrightarrow{f} Y] = [U \xrightarrow{f|_U} Y] + [Z \xrightarrow{f|_Z} Y]$$

for $U \subset X$ an open subscheme with closed complement Z .

- (2) We impose the relation

$$[V \rightarrow X \rightarrow Y] = [X \times \mathbb{A}_{\mathbb{C}}^n \xrightarrow{f \circ \pi_X} Y]$$

if $V \rightarrow X$ is the projection from the total space of a rank n G -equivariant vector bundle.

We call sums of elements as in (8) *effective*. General elements in $\hat{K}_0^G(\text{Var}/Y)$ and variants of this ring can be written as $A - B$ where A and B are effective. Let $d \in \mathbb{Z}_{\geq 0}$, and give $Y \times \mathbb{A}_{\mathbb{C}}^1$ the \mathbb{G}_m -action $z \cdot (y, t) = (y, z^d t)$, thus defining the group $\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y \times \mathbb{A}_{\mathbb{C}}^1)$.

We denote by $\mu_d \subset \mathbb{C}^*$ the group of d th roots of unity. We define the groups $\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y \times \mathbb{G}_m)$ and $\hat{K}_0^{\mu_d}(\text{Var}/Y)$ similarly, giving Y the trivial μ_d -action in the second case. There is an isomorphism of groups

$$\begin{aligned} \hat{K}_0^{\mu_d}(\text{Var}/Y) &\rightarrow \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y \times \mathbb{G}_m) \\ [X \xrightarrow{f} Y] &\mapsto - [X \times_{\mu_d} \mathbb{G}_m \xrightarrow{(x, z) \mapsto (f(x), z^d)} Y \times \mathbb{G}_m]. \end{aligned}$$

This isomorphism sends a variety with a μ_d -action to a variety over \mathbb{G}_m , locally constant in the étale topology, with monodromy given by the μ_d -action. The \mathbb{G}_m -equivariant inclusion $\mathbb{G}_m \hookrightarrow \mathbb{A}_{\mathbb{C}}^1$ induces an inclusion of Abelian groups

$$\iota_d: \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y \times \mathbb{G}_m) \rightarrow \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y \times \mathbb{A}_{\mathbb{C}}^1)$$

via composition. The structure morphism for $\mathbb{A}_{\mathbb{C}}^1$ induces the map

$$\begin{aligned} \nu_d^*: \hat{K}_0(\text{Var}/Y) &\rightarrow \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y \times \mathbb{A}_{\mathbb{C}}^1) \\ [X \xrightarrow{f} Y] &\mapsto [X \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{f \times \text{id}} Y \times \mathbb{A}_{\mathbb{C}}^1] \end{aligned}$$

where the \mathbb{G}_m -action on the target is defined by $z \cdot (y, z') = (y, z^d z')$. The group $\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)$ carries a ring structure defined by

$$(9) \quad [X_1 \xrightarrow{f_1} \mathbb{A}_{\mathbb{C}}^1] \cdot [X_2 \xrightarrow{f_2} \mathbb{A}_{\mathbb{C}}^1] := [X_1 \times X_2 \xrightarrow{+(f_1 \times f_2)} \mathbb{A}_{\mathbb{C}}^1]$$

and $\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/Y)$ carries a $\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\text{pt})$ module structure defined in the same way. If $f_1 = f_2$ there is an extra \mathfrak{S}_2 action on the right hand side of (9). We define

$$(10) \quad [X \xrightarrow{f_1} \mathbb{A}_{\mathbb{C}}^1]^{\square n} = [X^n \xrightarrow{+ \circ f^{\times n}} \mathbb{A}_{\mathbb{C}}^1] \in \hat{K}_0^{\mathfrak{S}_n \times \mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)$$

where the symmetric group \mathfrak{S}_n acts trivially on $\mathbb{A}_{\mathbb{C}}^1$. For $n \geq 0$ we define operations σ^n on effective classes via

$$(11) \quad \sigma^n [X \xrightarrow{f} \mathbb{A}_{\mathbb{C}}^1] := [\text{Sym}^n(X) \xrightarrow{+ \circ \text{Sym}^n(f)} \mathbb{A}_{\mathbb{C}}^1] = \pi_{(n)} \left([X \xrightarrow{f} \mathbb{A}_{\mathbb{C}}^1]^{\square n} \right)$$

where we slightly abuse notation and denote by the same symbol the morphisms

$$\begin{aligned} +: \mathbb{A}_{\mathbb{C}}^1 \times \mathbb{A}_{\mathbb{C}}^1 &\rightarrow \mathbb{A}_{\mathbb{C}}^1 \\ +: \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^1) &\rightarrow \mathbb{A}_{\mathbb{C}}^1 \end{aligned}$$

provided by addition. The map $\pi_{(n)}$ in (11) is defined by

$$(12) \quad \begin{aligned} \pi_{(n)}: \hat{K}_0^{\mathfrak{S}_n \times \mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1) &\rightarrow \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1) \\ [X \rightarrow \mathbb{A}_{\mathbb{C}}^1] &\mapsto [X/\mathfrak{S}_n \rightarrow \mathbb{A}_{\mathbb{C}}^1]. \end{aligned}$$

The operations σ^n are extended to all classes in $\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)$ via the relation

$$(13) \quad \sigma^n(A + B) = \sum_{i=0}^n \sigma^i(A) \sigma^{n-i}(B).$$

For the proof that σ^n can be extended in this way see [20]. Obviously (11) and (13) then determine the operations σ^n uniquely.

The subgroup $\nu_d^*(\hat{K}_0(\text{Var}/\text{pt})) \subset \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)$ is a λ -ideal, and so the quotient

$$\hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)/\hat{K}_0(\text{Var}/\text{pt}) \cong \hat{K}_0^{\mu_d}(\text{Var}/\text{pt})$$

acquires the structure of a pre- λ -ring. For $d'|d$ there is a natural inclusion of pre- λ -rings

$$\hat{K}_0^{\mu_{d'}}(\text{Var}/\text{pt}) \hookrightarrow \hat{K}_0^{\mu_d}(\text{Var}/\text{pt})$$

given by the morphism

$$\begin{aligned} \mu_d &\rightarrow \mu_{d'} \\ \zeta &\mapsto \zeta^{d/d'} \end{aligned}$$

and we define by $\hat{K}_0^{\hat{\mu}}(\text{Var}/\text{pt})$ the pre- λ -ring obtained as the limit of these inclusions. In particular, there is an inclusion

$$\hat{K}_0(\text{Var}/\text{pt}) = \hat{K}_0^{\mu_1}(\text{Var}/\text{pt}) \subset \hat{K}_0^{\hat{\mu}}(\text{Var}/\text{pt})$$

of *monodromy-free motives*, which form a sub-pre- λ -ring.

Equivalently, there is an embedding

$$\begin{aligned} \hat{K}_0^{\mathbb{G}_m, d'}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)/\hat{K}_0(\text{Var}/\text{pt}) &\hookrightarrow \hat{K}_0^{\mathbb{G}_m, d}(\text{Var}/\mathbb{A}_{\mathbb{C}}^1)/\hat{K}_0(\text{Var}/\text{pt}) \\ [X \xrightarrow{f} \mathbb{A}_{\mathbb{C}}^1] &\mapsto [X \xrightarrow{(t \mapsto t^{d/d'}) \circ f} \mathbb{A}_{\mathbb{C}}^1]. \end{aligned}$$

and we define $\hat{K}_0^{\text{mon}}(\text{Var}/\text{pt})$ to be the limit of these embeddings. We define the isomorphic groups

$$(14) \quad \Xi: \hat{K}_0^{\hat{\mu}}(\text{Var}/Y) \cong \hat{K}_0^{\text{mon}}(\text{Var}/Y)$$

as a limit of quotients in the same way. As above, there is an embedding

$$\begin{aligned} \hat{K}_0(\text{Var}/Y) &\rightarrow \hat{K}_0^{\text{mon}}(\text{Var}/Y) \\ [X \xrightarrow{f} Y] &\mapsto [X \xrightarrow{f \times 0} Y \times \mathbb{A}_{\mathbb{C}}^1] \end{aligned}$$

and we will generally abuse notation and consider elements $[X]$ and $[X \xrightarrow{f} Y]$ as elements of $\hat{K}_0^{\text{mon}}(\text{Var}/\text{pt})$ and $\hat{K}_0^{\text{mon}}(\text{Var}/Y)$ via these embeddings.

Given a morphism $h: Y \rightarrow Y'$ of varieties we define the operations

$$h_!: \hat{K}_0^{\text{mon}}(\text{Var}/Y) \rightarrow \hat{K}_0^{\text{mon}}(\text{Var}/Y')$$

and

$$h^*: \hat{K}_0^{\text{mon}}(\text{Var}/Y') \rightarrow \hat{K}_0^{\text{mon}}(\text{Var}/Y)$$

via composition and fibre product, respectively. We define $f: \hat{K}_0^{\text{mon}}(\text{Var}/Y) \rightarrow \hat{K}_0^{\text{mon}}(\text{Var}/\text{pt})$ via

$$\int := (Y \rightarrow \text{pt})_!.$$

For $Y' \subset Y$ we define $(Y' \cap \bullet) := (Y' \hookrightarrow Y)_!(Y' \hookrightarrow Y)^*$. Given varieties Y_1 and Y_2 we define an external tensor product

$$\begin{aligned} \boxtimes: \hat{K}_0^{\text{mon}}(\text{Var}/Y_1) \times \hat{K}_0^{\text{mon}}(\text{Var}/Y_2) \\ \left([X_1 \xrightarrow{f_1} Y_1 \times \mathbb{A}_{\mathbb{C}}^1], [X_2 \xrightarrow{f_2} Y_2 \times \mathbb{A}_{\mathbb{C}}^1] \right) &\mapsto [X_1 \times X_2 \xrightarrow{p} Y_1 \times Y_2 \times \mathbb{A}_{\mathbb{C}}^1] \end{aligned}$$

where $p = (\text{Id}_{Y_1 \times Y_2} \times +) \circ (f_1 \times f_2)$.

The element $\mathbb{L} := [\mathbb{A}_{\mathbb{C}}^1] \in \hat{K}_0(\text{Var}/\text{pt})$ has a square root $\mathbb{L}^{1/2} = [\mathbb{A}_{\mathbb{C}}^1 \xrightarrow{t \mapsto t^2} \mathbb{A}_{\mathbb{C}}^1]$ in $\hat{K}_0^{\text{mon}}(\text{Var}/\text{pt})$, and we define the localized $\hat{K}_0(\text{Var}/\text{pt})$ -module

$$\mathcal{M}_Y^{\text{mon}} := \hat{K}_0^{\text{mon}}(\text{Var}/Y)[\mathbb{L}^{-1/2}, (1 - \mathbb{L}^d)^{-1} | d > 0].$$

We denote by $\mathcal{M}_Y \subset \mathcal{M}_Y^{\text{mon}}$ the submodule

$$(15) \quad \hat{K}_0(\text{Var}/Y)[\mathbb{L}^{-1/2}, (1 - \mathbb{L}^d)^{-1} | d > 0].$$

We set $\mathcal{M}_{\text{pt}}^{\text{mon}} = \mathcal{M}_{\text{pt}}^{\text{mon}}$. Note that $\sigma^2(\mathbb{L}^{1/2}) = 0$, $\sigma^2(-\mathbb{L}^{1/2}) = \mathbb{L}$ and more generally $\sigma^m((-\mathbb{L}^{1/2})^n) = (-\mathbb{L}^{1/2})^{mn}$. By the results of [12] the operations (9, 11) define a

pre- λ -ring structure on the limit of quotients $\hat{K}_0^{\text{mon}}(\text{Var}/\text{pt})$ which extends uniquely to \mathcal{M} .

We define $\hat{K}_0(\text{Sta}/Y)$ to be group defined by symbols (8), where now X is a finite type Artin stack with geometric affine stabilizers, and with relations defined as before. One can show that $[\text{BGL}_n] = [\text{GL}_n]^{-1}$ inside $\hat{K}_0(\text{Sta}/\text{pt})$ and so there is a morphism

$$\mathcal{M}_Y \rightarrow \hat{K}_0(\text{Sta}/Y)[\mathbb{L}^{-1/2}]$$

which is moreover an isomorphism [18, Thm.1.2]. In particular, we will be able to consider global quotient stacks X/G over Y as elements of \mathcal{M}_Y . Moreover if G is *special* in the sense that étale locally trivial G bundles are Zariski locally trivial, then we have

$$[X/G \xrightarrow{f} Y] = [X \xrightarrow{f \circ p} Y] \cdot [G]^{-1} \in \mathcal{M}_Y$$

where $p: X \rightarrow X/G$ is the quotient map.

We define the plethystic exponential

$$\begin{aligned} \text{EXP}: \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]_+ &\rightarrow \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]] \\ \alpha &\mapsto \sum_{j \geq 0} \sigma^j(\alpha) \end{aligned}$$

where $\mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]_+ \subset \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]$ is the ideal generated by (T_1, \dots, T_n) . This morphism is an isomorphism onto its image $1 + \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]_+$, with inverse

$$\text{LOG}: 1 + \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]_+ \xrightarrow{\cong} \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]_+.$$

More generally, consider the scheme $Y = \coprod_{\gamma \in \mathbb{Z}_{\geq 0}^n} Y_\gamma$, an infinite disjoint union of varieties. Then we define

$$\mathcal{M}_Y^{\text{mon}} = \prod_{\gamma \in \mathbb{Z}_{\geq 0}^n} \mathcal{M}_{Y_\gamma}^{\text{mon}}$$

We frequently abuse notation by denoting

$$\begin{aligned} ([Z_\gamma \xrightarrow{f_\gamma} Y_\gamma \times \mathbb{A}_{\mathbb{C}}^1])_{\gamma \in \mathbb{Z}_{\geq 0}^n} &= \left[\prod_{\gamma \in \mathbb{Z}_{\geq 0}^n} Z_\gamma \xrightarrow{\coprod_{\gamma \in \mathbb{Z}_{\geq 0}^n} f_\gamma} Y_\gamma \times \mathbb{A}_{\mathbb{C}}^1 \right] \\ &= \sum_{\gamma \in \mathbb{Z}_{\geq 0}^n} [Z_\gamma \xrightarrow{f_\gamma} Y_\gamma \times \mathbb{A}_{\mathbb{C}}^1] \end{aligned}$$

For instance we can make the set $\mathbb{Z}_{\geq 0}^n$ into a scheme, with an isolated closed point for every $\gamma \in \mathbb{Z}_{\geq 0}^n$, and there is a natural isomorphism

$$\tau: \mathcal{M}_{\mathbb{Z}_{\geq 0}^n}^{\text{mon}} \cong \mathcal{M}^{\text{mon}}[[T_1, \dots, T_n]]$$

sending

$$\alpha \mapsto \sum_{\gamma \in \mathbb{Z}_{\geq 0}^n} (\{\gamma\} \times \mathbb{A}_{\mathbb{C}}^1 \hookrightarrow \mathbb{Z}_{\geq 0}^n \times \mathbb{A}_{\mathbb{C}}^1)^* \alpha T^\gamma.$$

Now assume that Y carries a finite type monoid map $\mu: Y \times Y \rightarrow Y$ such that $\mu(Y_\gamma \times Y_{\gamma'}) \subset Y_{\gamma+\gamma'}$. We define the product on $\hat{K}_0^{\text{mon}}(\text{Var}/\mathbb{Z}_{\geq 0}^n)$:

$$[X_1 \xrightarrow{f_1} Y \times \mathbb{A}_{\mathbb{C}}^1] \cdot_\mu [X_2 \xrightarrow{f_2} Y \times \mathbb{A}_{\mathbb{C}}^1] = [X_1 \times X_2 \xrightarrow{(\mu \times +) \circ (f_1 \times f_2)} Y \times Y \times \mathbb{A}_{\mathbb{C}}^1]$$

extending by linearity to give a product on $\mathcal{M}_Y^{\text{mon}}$. This product is commutative if μ is. Likewise, if μ is commutative we define operations σ_μ^n on effective classes via

$$\sigma_\mu^n [X \xrightarrow{f} Y \times \mathbb{A}_{\mathbb{C}}^1] := [\text{Sym}^n(X) \xrightarrow{(\mu \times +) \circ \text{Sym}^n(f)} Y \times \mathbb{A}_{\mathbb{C}}^1]$$

and extend to all classes as in (13), defining operations σ^n on $\mathcal{M}_Y^{\text{mon}}$. Set

$$Y_+ := \coprod_{0 \neq \gamma \in \mathbb{Z}_{\geq 0}^n} Y_\gamma.$$

We define

$$\begin{aligned} \text{EXP}_\mu: \mathcal{M}_{Y_+}^{\text{mon}} &\rightarrow \mathcal{M}_Y^{\text{mon}} \\ \alpha &\mapsto \sum_{j \geq 0} \sigma_j(\alpha). \end{aligned}$$

In the case in which $Y = \mathbb{Z}_{\geq 0}^n$, given the finite type commutative monoid structure arising from addition, this recovers the previous definition of the plethystic exponential via τ .

For example, let P be a variety. Then we set $\text{Sym}(P) = \coprod_{i \in \mathbb{Z}_{\geq 0}} \text{Sym}^i(P)$, the configuration space of unordered points on P . We set $\text{Sym}^0(P) = \text{pt}$. There is a union map

$$\cup: \text{Sym}^i(P) \times \text{Sym}^j(P) \rightarrow \text{Sym}^{i+j}(P)$$

making $\text{Sym}(P)$ into a commutative monoid. By the above definitions, there is a plethystic exponential

$$\text{EXP}_\cup: \mathcal{M}_{\text{Sym}(P)_+}^{\text{mon}} \rightarrow \mathcal{M}_{\text{Sym}(P)}^{\text{mon}}$$

along with an inverse isomorphism from the image

$$\text{LOG}_\cup: 1 + \mathcal{M}_{\text{Sym}(P)_+}^{\text{mon}} \xrightarrow{\cong} \mathcal{M}_{\text{Sym}(P)_+}^{\text{mon}}.$$

2.2. The motivic version of the main theorem. Let $g: Y \rightarrow \mathbb{C}$ be a regular function on a smooth variety Y , and let Y_0 be the reduced zero locus of g . In [14] Denef and Loeser define the relative motivic vanishing cycle $[\phi_g] \in \hat{K}_0^{\text{fl}}(\text{Var}/Y_0)$, we denote by $[\phi_g^{\text{mon}}] \in \hat{K}_0^{\text{mon}}(\text{Var}/Y_0)$ its image under the isomorphism (14). There is not a consensus in the literature regarding the normalizing factor for vanishing cycles. We pick the normalization so that, if $g = 0$,

$$[\phi_g^{\text{mon}}] = \mathbb{L}^{-\dim Y/2} [Y \xrightarrow{\text{id}_Y} Y].$$

Let X and Z be varieties, and assume that Z is a subvariety of the critical locus of the function $g \in \Gamma(Y)$. Let $f: Z \rightarrow X$ is a morphism of varieties, we write

$$[Z \xrightarrow{f} X]_{\text{vir}} := f_!(Z \hookrightarrow Y_0)^* [\phi_g^{\text{mon}}] \in \hat{K}_0^{\text{mon}}(\text{Var}/X).$$

Let Y be a G -equivariant variety, for G a special algebraic group, let $Z \subset X$ be a G -equivariant subvariety, let X be a variety, and let $\bar{f}: Z \rightarrow X$ be a G invariant morphism inducing a morphism of stacks $f: Z/G \rightarrow X$. If $\bar{g} \in \Gamma(Y)^G$ is a G -invariant function, we extend the above definition by setting

$$[Z/G \xrightarrow{f} X]_{\text{vir}} := \bar{f}_!(Z \hookrightarrow Y_0)^* [\phi_{\bar{g}}^{\text{mon}}] \mathbb{L}^{\dim(G)/2} / [G].$$

In general, motivic vanishing cycles can be hard to calculate, but there is a situation in which things become easier to write down, thanks to a theorem of Nicaise and Payne.

Theorem 2.1. [26, Thm.4.1.1+Prop.5.1.5] *Let $\tilde{X} = X \times \mathbb{A}_{\mathbb{C}}^n$ be a \mathbb{G}_m -equivariant variety, with action*

$$z \cdot (x, z_1, \dots, z_n) = (x, z^{d_1} z_1, \dots, z^{d_n} z_n)$$

with all the d_i positive. Assume furthermore that $g: \tilde{X} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ is \mathbb{G}_m -equivariant, with weight $d > 0$ action on the target. Then there is an equality

$$\pi_{X,1}[\phi_g^{\text{mon}}] = \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} [\tilde{X} \xrightarrow{\pi_X \times g} X \times \mathbb{A}_{\mathbb{C}}^1] \in \hat{K}_0^{\text{mon}}(\text{Var}/X).$$

In particular, if $X' \subset X$ is a subvariety and $\tilde{X}' = \pi_X^{-1}(X')$, there is an equality

$$\int (\tilde{X}' \cap [\phi_g^{\text{mon}}]) = \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} [\tilde{X}' \xrightarrow{g|_{\tilde{X}'}} \mathbb{A}_{\mathbb{C}}^1] \in \hat{K}_0^{\text{mon}}(\text{Var}/\text{pt}).$$

The motivic incarnation of our main theorem is a consequence of the Nicaise–Payne theorem.

Theorem 2.2. *Let $g \in \Gamma(X \times \mathbb{A}_{\mathbb{C}}^n)$ satisfy the conditions of Theorem 2.1, and assume that there is a decomposition $\mathbb{A}_{\mathbb{C}}^n = \mathbb{A}_{\mathbb{C}}^m \times \mathbb{A}_{\mathbb{C}}^{n-m}$ such that we can write*

$$g = \sum_{j=1}^m g_j t_j + g_0.$$

with g_0, \dots, g_m pulled back from $X \times \mathbb{A}_{\mathbb{C}}^{n-m}$, and t_1, \dots, t_m coordinates for $\mathbb{A}_{\mathbb{C}}^m$. Let $\tilde{Z} = Z(g_1, \dots, g_m)$ be the vanishing locus of the functions g_1, \dots, g_m , and let h be the restriction of g_0 to \tilde{Z} . Then there is an equality

$$\pi_{X,*}[\phi_g^{\text{mon}}] = \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} [\tilde{Z} \xrightarrow{\pi_X \times h} X \times \mathbb{A}_{\mathbb{C}}^1] \in \hat{K}_0^{\text{mon}}(\text{Var}/X).$$

In particular, if $X' \subset X$ is a subvariety, and $\tilde{X}' = \pi_X^{-1}(X')$, there is an equality

$$\int (\tilde{X}' \cap [\phi_g^{\text{mon}}]) = \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} [\tilde{X}' \cap \tilde{Z} \xrightarrow{h|_{\tilde{X}' \cap \tilde{Z}}} \mathbb{A}_{\mathbb{C}}^1] \in \hat{K}_0^{\text{mon}}(\text{Var}/\text{pt}).$$

Proof. Set $V = X \times \mathbb{A}_{\mathbb{C}}^{n-m}$, and for $0 \leq j \leq m$ let $h_j \in \Gamma(V)$ satisfy $h_j \circ (\tilde{X} \xrightarrow{\pi_V} V) = g_j$. We stratify V by setting

$$V_k = \{x \in V \mid h_l(x) = 0 \text{ for } 1 \leq l < k, \text{ and } h_k(x) \neq 0\}$$

for $k = 1, \dots, m$ and $V_{m+1} = Z(h_1, \dots, h_m)$. We denote by \tilde{X}_k the preimage of V_k under the natural projection. Then $\tilde{Z} = \tilde{X}_{m+1}$. By Theorem (2.1) we have

$$\begin{aligned} \pi_{X,*}[\phi_g^{\text{mon}}] &= \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} (\tilde{X} \rightarrow X)_! [\tilde{X} \xrightarrow{\text{id}_{\tilde{X}} \times g} \tilde{X} \times \mathbb{A}_{\mathbb{C}}^1] \\ &= \mathbb{L}^{-\frac{\dim(\tilde{X})}{2}} (V \rightarrow X)_! (\tilde{X} \rightarrow V)_! [\tilde{X} \xrightarrow{\text{id}_{\tilde{X}} \times g} \tilde{X} \times \mathbb{A}_{\mathbb{C}}^1] \\ &= \sum_{k=1}^{m+1} (V_k \rightarrow X)_! \left((V_k \cap (\tilde{X} \rightarrow V))_! [\tilde{X} \xrightarrow{\text{id}_{\tilde{X}} \times g} \tilde{X} \times \mathbb{A}_{\mathbb{C}}^1] \right) \\ &= \sum_{k=1}^{m+1} (V_k \rightarrow X)_! \left[\tilde{X}_k \xrightarrow{\pi_{V_k} \times g|_{\tilde{X}_k}} V_k \times \mathbb{A}_{\mathbb{C}}^1 \right] \end{aligned}$$

and the theorem follows from the claim that all the terms in the final sum are zero, apart from the $k = m + 1$ term. Let $1 \leq k \leq m$ and consider the commutative diagram

$$\begin{array}{ccc} V_k \times \mathbb{A}_{\mathbb{C}}^m & \xrightarrow{\zeta} & V_k \times \mathbb{A}_{\mathbb{C}}^{m-1} \times \mathbb{A}_{\mathbb{C}}^1 \\ & \searrow \pi_{V_k} \times g & \swarrow \pi_{V_k} \times \mathbb{A}_{\mathbb{C}}^1 \\ & & V_k \times \mathbb{A}_{\mathbb{C}}^1 \end{array}$$

where the map

$$\zeta(v, t_1, \dots, t_m) = (v, t_1, \dots, \hat{t}_k, \dots, t_m, g(v, t_1, \dots, t_m))$$

has an inverse given by

$$\zeta^{-1}(v, t_1, \dots, \hat{t}_k, \dots, t_m, s) = (v, t_1, \dots, t_{k-1}, (s - g_0(v) - \sum_{j \neq k} t_j g_j(v)) / g_k(v), t_{k+1}, \dots, t_m)$$

It follows by definition that $[V_k \times \mathbb{A}_{\mathbb{C}}^{m-1} \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{\pi_{V_k \times \mathbb{A}_{\mathbb{C}}^1}} V_k \times \mathbb{A}_{\mathbb{C}}^1]$ is zero in the quotient group $\hat{K}_0^{\text{mon}}(\text{Var}/V_k)$, and so we deduce that

$$[\tilde{X}_k = V_k \times \mathbb{A}_{\mathbb{C}}^m \xrightarrow{\pi_{V_k \times \mathbb{A}_{\mathbb{C}}^g}} V_k \times \mathbb{A}_{\mathbb{C}}^1] = 0$$

as required. \square

3. DONALDSON–THOMAS THEORY FOR QUIVERS WITH POTENTIAL

Let Q be a quiver, by which we mean two sets Q_1 and Q_0 that we always assume to be finite, along with two maps

$$s, t: Q_1 \rightarrow Q_0.$$

The sets Q_0 and Q_1 should be thought of as the set of vertices and arrows, respectively, while the maps s and t take an arrow to its source and target, respectively. For simplicity, in this paper, we will only ever consider symmetric quivers, i.e. those quivers such that for all pairs of vertices i, i' there are as many arrows from i to i' as from i' to i . Let $\gamma \in \mathbb{Z}_{\geq 0}^{Q_0}$ be a dimension vector. A representation of Q with dimension vector γ is given by a set of vector spaces $(V_i)_{i \in Q_0}$ with $\dim(V_i) = \gamma(i)$ and linear maps $\rho_a: V_{s(a)} \rightarrow V_{t(a)}$ for each $a \in Q_1$. We define the affine space

$$\mathbb{X}_{\gamma}(Q) := \prod_{a \in Q_1} \text{Hom}(V_{s(a)}, V_{t(a)}),$$

which is acted on by the gauge group

$$\text{GL}_{\gamma} := \prod_{i \in Q_0} \text{GL}(V_i)$$

by change of basis. The stack-theoretic quotient $\mathfrak{M}_{\gamma}(Q) := \mathbb{X}_{\gamma}(Q)/\text{GL}_{\gamma}$ is the stack of γ -dimensional Q -representations, which we identify throughout this paper with left $\mathbb{C}Q$ -modules. We denote by

$$\mathcal{M}_{\gamma}(Q) := \text{Spec}(\Gamma(\mathbb{X}_{\gamma}(Q))^{\text{GL}_{\gamma}}),$$

the affinization of this stack, and consider the canonical map:

$$p_{Q, \gamma}: \mathbb{X}_{\gamma}(Q)/\text{GL}_{\gamma} \rightarrow \mathcal{M}_{\gamma}(Q).$$

At the level of geometric points, this is the map that takes a Q -representation over a field extension $K \supset \mathbb{C}$ to its semisimplification.

Let

$$W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]_{\text{vect}}$$

be a linear combination of cyclic paths in Q , i.e. a *potential*. Evaluating $\text{Tr}(W)$ at the representations defined by points of $\mathbb{X}_{\gamma}(Q)$ provides a GL_{γ} -invariant function $\text{Tr}(W)_{\gamma}$ on $\mathbb{X}_{\gamma}(Q)$, and so a function $\mathfrak{Tr}(W)_{\gamma}$ on $\text{Rep}_{\gamma}(Q)$, pulled back from the induced function $\mathcal{T}r(W)_{\gamma}$ on $\mathcal{M}_{\gamma}(Q)$. We define

$$\mathbb{X}_{\gamma}(Q, W) := \text{crit}(\text{Tr}(W)_{\gamma})$$

and we will use the notation:

$$[\mathbb{X}_{\gamma}(Q, W)]_{\text{vir}} := \int [\phi_{\text{Tr}(W)_{\gamma}}^{\text{mon}}] \in \hat{K}_0^{\text{mon}}(\text{Var}/\text{pt}).$$

Consider also the stack $\text{Rep}_{\gamma}(Q, W) := \mathbb{X}_{\gamma}(Q, W)/\text{GL}_{\gamma}$. For K a field, we define the Jacobi algebra

$$K(Q, W) = KQ/\langle \partial W/\partial a \mid a \in Q_1 \rangle$$

where $\partial W/\partial a$ is the noncommutative derivative of W with respect to a . The critical locus of $\mathfrak{Tr}(W)_{\gamma}$ and the stack of γ -dimensional $\mathbb{C}(Q, W)$ -modules are equal as substacks of $\mathfrak{M}_{\gamma}(Q)$.

Fix a bijection $Q_0 \cong \{1, \dots, n\}$. For $\gamma \in \mathbb{N}^{Q_0}$ we write

$$T^\gamma = T_1^{\gamma(1)} \cdots T_n^{\gamma(n)}.$$

Then we define the *motivic DT partition function*

$$\mathcal{Z}_{Q,W}(T) := \sum_{\gamma \in \mathbb{N}^{Q_0}} \mathbb{L}^{\frac{\gamma \cdot \gamma}{2}} [\mathrm{GL}_\gamma]^{-1} [\mathbb{X}_\gamma(Q, W)]_{\mathrm{vir}} T^\gamma \in 1 + \mathcal{M}^{\mathrm{mon}}[[T_1, \dots, T_n]]_+$$

and the *motivic DT invariants* $\Omega_{Q,W,\gamma}$ via the formula

$$(16) \quad \mathrm{EXP} \left(\sum_{0 \neq \gamma \in \mathbb{N}^{Q_0}} \Omega_{Q,W,\gamma} (\mathbb{L}^{1/2} - \mathbb{L}^{-1/2})^{-1} T^\gamma \right) = \mathcal{Z}_{Q,W}(T).$$

A priori, the elements $\Omega_{Q,W,\gamma}$ are only defined as elements of $\mathcal{M}^{\mathrm{mon}}$. The integrality conjecture states that after quotienting out by certain extra relations, the elements $\Omega_{Q,W,\gamma}$ are in the image of the natural map from $\hat{K}_0^{\mathrm{mon}}(\mathrm{Var}/\mathrm{pt})$. See [23, 11] for proofs of variants of this conjecture. The proof of the Cazzaniga–Morrison–Pym–Szendrői conjecture below, in particular, implies the integrality conjecture for the quivers with potential that we consider in this paper.

4. MOTIVIC DT INVARIANTS FOR THE DEFORMED WEYL POTENTIAL

We come now to the main application of Theorem 2.2 in this paper. Let Q be the three loop quiver, i.e. $|Q_0| = 1$ and $Q_1 = \{a, b, c\}$. Let Q' be the two loop quiver obtained by removing a from Q_1 . Set

$$(17) \quad W = [a, b]c.$$

For $d \geq 2$ we define

$$(18) \quad W_d = [a, b]c + c^d.$$

We let \mathbb{G}_m act on triples of matrices in $\mathbb{X}_n(Q)$ via

$$z \cdot (A, B, C) = (z^{d-1}A, B, zC).$$

Then $\mathrm{Tr}(W_d)_n$ is a \mathbb{G}_m -equivariant function on $\mathbb{X}_n(Q)$, of weight d , and we can write

$$\begin{aligned} \mathrm{Tr}(W_d)_n(\rho) &= \mathrm{Tr}(W)(\rho(a), \rho(b), \rho(c)) + \mathrm{Tr}(\rho(c)^d) \\ &= \left(\sum_{(i,j) \in \{1, \dots, n\}^2} \rho(a)_{ij} [\rho(b), \rho(c)]_{ji} \right) + \mathrm{Tr}(\rho(c)^d). \end{aligned}$$

In the terminology of Theorem 2.2, we set $g_0 = \mathrm{Tr}(\rho(c)^d)$ and we set g_1, \dots, g_{n^2} to be the functions recording the matrix entries of $[\rho(b), \rho(c)]$, so that Z is the locus on which $\rho(b)$ and $\rho(c)$ commute, which we denote

$$\mathbf{C}_n = \{(B, C) \in \mathrm{Mat}_{n \times n}(\mathbb{C})^{\times 2} \mid [B, C] = 0\} \subset \mathrm{Mat}_{n \times n}(\mathbb{C})^{\times 2}.$$

Theorem 2.2 gives the equality in $\hat{K}_0^{\mathrm{mon}}(\mathrm{Var}/\mathrm{pt})$

$$(19) \quad [\mathbb{X}_n(Q, W_d)]_{\mathrm{vir}} = \mathbb{L}^{-\frac{n^2}{2}} \cdot [\mathbf{C}_n \xrightarrow{(B,C) \mapsto \mathrm{Tr}(C^d)} \mathbb{A}_{\mathbb{C}}^1].$$

We define a map

$$\lambda_n: \mathbf{C}_n \rightarrow \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^2)$$

sending a pair of commuting matrices a, b to the generalized eigenvalues (with multiplicities) of the two matrices. I.e. if $\mathbb{C}^n \cong \bigoplus_{i \in I} V_i$ is a decomposition of \mathbb{C}^n into subspaces preserved by both a and b , for which the generalized eigenvalues are $\alpha_{i,1}$ and $\alpha_{i,2}$ respectively, and such that for all $i \neq j$ either $\alpha_{i,1} \neq \alpha_{j,1}$ or $\alpha_{i,2} \neq \alpha_{j,2}$, then (a, b) is mapped to the n -tuple of points in $\mathbb{A}_{\mathbb{C}}^2$ containing $(\alpha_{i,1}, \alpha_{i,2})$ $\dim(V_i)$

times. As a map of schemes, this is induced from the inclusion of the subring of GL_n -invariant functions on \mathbf{C}_n .

Denote by Q^{fr} the quiver obtained by adding a vertex ∞ to Q and an extra arrow j from ∞ to the original vertex of Q . We consider the potential W also as a potential for Q^{fr} . Consider

$$\mathrm{ncHilb}_n \subset \mathbb{X}_{(1,n)}(Q^{\mathrm{fr}}) / \mathrm{GL}_n$$

the open substack corresponding to Q^{fr} -representations ρ for which the image of $\rho(a)$ generates ρ . This is actually a fine moduli scheme called the noncommutative Hilbert variety. We define the map

$$\tilde{\lambda}_n : \mathbb{X}_n(Q, W) \rightarrow \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)$$

the same way as λ_n , and denote by the same symbol $\tilde{\lambda}_n$ the extensions/restrictions to moduli stacks of Q^{fr} -representations lying in the critical locus of $\mathrm{Tr}(W)$. For a stack \mathcal{X} with a regular function $f : \mathcal{X} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ and a morphism $\tilde{\lambda}_n : \mathcal{X} \rightarrow \mathrm{Sym}^n \mathbb{A}_{\mathbb{C}}^3$, we will use the notation

$$[\mathcal{X} \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n \mathbb{A}_{\mathbb{C}}^3]_{\mathrm{vir}} := (\mathcal{X} \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n \mathbb{A}_{\mathbb{C}}^3)_{!} [\Phi_{\mathrm{Tr}(W), \gamma}^{\mathrm{mon}}].$$

The following application of wall crossing formula is standard, but we will indicate the proof for completeness:

Proposition 4.1. *There is an equality in $\mathcal{M}_{\mathrm{Sym}(\mathbb{A}_{\mathbb{C}}^3)}^{\mathrm{mon}}$:*

$$\begin{aligned} \sum_{n \geq 0} [\mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}}, W) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} = \\ \sum_{n \geq 0} [\mathrm{Rep}_n(Q, W) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} \cdot \sum_{n \geq 0} [\mathrm{ncHilb}_n(Q, W) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} \cdot (\mathbb{L} - 1)^{-1}. \end{aligned}$$

Proof. There is a decomposition into locally closed substacks

$$\mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}}) = \coprod_{i=0}^n \mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}})_{[i]}$$

where

$$\mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}})_{[i]} \subset \mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}})$$

is the substack of Q^{fr} -representations ρ for which the subspace generated by the image of $\rho(j)$ under the action of $\rho(a), \rho(b), \rho(c)$ is i -dimensional. We define

$$\iota_{[i]} : \mathbb{X}_{(1,n)}(Q^{\mathrm{fr}})_{[i]} \hookrightarrow \mathbb{X}_{(1,n)}(Q^{\mathrm{fr}})$$

to be the inclusion of the subset for which the image of $\rho(j)$ lies in the first summand of the decomposition $\mathbb{C}^n = \mathbb{C}^i \oplus \mathbb{C}^{n-i}$, and generates it, and for which $\rho(a), \rho(b), \rho(c)$ also preserve this summand. We define $\mathrm{GL}_{i,n-i} \subset \mathrm{GL}(\mathbb{C}^n)$ to be the subgroup of automorphisms preserving the filtration $0 \subset \mathbb{C}^i \subset \mathbb{C}^n$. Then

$$\mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}})_{[i]} \cong \mathbb{X}_{(1,n)}(Q^{\mathrm{fr}})_{[i]} / (\mathrm{GL}(\mathbb{C}) \times \mathrm{GL}_{i,n-i})$$

where the first factor of the gauge group acts by

$$z \cdot (\rho(a), \rho(b), \rho(c), \rho(j)) = (\rho(a), \rho(b), \rho(c), z^{-1}\rho(j)).$$

We may identify

$$\mathrm{ncHilb}_n(Q) = \mathbb{X}_{(1,n)}(Q^{\mathrm{fr}})_{[n]} / \mathrm{GL}_n.$$

The decomposition $\mathbb{C} = \mathbb{C}^i \oplus \mathbb{C}^{n-i}$ induces a decomposition

$$\mathbb{X}_{(1,n)}(Q^{\mathrm{fr}})_{[i]} = V_0 \times V_1 \times V_{-1}$$

with

$$\begin{aligned} V_0 &= \mathbb{X}_{(1,i)}(Q^{\text{fr}})_{[i]} \times \mathbb{X}_{n-i}(Q) \\ V_1 &= \text{Hom}(\mathbb{C}^{n-i}, \mathbb{C}^i)^{\times 3} \\ V_{-1} &= \text{Hom}(\mathbb{C}^i, \mathbb{C}^{n-i})^{\times 3} \end{aligned}$$

and we have $\mathbb{X}_{(1,n)}(Q^{\text{fr}})_{[i]} = V_0 \oplus V_1$. Note that the diagram

$$\begin{array}{ccc} \mathbb{X}_{(1,n)}(Q^{\text{fr}})_{[i]} & \longrightarrow & V_0 \xrightarrow{\tilde{\lambda}_i \times \tilde{\lambda}_{n-i}} \text{Sym}^i(\mathbb{A}_{\mathbb{C}}^3) \times \text{Sym}^{n-i}(\mathbb{A}_{\mathbb{C}}^3) \\ \downarrow \iota_{[i]} & & \downarrow \cup \\ \mathbb{X}_{(1,n)}(Q^{\text{fr}}) & \xrightarrow{\tilde{\lambda}_n} & \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^3) \end{array}$$

commutes. Let \mathbb{C}^* act on each V_k with weight k . Denote by $f_{[i]}$ the restriction of $\text{Tr}(W)_n$ to V_0 . Then $\text{Tr}(W)_n$ is \mathbb{C}^* -invariant, and by the integral identity [33] we have

$$(\mathbb{X}_{(1,n)}(Q^{\text{fr}})_{[i]} \rightarrow V_0)_{*} \iota_{[i]}^* [\Phi_{\text{Tr}(W)_n}^{\text{mon}}] = [\Phi_{f_{[i]}}^{\text{mon}}].$$

By the motivic Thom–Sebastiani theorem [15], we deduce that

$$[\Phi_{f_{[i]}}^{\text{mon}}] = [\Phi_{\text{Tr}(W)_{(1,i)}}^{\text{mon}}] \boxtimes [\Phi_{\text{Tr}(W)_{n-i}}^{\text{mon}}] \in \hat{\mathbf{K}}_0^{\text{mon}}(\text{Var}/V_0).$$

Finally, setting $H_i = ([\text{GL}(\mathbb{C}) \times \text{GL}(\mathbb{C}^i) \times \text{GL}(\mathbb{C}^{n-i})]) \in \hat{\mathbf{K}}_0(\text{Var}/\text{pt})$ we calculate

$$\begin{aligned} & [\text{Rep}_{(1,n)}(Q^{\text{fr}}, W) \xrightarrow{\tilde{\lambda}_n} \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\text{vir}} = \\ & \sum_{i=0}^n (\mathbb{X}_{(1,n)}(Q^{\text{fr}}, W)_{[i]} \xrightarrow{\tilde{\lambda}_n} \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^3))_{*} \iota_{[i]}^* [\Phi_{\text{Tr}(W)_{(1,n)}}^{\text{mon}}] \mathbb{L}^{n^2/2} / [\text{GL}(\mathbb{C}) \times \text{GL}_{i,n-i}] = \\ & \sum_{i=0}^n [\mathbb{X}_{(1,i)}(Q^{\text{fr}}, W)_{[i]} \xrightarrow{\tilde{\lambda}_i} \text{Sym}^i(\mathbb{A}_{\mathbb{C}}^3)]_{\text{vir}} \cdot \cup [\mathbb{X}_{n-i}(Q, W) \xrightarrow{\tilde{\lambda}_{n-i}} \text{Sym}^{n-i}(\mathbb{A}_{\mathbb{C}}^3)]_{\text{vir}} \mathbb{L}^{i^2+(n-i)^2} / H_i = \\ & \sum_{i=0}^n [\text{ncHilb}_i(Q, W) \xrightarrow{\tilde{\lambda}_i} \text{Sym}^i(\mathbb{A}_{\mathbb{C}}^3)]_{\text{vir}} \cdot \cup [\text{Rep}_{n-i}(Q, W) \xrightarrow{\tilde{\lambda}_{n-i}} \text{Sym}^{n-i}(\mathbb{A}_{\mathbb{C}}^3)]_{\text{vir}} \cdot (\mathbb{L} - 1)^{-1} \end{aligned}$$

as required. \square

The next proposition is essentially a refinement of a classical result of Feit–Fine, who proved the counting result analogous to the identity

$$\sum_{n \geq 0} [\mathfrak{C}_n] T^n = \text{EXP} \left(\sum_{n \geq 1} \mathbb{L}^2 T^n \right)$$

obtained by pushing forward both the left and the right hand side of (20) to absolute motives.

Proposition 4.2. *There is an equality of generating series in $\mathcal{M}_{\text{Sym}(\mathbb{A}_{\mathbb{C}}^2)}$:*

$$(20) \quad \sum_{n \geq 0} [\mathfrak{C}_n \xrightarrow{\tilde{\lambda}_n} \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^2)] = \text{EXP}_{\cup} \left(\sum_{n \geq 1} [\mathbb{A}_{\mathbb{C}}^2 \xrightarrow{\Delta} \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^2)] / (\mathbb{L} - 1) \right)$$

Proof. Set $W = [A, B]C$. We define the projection

$$\begin{aligned} \pi_n &: \text{Mat}_{n \times n}(\mathbb{C})^{\times 3} \rightarrow \text{Mat}_{n \times n}(\mathbb{C})^{\times 2} \\ & (A, B, C) \mapsto (B, C) \end{aligned}$$

By the relative statement of the (undeformed) dimensional reduction theorem (Theorem 2.1) there are equalities of relative motives

$$\begin{aligned} \pi_{n,!}[\phi_{\mathrm{Tr}(W)_n}^{\mathrm{mon}}] &= [\mathrm{Mat}_{n \times n}(\mathbb{C})^{\times 3} \xrightarrow{g_n} Y_n \times \mathbb{A}_{\mathbb{C}}^1] \mathbb{L}^{-3n^2/2} && \in \hat{\mathbf{K}}_0^{\mathrm{mon}}(\mathrm{Var}/Y_n) \\ &= [\mathbf{C}_n \hookrightarrow Y_n] \mathbb{L}^{-n^2/2} && \in \hat{\mathbf{K}}_0(\mathrm{Var}/Y_n) \end{aligned}$$

where $g_n(A, B, C) = (B, C, \mathrm{Tr}([A, B]C))$ and $Y_n = \mathrm{Mat}_{n \times n}(\mathbb{C})^{\times 2}$. There is an obvious identity

$$[\mathrm{Rep}_{(1,n)}(Q^{\mathrm{fr}}) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)] = \mathbb{L}^n [\mathrm{Rep}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]$$

and so by Proposition 4.1 there is an identity

$$(21) \quad \sum_{n \geq 0} [\mathrm{ncHilb}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} = \sum_{n \geq 0} [\mathrm{Rep}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} \mathbb{L}^n \left(\sum_{n \geq 0} [\mathrm{Rep}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} \right)^{-1}$$

On the other hand, by [8, Prop.4.3, Cor.4.4] there is an equality

$$(22) \quad \sum_{n \geq 0} [\mathrm{ncHilb}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} = \mathrm{EXP}_{\cup} \left(\sum_{n \geq 1} [\mathbb{A}_{\mathbb{C}}^3 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)] \frac{\mathbb{L}^{-3/2}(\mathbb{L}^n - 1)}{\mathbb{L} - 1} \right)$$

Taking plethystic logarithms, we may write

$$\sum_{n \geq 0} [\mathrm{Rep}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} = \mathrm{EXP}_{\cup} \left(\sum_{n \geq 1} \Omega_n \frac{\mathbb{L}^{1/2}}{\mathbb{L} - 1} \right).$$

for $\Omega_n \in \mathcal{M}_{\mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)}$. Then (21) implies that

$$\sum_{n \geq 0} [\mathrm{ncHilb}_n(Q) \xrightarrow{\tilde{\lambda}_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]_{\mathrm{vir}} = \mathrm{EXP}_{\cup} \left(\sum_{n \geq 1} \Omega_n \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} \right)$$

and so $\Omega_n = \mathbb{L}^{-3/2} [\mathbb{A}_{\mathbb{C}}^3 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)]$ for all $n \geq 1$ by (22). Let $k: \mathrm{Sym}(\mathbb{A}_{\mathbb{C}}^3) \rightarrow \mathrm{Sym}(\mathbb{A}_{\mathbb{C}}^2)$ be the map induced on tuples of points by the projection $(x, y, z) \mapsto (y, z)$. Putting everything together,

$$\begin{aligned} \sum_{n \geq 0} [\mathbf{c}_n \xrightarrow{\lambda_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^2)] &= \sum_{n \geq 0} \lambda_{n,!} \pi_{n,!} [\phi_{\mathrm{Tr}(W)_n}^{\mathrm{mon}}] \mathbb{L}^{-n^2/2} / [\mathrm{GL}_n] \\ &= k_! \mathrm{EXP}_{\cup} \left(\sum_{n \geq 1} \mathbb{L}^{-3/2} [\mathbb{A}_{\mathbb{C}}^3 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^3)] \frac{\mathbb{L}^{1/2}}{\mathbb{L} - 1} \right) \\ &= \mathrm{EXP}_{\cup} \left(\sum_{n \geq 1} [\mathbb{A}_{\mathbb{C}}^2 \xrightarrow{\Delta_n} \mathrm{Sym}^n(\mathbb{A}_{\mathbb{C}}^2)] \frac{1}{\mathbb{L} - 1} \right) \end{aligned}$$

as required. \square

Theorem 4.3. *The motivic DT invariants for the quiver with potential (Q, W_d) are given by the formula*

$$\Omega_{Q, W_d, n} = [\mathbb{L}]^{1/2} [\mathbb{A}_{\mathbb{C}}^1 \xrightarrow{t \mapsto t^d} \mathbb{A}_{\mathbb{C}}^1] \in \hat{\mathbf{K}}_0^{\mathrm{mon}}(\mathrm{Var}/\mathrm{pt})$$

for all $n \geq 1$.

In the case $d = 3$, the above theorem is a verification of [4, Conj.3.3]. In [4] the equivalent (via isomorphism (14)) formulation $[\mathbb{L}]^{1/2}(1 - [\mu_d])$ was given for the motivic DT invariants, in the ring of $\hat{\mu}$ -equivariant motives. For $n = 1$ and general d the conjecture is trivial, for $n = 2$ and $d = 3$ its verification is already rather involved, but was successfully carried out in [3].

Proof. Let $\text{lt}: \coprod_{n \geq 0} \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^2) \rightarrow \mathbb{Z}_{\geq 0}$ be the morphism of monoids taking $\text{Sym}^n(\mathbb{A}_{\mathbb{C}}^2)$ to the point n . We define the morphism

$$k_n: \text{Sym}^n(\mathbb{A}_{\mathbb{C}}^2) \rightarrow \mathbb{A}_{\mathbb{C}}^1$$

$$((x_1, y_1), \dots, (x_n, y_n)) \mapsto \sum_{i=1}^n y_i^d.$$

Then $k = \coprod_{n \geq 0} k_n$ is a morphism of commutative monoids, so that k_i commutes with taking plethystic exponentials. Combining (19) with Proposition 4.2 we deduce that

$$\begin{aligned} \mathcal{Z}_{Q, W_d}(T) &:= \sum_{n \geq 0} \left([\mathbb{X}_n(Q, W_d)]_{\text{vir}} \mathbb{L}^{n^2/2} / [\text{GL}_n] \right) T^n \\ &= \sum_{n \geq 0} \left([\mathbf{C}_n \xrightarrow{(b,c) \mapsto \text{Tr}(c^d)} \mathbb{A}_{\mathbb{C}}^1] / [\text{GL}_n] \right) T^n && \text{Equation (19)} \\ &= \sum_{n \geq 0} (k_{n,!} \lambda_{n,!} [\mathbf{C}_n \rightarrow \mathbf{C}_n] / [\text{GL}_n]) T^n \\ &= k_! \text{EXP}_{\cup} \left(\sum_{n \geq 1} [\mathbb{A}^2 \xrightarrow{\Delta_n} \text{Sym}^n(\mathbb{A}^2)] / (\mathbb{L} - 1) \right) && \text{Proposition 4.2} \\ &= \text{EXP} \left(\sum_{n \geq 1} (k_{n,!} [\mathbb{A}^2 \xrightarrow{\Delta_n} \text{Sym}^n(\mathbb{A}^2)] / (\mathbb{L} - 1)) T^n \right) \\ &= \text{EXP} \left(\sum_{n \geq 1} \left([\mathbb{A}_{\mathbb{C}}^2 \xrightarrow{(x,y) \mapsto ny^d} \mathbb{A}_{\mathbb{C}}^1] / (\mathbb{L} - 1) \right) T^n \right). \end{aligned}$$

The result then follows by comparing with (16). \square

5. VANISHING CYCLES AND COHOMOLOGICAL DT THEORY

For the rest of the paper we leave behind the naive Grothendieck ring of motives, and work in the category of monodromic mixed Hodge structures and mixed Hodge modules, the natural home of *cohomological* DT theory. We introduce the key features here, for a fuller reference the reader is advised to consult [13, Sec.2].

5.1. The category MMHM. Let X be a complex variety. We define

$$\text{MMHM}(X) := \mathcal{B}_X / \mathcal{C}_X,$$

the Serre quotient of two abelian subcategories of $\text{MHM}(X \times \mathbb{A}_{\mathbb{C}}^1)$ the category of mixed Hodge modules on $X \times \mathbb{A}_{\mathbb{C}}^1$. This is as defined by Saito (see [30, 31, 32]). We denote by $\mathcal{D}(\text{MMHM}(X))$ the derived category of (not necessarily bounded) complexes of monodromic mixed Hodge modules. For a complex variety Y we denote by $\text{rat}_Y: \text{MHM}(Y) \rightarrow \text{Perv}(Y)$ the (faithful) forgetful functor to perverse sheaves on Y . We define \mathcal{B}_X to be the full subcategory with objects those mixed Hodge modules such that for each $x \in X$ the underlying complex of constructible sheaves of $(x \times \mathbb{G}_m \rightarrow X \times \mathbb{A}_{\mathbb{C}}^1)^* \mathcal{F}$ has locally constant cohomology, while \mathcal{C}_X is the full subcategory, the objects of which satisfy the stronger condition that each

$(x \times \mathbb{A}_{\mathbb{C}}^1 \rightarrow X \times \mathbb{A}_{\mathbb{C}}^1)^* \mathcal{F}$ has constant cohomology sheaves. We write $\text{MMHS} := \text{MMHM}(\text{pt})$. Objects in $\text{MMHM}(X)$ have a weight filtration inherited from Saito's weight filtration of objects in $\text{MHM}(X \times \mathbb{A}_{\mathbb{C}}^1)$, and we say that an object $\mathcal{F} \in \mathcal{D}(\text{MMHM}(X))$ is pure if $\mathcal{H}^j(\mathcal{F}) \in \text{MMHM}(X)$ is pure of weight j for all $j \in \mathbb{Z}$.

Fix a finite quiver Q . As in §3 we denote by

$$\mathcal{M}_{\gamma}(Q) := \text{Spec}(\Gamma(\mathbb{X}_{\gamma}(Q))^{\text{GL}_{\gamma}})$$

the coarse moduli space of γ -dimensional Q -representations. We denote by $\mathcal{M}_{\gamma}^{\text{st}}(Q) \subset \mathcal{M}_{\gamma}(Q)$ the smooth irreducible open subvariety of simple modules, which is dense if it is nonempty. The closed points of $\mathcal{M}_{\gamma}(Q)$ are in bijection with semisimple γ -dimensional $\mathbb{C}Q$ -modules. We set

$$\mathcal{M}(Q) := \prod_{\gamma \in \mathbb{N}^{Q_0}} \mathcal{M}_{\gamma}(Q),$$

a monoid in the category of schemes, with monoid structure denoted \oplus , as at the level of closed points it takes a pair of semisimple $\mathbb{C}Q$ -modules to their direct sum. The map \oplus is finite by [25, Lem.2.1]. Denote by $\mathcal{D}^{\leq, \text{lf}}(\text{MMHM}(\mathcal{M}(Q))) \subset \mathcal{D}(\text{MMHM}(\mathcal{M}(Q)))$ the full subcategory containing those objects such that for each γ , and each weight n , $\text{Gr}_W^n(\mathcal{F}|_{\mathcal{M}_{\gamma}})$ has bounded total cohomology, and for each $\gamma \in \mathbb{N}^{Q_0}$ there is an equality $\text{Gr}_W^n(\mathcal{F}|_{\mathcal{M}_{\gamma}}) = 0$ for $n \gg 0$. There is a symmetric monoidal product defined on $\mathcal{D}^{\leq, \text{lf}}(\text{MMHM}(\mathcal{M}(Q)))$ by

$$\mathcal{F} \boxtimes_{\oplus} \mathcal{G} := \left(\mathcal{M} \times \mathbb{A}_{\mathbb{C}}^1 \times \mathcal{M} \times \mathbb{A}_{\mathbb{C}}^1 \xrightarrow{(\rho, t, \rho', t') \mapsto (\rho \oplus \rho', t + t')} \mathcal{M} \times \mathbb{A}_{\mathbb{C}}^1 \right)_* (\mathcal{F} \boxtimes \mathcal{G})$$

which is exact, and preserves weights by [13, Prop.3.5].

Let $z: \{0\} \rightarrow \mathbb{A}_{\mathbb{C}}^1$ be the inclusion of the origin. Then the functor $(\text{id}_{\mathcal{M}(Q)} \times z)_*$ provides an embedding of symmetric monoidal categories

$$\text{MHM}(\mathcal{M}(Q)) \hookrightarrow \text{MMHM}(\mathcal{M}(Q))$$

which moreover preserves weights. In this way we consider $\text{MHM}(\mathcal{M}(Q))$ as a full subcategory of $\text{MMHM}(\mathcal{M}(Q))$.

Let $\mathfrak{L} := z_* \text{H}_c(\mathbb{A}_{\mathbb{C}}^1, \mathbb{Q})$. Then \mathfrak{L} is concentrated in cohomological degree 2, and its second cohomology is a pure weight 2 one-dimensional monodromic mixed Hodge structure. Moreover $(\mathbb{A}_{\mathbb{C}}^1 \xrightarrow{x \mapsto x^2} \mathbb{A}_{\mathbb{C}}^1)_* \mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^1}$ provides a tensor square root to \mathfrak{L} , denoted $\mathfrak{L}^{1/2}$. Note that there is no tensor square root for \mathfrak{L} inside $\mathcal{D}^{\leq, \text{lf}}(\text{MHS})$, since a pure odd-weight Hodge structure must have even dimension.

5.2. Vanishing cycles. Let Y be a smooth variety, and let $f: Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ be a regular function. Let $i: Y_0 \rightarrow Y$ be the inclusion of the zero fiber of f , and consider the pullback diagram induced by $\text{exp}: \mathbb{A}_{\mathbb{C}}^1 \rightarrow \mathbb{A}_{\mathbb{C}}^1$:

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & \mathbb{A}_{\mathbb{C}}^1 \\ \downarrow p & & \downarrow \text{exp} \\ Y & \xrightarrow{f} & \mathbb{A}_{\mathbb{C}}^1. \end{array}$$

Define the nearby cycle functor $\psi_f: \mathcal{D}^b(\text{Perv}(Y)) \rightarrow \mathcal{D}^b(\text{Perv}(Y))$ by the formula:

$$\psi_f := i^* p_* p^*.$$

The vanishing cycle functor $\phi_f[-1]: \text{Perv}(Y) \rightarrow \text{Perv}(Y)$ maps perverse sheaves on Y to perverse sheaves supported on Y_0 . For $F \in \text{Perv}(Y)$, $\phi_f F$ fits in the distinguished triangle

$$i^* F \rightarrow \psi_f F \rightarrow \phi_f F \rightarrow$$

in $\mathcal{D}^b(\text{Perv}(Y))$. In [30, 32] Saito defines an upgrade of the nearby and vanishing cycles functors to functors

$$\psi_f[-1], \phi_f[-1]: \text{MHM}(Y) \rightarrow \text{MHM}(Y)$$

for the category of mixed Hodge modules. It is an upgrade in the sense that there are natural isomorphisms $\text{rat}_Y \circ \psi_f[-1] \cong \psi_f[-1] \circ \text{rat}_Y$ and $\text{rat}_Y \circ \phi_f[-1] \cong \phi_f[-1] \circ \text{rat}_Y$. We denote by the same symbol the functor $\phi_f: \mathcal{D}(\text{MHM}(Y)) \rightarrow \mathcal{D}(\text{MHM}(Y))$. We define

(23)

$$\phi_f^{\text{mon}} := (Y \times \mathbb{G}_m \rightarrow Y \times \mathbb{A}^1)_! \phi_{f/u}(Y \times \mathbb{G}_m \xrightarrow{\pi_Y} Y)^*: \text{MHM}(Y) \rightarrow \text{MMHM}(Y)$$

where $\mathbb{G}_m = \text{Spec}(\mathbb{C}[u^{\pm 1}])$. Since there is a natural isomorphism

(24)

$$\phi_f^{\text{mon}}(\mathcal{F} \otimes \mathcal{L}^{n/2}) \cong \phi_f^{\text{mon}} \mathcal{F} \otimes \mathcal{L}^{n/2}$$

for n even, we can use the right hand side of (24) to define the left hand side when n is odd.

There is a forgetful functor $\text{forg} : \text{MMHM}(X) \rightarrow \text{Perv}(X)$ defined as follows. Denote by t the coordinate for $\mathbb{A}_{\mathbb{C}}^1$. We denote by the same symbol the induced function on $X \times \mathbb{A}_{\mathbb{C}}^1$. Then the vanishing cycles functor restricts to give a functor

$$\phi_t[-1]: \mathcal{B}_X \rightarrow \text{MHM}(X)$$

where we have identified X with the zero locus of t . Since all objects in \mathcal{C}_X are sent to the zero object by this functor, there is a unique functor $\overline{\phi}_t[-1]: \text{MMHM}(X) \rightarrow \text{MHM}(X)$ from the Serre quotient, through which $\phi_t[-1]$ factors. Composing with the faithful forgetful functor $\text{rat}_X: \text{MHM}(X) \rightarrow \text{Perv}(X)$, we obtain the faithful functor

$$\text{forg}_X = \text{rat}_X \circ \overline{\phi}_t[-1]: \text{MMHM}(X) \rightarrow \text{Perv}(X),$$

see [13, Sec.2.1] for further details.

We next explain how to extend the definition of vanishing cycles to smooth quotient stacks. Let Y be a smooth variety with an action of a reductive group $G \subset \text{GL}(n)$, and consider a G -invariant function $f: Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$. For $N \geq n$, define $\text{Fr}(n, N) \subset (\mathbb{A}_{\mathbb{C}}^N)^n$ the open set of n -tuples of linearly independent vectors in $\mathbb{A}_{\mathbb{C}}^N$. The action of G on $\text{Fr}(n, N)$ is free, so $Y \times_G \text{Fr}(n, N)$ is a smooth variety. Then there are isomorphisms, which we can take as a definition for the purposes of this paper:

$$(25) \quad H_c^j(Y/G, \phi_f^{\text{mon}} \mathbb{Q}) \cong \lim_{N \rightarrow \infty} H_c^j(Y \times_G \text{Fr}(n, N), \phi_f^{\text{mon}} \mathbb{Q} \otimes \mathcal{L}^{-nN}).$$

5.3. Cohomological BPS invariants. Let Q be a quiver and let $W \in \mathbb{C}Q/[\mathbb{C}Q, \mathbb{C}Q]$ be a potential. Let $N \in \mathbb{Z}_{\geq 0}$, and define the quiver Q_N to be the quiver obtained from Q by adding one extra vertex, labelled ∞ , and N arrows from ∞ to i for each $i \in Q_0$. We have

$$\mathbb{Z}_{\geq 0}^{(Q_N)_0} = \mathbb{Z}_{\geq 0} \oplus \mathbb{Z}_{\geq 0}^{Q_0}$$

Fix a dimension vector $\gamma \in \mathbb{Z}_{\geq 0}^{Q_0}$. Let

$$\mathbb{X}_{(1, \gamma)}^{\text{st}}(Q_N) \subset \mathbb{X}_{(1, \gamma)}(Q_N)$$

be the open subvariety, the closed points of which correspond to $\mathbb{C}Q_N$ -modules ρ satisfying the condition that there are no proper submodules ρ' with $\dim(\rho')_{\infty} = 1$. The GL_{γ} -action on this variety is scheme-theoretically free, and we define

$$\mathcal{M}_{N, \gamma}^{\text{fr}}(Q) := \mathbb{X}_{(1, \gamma)}^{\text{st}}(Q_N) / \text{GL}_{\gamma},$$

the fine moduli space of γ -dimensional stable N -framed modules. This space is smooth, and the forgetful map

$$q_{N,\gamma} : \mathcal{M}_{N,\gamma}^{\text{fr}}(Q) \rightarrow \mathcal{M}_\gamma(Q)$$

is proper. We write $\mathcal{T}r(W)_\gamma$ for the function defined on $\mathcal{M}_\gamma(Q)$, and $\mathcal{T}r(W)_{N,\gamma}$ for the function defined on $\mathcal{M}_{N,\gamma}^{\text{fr}}(Q)$. There are isomorphisms:

$$\begin{aligned} \mathcal{H}^j(p! \phi_{\mathcal{T}r(W)_\gamma}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_\gamma(Q)}) &\cong \lim_{N \rightarrow \infty} \mathcal{H}^j \left(q_{N,\gamma}! \phi_{\mathcal{T}r(W)_{N,\gamma}}^{\text{mon}} \mathbb{Q}_{\mathcal{M}_{N,\gamma}^{\text{fr}}(Q)} \otimes \mathfrak{L}^{-N \cdot \sum_{i \in Q_0} \gamma_i} \right) \\ \mathbb{H}_c^j(\mathfrak{M}_\gamma(Q), \phi_{\mathcal{T}r(W)_\gamma}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_\gamma(Q)}) &\cong \lim_{N \rightarrow \infty} \mathbb{H}_c^j \left(\mathcal{M}_{N,\gamma}^{\text{fr}}(Q), \phi_{\mathcal{T}r(W)_{N,\gamma}}^{\text{mon}} \mathbb{Q}_{\mathcal{M}_{N,\gamma}^{\text{fr}}(Q)} \otimes \mathfrak{L}^{-N \cdot \sum_{i \in Q_0} \gamma_i} \right). \end{aligned}$$

As a very special case, letting Q be the quiver with one vertex, and no loops, and taking the dimension vector (1), we calculate

$$\begin{aligned} \mathbb{H}_c(\text{pt}/\mathbb{C}^*) &= \lim_{N \rightarrow \infty} (\mathbb{H}(\mathbb{C}\mathbb{P}^{N-1}, \mathbb{Q}) \otimes \mathfrak{L}^{-N}) \\ &= \bigoplus_{j \leq -1} \mathfrak{L}^j. \end{aligned}$$

It follows [13, Prop.4.4] from the properness of the maps $q_{N,\gamma}$ that there is an isomorphism

$$(26) \quad \mathbb{H}_c^j(\mathfrak{M}_\gamma(Q), \phi_{\mathcal{T}r(W)_\gamma}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_\gamma(Q)}) \cong \mathbb{H}_c^j \left(\mathcal{M}_\gamma, \mathcal{H} \left(p! \phi_{\mathcal{T}r(W)}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_\gamma} \right) \right).$$

By the cohomological integrality theorem [13, Thm.A] there is an isomorphism of monodromic mixed Hodge modules

$$\bigoplus_{\gamma \in \mathbb{N}^{Q_0}} \mathcal{H} \left(p! \phi_{\mathcal{T}r(W)}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_\gamma} \right) \otimes \mathfrak{L}^{(\gamma,\gamma)/2} \cong \text{Sym}_{\boxplus} \left(\mathcal{BPS}_{Q,W,\gamma} \otimes \mathbb{H}_c(\text{pt}/\mathbb{C}^*) \otimes \mathfrak{L}^{1/2} \right)$$

where

$$\mathcal{BPS}_{Q,W,\gamma} := \begin{cases} \phi_{\mathcal{T}r(W)_\gamma}^{\text{mon}} \overline{\mathcal{IC}}_{\mathcal{M}_\gamma(Q)}(\mathbb{Q}) \otimes \mathfrak{L}^{-\dim(\mathcal{M}_\gamma(Q))/2} & \text{if } \mathcal{M}_\gamma^{\text{st}}(Q) \neq \emptyset \\ 0 & \text{otherwise} \end{cases}$$

where $\overline{\mathcal{IC}}_{\mathcal{M}_\gamma(Q)}(\mathbb{Q})$ is (up to shifting cohomological degree down by $\dim(\mathcal{M}_\gamma(Q))$) the intersection cohomology mixed Hodge module on $\mathcal{M}_\gamma(Q)$ obtained by taking the intermediate extension of the constant mixed Hodge module on $\mathcal{M}_\gamma^{\text{st}}(Q)$. It then follows from (26) that

$$\bigoplus_{\gamma \in \mathbb{N}^{Q_0}} \mathbb{H}_c^j(\mathfrak{M}_\gamma(Q), \phi_{\mathcal{T}r(W)_\gamma}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_\gamma(Q)}) \otimes \mathfrak{L}^{(\gamma,\gamma)/2} \cong \text{Sym} \left(\mathcal{BPS}_{Q,W,\gamma} \otimes \mathbb{H}_c(\text{pt}/\mathbb{C}^*) \otimes \mathfrak{L}^{1/2} \right)$$

where $\mathcal{BPS}_{Q,W,\gamma} := \mathbb{H}_c(\mathcal{M}_\gamma(Q), \mathcal{BPS}_{Q,W,\gamma})$.

Finally, for the connection to the preceding sections of the paper, there is a ring homomorphism

$$\begin{aligned} \chi_{\text{MMHS}} : \hat{\mathbf{K}}_0^{\text{fr}}(\text{Var}/\text{pt}) &\rightarrow \mathbf{K}_0(\text{MMHS}) \\ [X] &\mapsto -[(X \times_{\mu_d} \mathbb{G}_m \xrightarrow{(x,t) \mapsto t^d} \mathbb{A}_{\mathbb{C}}^1)! \mathbb{Q}_{X \times_{\mu_d} \mathbb{G}_m}] \end{aligned}$$

taking the motivic DT invariants to the Hodge theoretic DT invariants, and we have

$$\chi_{\text{MMHS}}(\Omega_{Q,W,\gamma}) = [\mathcal{BPS}_{Q,W,\gamma}]_{\mathbf{K}_0}.$$

See [7, Sec.2.7] for details.

6. COHOMOLOGICAL DT INVARIANTS FOR THE DEFORMED WEYL POTENTIAL

Now set Q to be the three loop quiver, with potential $W_d = [a, b]c + c^d$ as in (18). In this section we describe the BPS sheaves $\mathcal{BPS}_{Q, W_d, n} \subset \text{MMHM}(\mathcal{M}_n(Q))$ along with the BPS cohomology $\text{BPS}_{Q, W_d, n}$ defined in the last section. We follow the strategy of [6]; we prove that the monodromic mixed Hodge modules $\mathcal{BPS}_{Q, W_d, n}$ are pure, have very restricted support, and are moreover constant on their support. The result then follows from the motivic calculations, specifically Theorem 4.3.

Lemma 6.1. *Let ρ be a representation of $\mathbb{C}(Q, W_d)$. Then each of the operators $\rho(a), \rho(b), \rho(c)$ preserve the generalized eigenspaces of each of the others. Moreover, the only nontrivial generalized eigenspace for $\rho(c)$ is for the generalized eigenvalue zero.*

Proof. The operator $\rho(c)$ is nilpotent, following the proof of [4, Lem.3.7+Lem.3.9], and so $\rho(c)$ has only one generalized eigenspace, which is trivially preserved by $\rho(a)$ and $\rho(b)$. The Jacobi relations include the relations

$$\begin{aligned}\partial W_d / \partial a &= [b, c] \\ \partial W_d / \partial b &= [c, a]\end{aligned}$$

and so it follows that $\rho(c)$ preserves the generalized eigenspaces of $\rho(a)$ and $\rho(b)$ as well. Let v be a generalized eigenvector of $\rho(b)$, with generalised eigenvalue λ . Define $\beta = (\rho(b) - \lambda \cdot)$. Since $[\rho(a), \beta] = d\rho(c)^{d-1}$, it follows that $[\rho(a), \beta]$ commutes with β and so for $m \gg 0$

$$\begin{aligned}\beta^m \rho(a)v &= \rho(a)\beta^m v + m[\beta, \rho(a)]\beta^{m-1}v \\ &= 0\end{aligned}$$

and so $\rho(a)v$ has generalised eigenvalue λ for the operator $\rho(b)$. The same argument, swapping $\rho(a)$ and $\rho(b)$, shows that $\rho(b)$ preserves the generalized eigenspaces of $\rho(a)$. □

Corollary 6.2. *Every finite-dimensional $\mathbb{C}(Q, W_d)$ -module ρ admits a canonical decomposition into nonzero $\mathbb{C}(Q, W_d)$ -modules*

$$\rho \cong \bigoplus_{s \in S} \rho_s$$

where $S \subset \mathbb{C}^2$ is a finite subset, and for $s = (s_1, s_2)$ the generalized eigenvalues of $\rho(a), \rho(b)$ and $\rho(c)$ restricted to ρ_s are given by s_1, s_2 and 0 respectively.

Definition 6.3. *For ρ a $\mathbb{C}(Q, W_d)$ -module, we call the set S in Corollary 6.2 the set of generalized (a, b) -eigenvalues of ρ .*

Lemma 6.4. *Let $\mathcal{M}_n^{\text{c-nilp}} \subset \mathcal{M}_n(Q)$ be the closed subvariety corresponding to those $\mathbb{C}Q$ -modules for which c acts via the zero map. Then $\text{supp}(\mathcal{BPS}_{Q, W_d, n}) \subset \mathcal{M}_n^{\text{c-nilp}}$.*

Proof. By definition, there is an inclusion

$$\mathcal{BPS}_{Q, W_d, n} \otimes \mathfrak{L}^{1/2} \hookrightarrow \mathcal{H} \left(p_n, !\Phi_{\mathfrak{Tr}(W_d)_n}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}_n(Q)} \otimes \mathfrak{L}^{(\gamma, \gamma)/2} \right),$$

and so $\text{supp}(\mathcal{BPS}_{Q, W_d, n}) \subset p_n(\text{crit } \mathfrak{Tr}(W_d)_n)$. In particular, for a $\mathbb{C}Q$ -module ρ corresponding to a point in $\text{supp}(\mathcal{BPS}_{Q, W_d, n})$, $\rho(c)$ acts nilpotently, and commutes with the action of $\rho(a)$ and $\rho(b)$. On the other hand, such modules are semisimple (as they correspond to points of $\mathcal{M}_n(Q)$), and so it follows that $\rho(c)$ acts via the zero map. □

Define

$$\begin{aligned} \Delta_n: \mathbb{A}_{\mathbb{C}}^2 &\hookrightarrow \mathcal{M}_n(Q) \\ (t_1, t_2) &\mapsto (t_1 \cdot \text{Id}_{n \times n}, t_2 \cdot \text{Id}_{n \times n}, 0) \end{aligned}$$

Lemma 6.5. *There is an inclusion $\text{supp}(\mathcal{BPS}_{W_d, n}) \subset \Delta_n(\mathbb{A}_{\mathbb{C}}^2)$, and furthermore, there exists a $\mathcal{G}_n \in \text{MMHS}$ such that*

$$\mathcal{BPS}_{W_d, n} \cong \Delta_{n, *}\mathcal{G}_n \otimes \mathcal{L}^{-1}.$$

The proof of this lemma is essentially the same as the proof of Lemma [6, Lem.4.1], we give an abridged version.

Proof. By Lemma 6.1, any representation ρ of $\mathbb{C}(Q, W_d)$ splits canonically as a direct sum of nonzero representations

$$(27) \quad \rho = \bigoplus_{s \in S} \rho_s$$

where $S \subset \mathbb{C}$ is a finite subset and the generalized eigenvalue of the operators $\rho(a)|_{\rho(\lambda_1, \lambda_2)}$ and $\rho(b)|_{\rho(\lambda_1, \lambda_2)}$ are λ_1 and λ_2 , respectively. If we assume moreover that ρ is semisimple, then $\rho(c) = 0$, since $\rho(a)$ and $\rho(b)$ preserve $\ker(\rho(c))$. It follows that $\rho(a)$ and $\rho(b)$ commute, and so since ρ is semisimple, $\rho(a)$ and $\rho(b)$ are simultaneously diagonalizable and

$$\text{supp} \left(\mathcal{H} \left(p! \Phi_{\text{Tr}(W_d)}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}(Q)} \right) \right) \subset \text{Sym} \left(\prod_{n \geq 1} \Delta_n(\mathbb{A}_{\mathbb{C}}^2) \right).$$

For an analytic open subset $U \subset \mathbb{A}_{\mathbb{C}}^2$ let $\mathcal{M}^U \subset \mathcal{M}(Q)$ be the open analytic subspace of semisimple representations ρ such that in the (minimal) decomposition (27), $S \subset U$. Let U_1 and U_2 be disjoint open analytic subsets of $\mathbb{A}_{\mathbb{C}}^2$. Then $p^{-1}\mathcal{M}^{U_1} \amalg U_2 = p^{-1}\mathcal{M}^{U_1} \times p^{-1}\mathcal{M}^{U_2}$, and via the Thom–Sebastiani isomorphism there is a natural isomorphism

$$\mathcal{H} \left(p! \Phi_{\text{Tr}(W_d)}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}(Q)} \right) |_{\mathcal{M}^{U_1} \amalg U_2} \cong \mathcal{H} \left(p! \Phi_{\text{Tr}(W_d)}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}(Q)} \right) |_{\mathcal{M}^{U_1}} \boxtimes_{\oplus} \mathcal{H} \left(p! \Phi_{\text{Tr}(W_d)}^{\text{mon}} \mathbb{Q}_{\mathfrak{M}(Q)} \right) |_{\mathcal{M}^{U_2}}$$

and so an isomorphism (via the cohomological integrality theorem)

$$\begin{aligned} &\text{Sym}_{\boxtimes_{\oplus}} \left(\bigoplus_{n \geq 1} \mathcal{BPS}_{Q, W_d, n} |_{\mathcal{M}^{U_1} \amalg U_2} \otimes \text{H}_c(\text{pt}/\mathbb{C}^{\times}) \otimes \mathcal{L}^{1/2} \right) \cong \\ &\text{Sym}_{\boxtimes_{\oplus}} \left(\bigoplus_{n \geq 1} \mathcal{BPS}_{Q, W_d, n} |_{\mathcal{M}^{U_1}} \otimes \text{H}_c(\text{pt}/\mathbb{C}^{\times}) \otimes \mathcal{L}^{1/2} \right) \boxtimes_{\oplus} \\ &\text{Sym}_{\boxtimes_{\oplus}} \left(\bigoplus_{n \geq 1} \mathcal{BPS}_{Q, W_d, n} |_{\mathcal{M}^{U_2}} \otimes \text{H}_c(\text{pt}/\mathbb{C}^{\times}) \otimes \mathcal{L}^{1/2} \right) \end{aligned}$$

from which it follows that $\mathcal{BPS}_{Q, W_d, n} |_{\mathcal{M}^{U_1} \amalg U_2} \cong \mathcal{BPS}_{Q, W_d, n} |_{\mathcal{M}^{U_1}} \oplus \mathcal{BPS}_{Q, W_d, n} |_{\mathcal{M}^{U_2}}$.

Unravelling this a little: if ρ is a semisimple module lying in the support of $\mathcal{BPS}_{Q, W_d, n}$, for which all of the generalized (a, b) -eigenvalues lie in $U_1 \amalg U_2$, then either all of the generalized (a, b) -eigenvalues lie in U_1 , or they all lie in U_2 . It follows that it is not possible to separate the generalized (a, b) -eigenvalues of any ρ lying in the support of $\mathcal{BPS}_{Q, W_d, n}$ into two open sets in \mathbb{C}^2 , and so in fact they must all be the same, i.e. the decomposition (27) can have only one summand, which is the part of the lemma regarding support.

For the second part of the lemma, let $\text{Mat}_{n \times n}^0(\mathbb{C}) \subset \text{Mat}_{n \times n}(\mathbb{C})$ denote the subspace of trace-free matrices, and set

$$\mathbb{X}_n^0 := \text{Mat}_{n \times n}^0(\mathbb{C})^2 \times \text{Mat}_{n \times n}(\mathbb{C}) \subset \mathbb{X}_n(Q).$$

There is a GL_n -equivariant isomorphism

$$\begin{aligned} \mathbb{A}_{\mathbb{C}}^2 \times \mathbb{X}_n^0 &\rightarrow \mathbb{X}_n(Q) \\ (t_1, t_2, A, B, C) &\mapsto (t_1 \cdot \text{Id}_{n \times n} + A, t_2 \cdot \text{Id}_{n \times n} + B, C). \end{aligned}$$

The GL_n -action on $\mathbb{A}_{\mathbb{C}}^2$ is trivial, and is the conjugation action on all of the other factors. It follows that

$$\mathcal{M}_n(Q) \cong \mathbb{A}_{\mathbb{C}}^2 \times \mathcal{M}_n^0$$

where

$$\mathcal{M}_n^0 := \text{Spec}(\Gamma(\mathbb{X}_n^0)^{\text{GL}_n}),$$

and

$$\begin{aligned} \overline{\text{IC}}_{\mathcal{M}_n(Q)}(\mathbb{Q}) &\cong_{\mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2}} \boxtimes \overline{\text{IC}}_{\mathcal{M}_n^0}(\mathbb{Q}) \\ \phi_{\mathcal{T}r(W_d)_n}^{\text{mon}} \text{IC}_{\mathcal{M}_n(Q)}(\mathbb{Q}) &\cong_{\mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2}} \boxtimes \left(\phi_{\mathcal{T}r(W_d)_n}^{\text{mon}} \text{IC}_{\mathcal{M}_n^0}(\mathbb{Q}) \right) \end{aligned}$$

where the second isomorphism follows from the fact that the function $\mathcal{T}r(W_d)_n \in \Gamma(\mathcal{M}_n(Q))$ factors through the projection to \mathbb{X}_n^0 . From the condition on the support of \mathcal{BPS}_{Q, W_d} it follows that $\left(\phi_{\mathcal{T}r(W_d)_n}^{\text{mon}} \text{IC}_{\mathcal{M}_n^0}(\mathbb{Q}) \right)$ is supported at the origin of $\mathbb{A}_{\mathbb{C}}^0$, and the second part of the lemma follows. \square

Lemma 6.6. *For all n , the monodromic mixed Hodge module $\mathcal{BPS}_{Q, W_d, n}$ is pure.*

Proof. From the proof of the previous lemma, it is enough to show that $\phi_{\text{Tr}(W_d)_n}^{\text{mon}} \overline{\text{IC}}_{\mathcal{M}_n^0}(\mathbb{Q})$ is pure, or (as this complex of monodromic mixed Hodge modules is supported at a point) that $\text{H}\left(\mathcal{M}_n^0, \phi_{\text{Tr}(W_d)_n}^{\text{mon}} \text{IC}_{\mathcal{M}_n^0}(\mathbb{Q})\right)$ is pure. For this we use the main geometric result of [9]: $\text{IC}_{\mathcal{M}_n^0}(\mathbb{Q})$ is a pure complex of mixed Hodge modules, and $\text{Tr}(W_d)_n: \mathbb{X}_n^0 \rightarrow \mathbb{C}$ is a \mathbb{C}^* -equivariant function, and the support of $\phi_{\text{Tr}(W_d)_n}^{\text{mon}} \text{IC}_{\mathcal{M}_n^0}(\mathbb{Q})$ is proper (since it is a point), so that the cohomology is a pure complex of monodromic mixed Hodge structures by [9, Thm.3.1]. \square

We can now prove our main theorem regarding the cohomological DT theory for the potential W_d . It provides a categorical refinement of the Cazzanigga–Morrison–Pym–Szendrői conjecture, and is a consequence of the motivic version of our main theorem regarding deformed dimensional reduction.

Theorem 6.7. *For all n there is an isomorphism in $\text{MMHM}(\mathcal{M}_n)$*

$$\mathcal{BPS}_{Q, W_d, n} \cong \Delta_{n, *} \mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2} \otimes \text{H}(\mathbb{A}_{\mathbb{C}}^1, \phi_{x^d}^{\text{mon}} \mathbb{Q}) \otimes \mathfrak{L}^{-3/2}$$

as well as an isomorphism in MMHS

$$(28) \quad \text{BPS}_{Q, W_d, n} \cong \text{H}(\mathbb{A}_{\mathbb{C}}^1, \phi_{x^d}^{\text{mon}} \mathbb{Q}) \otimes \mathfrak{L}^{1/2}.$$

Proof. The second claim follows from the first, and due to Lemma (6.5) is in fact equivalent to it. By Lemma 6.6, $\text{BPS}_{Q, W_d, n}$ is determined by its class in the Grothendieck group of monodromic mixed Hodge structures, which is equal to

$$\chi^{\text{MMHS}}([\mathbb{L}]^{1/2}[\mathbb{A}_{\mathbb{C}}^1 \xrightarrow{t \mapsto t^d} \mathbb{A}_{\mathbb{C}}^1]) = [\text{H}(\mathbb{A}_{\mathbb{C}}^1, \phi_{x^d}^{\text{mon}} \mathbb{Q}) \otimes \mathfrak{L}^{1/2}]_{\text{K}_0}$$

as required, by Theorem 4.3. \square

7. COHOMOLOGICAL DEFORMED DIMENSIONAL REDUCTION THEOREM

In this section we prove Theorem 1.2, so we assume that we have a function $g \in \Gamma(X \times \mathbb{A}_{\mathbb{C}}^n)$ and a decomposition $\mathbb{A}_{\mathbb{C}}^n = \mathbb{A}_{\mathbb{C}}^m \times \mathbb{A}_{\mathbb{C}}^{n-m}$ satisfying the \mathbb{G}_m -equivariance assumptions of Theorem 1.2.

We assume that $m = 1$, since the general case follows from this. Let $X' := X \times \mathbb{A}_{\mathbb{C}}^{n-1}$, and for $z \in \mathbb{C}$ we define

$$(X' \times \mathbb{A}_{\mathbb{C}}^1)_z := g^{-1}(z).$$

Decompose $g = g_0 + tg_1$ where t is the coordinate on $\mathbb{A}_{\mathbb{C}}^m = \mathbb{A}_{\mathbb{C}}^1$, and g_0 and g_1 are functions pulled back from X' . Let $Z \subset X'$ be the zero locus of g_1 , and let $W \subset Z$ be the zero locus of

$$h := g_0|_Z.$$

Further, define

$$(Z \times \mathbb{A}_{\mathbb{C}}^1)_z := h^{-1}(z) \subset Z \times \mathbb{A}_{\mathbb{C}}^1.$$

and consider the inclusions for $z \in \mathbb{C}$:

$$\begin{aligned} i: Z \times \mathbb{A}_{\mathbb{C}}^1 &\rightarrow X' \times \mathbb{A}_{\mathbb{C}}^1 \\ i_z: (Z \times \mathbb{A}_{\mathbb{C}}^1)_z &\rightarrow (X' \times \mathbb{A}_{\mathbb{C}}^1)_z \\ \iota_z: (Z \times \mathbb{A}_{\mathbb{C}}^1)_z &\rightarrow Z \times \mathbb{A}_{\mathbb{C}}^1 \\ \kappa_z: (X' \times \mathbb{A}_{\mathbb{C}}^1)_z &\rightarrow X' \times \mathbb{A}_{\mathbb{C}}^1. \end{aligned}$$

Let $\mathcal{F} \in \text{MHM}(X)$. The natural map $\pi^*\mathcal{F} \rightarrow i_*i^*\pi^*\mathcal{F}$ induces a map:

$$(29) \quad \pi_!\phi_g^{\text{mon}}\pi^*\mathcal{F} \rightarrow \pi_!i_*\phi_h^{\text{mon}}i^*\pi^*\mathcal{F}$$

of monodromic mixed Hodge modules. By faithfulness of the forgetful functor forg_X , it is sufficient to show that the morphism

$$(30) \quad \pi_!\varphi_g\pi^*\mathcal{F} \rightarrow \pi_!i_*\varphi_hi^*\pi^*\mathcal{F}$$

is an isomorphism, considered as a morphism in the derived category of constructible sheaves.

Taking duals, this is equivalent to showing that the following map of complexes of constructible sheaves is an isomorphism:

$$(31) \quad \pi_*i_*\varphi_hi^!\pi^*\mathcal{E} \rightarrow \pi_*\varphi_g\pi^*\mathcal{E},$$

where $\mathcal{E} = \mathbb{D}\mathcal{F}[2n]$.

We introduce the notations that will be used in this section. Consider the diagram of Cartesian squares

$$\begin{array}{ccccc} \widetilde{Z \times \mathbb{A}_{\mathbb{C}}^1} & \xrightarrow{\widetilde{i}} & \widetilde{X' \times \mathbb{A}_{\mathbb{C}}^1} & \longrightarrow & \mathbb{A}_{\mathbb{C}}^1 \\ \downarrow \widetilde{p} & & \downarrow p & & \downarrow \text{exp} \\ Z \times \mathbb{A}_{\mathbb{C}}^1 & \xrightarrow{i} & X' \times \mathbb{A}_{\mathbb{C}}^1 & \xrightarrow{g} & \mathbb{A}_{\mathbb{C}}^1 \end{array}$$

in which the rightmost square is the diagram used to define the vanishing cycles functor for the function g . Consider the commutative diagram:

$$\begin{array}{ccc}
(Z \times \mathbb{A}_{\mathbb{C}}^1)_0 & \xrightarrow{i_0} & (X' \times \mathbb{A}_{\mathbb{C}}^1)_0 \\
\downarrow \iota_0 & & \downarrow \kappa_0 \\
Z \times \mathbb{A}_{\mathbb{C}}^1 & \xrightarrow{i} & X' \times \mathbb{A}_{\mathbb{C}}^1 \\
\iota_1 \uparrow & & \kappa_1 \uparrow \\
(Z \times \mathbb{A}_{\mathbb{C}}^1)_1 & \xrightarrow{i_1} & (X' \times \mathbb{A}_{\mathbb{C}}^1)_1.
\end{array}$$

Before we start the proof of Theorem 1.2, we need some preliminary results. We say that a sheaf \mathcal{F} on a space with a \mathbb{C}^* -action is *locally constant on \mathbb{C}^* -orbits* if for any \mathbb{C}^* -orbit O , the restriction $\mathcal{F}|_O$ is locally constant.

Proposition 7.1. *Let $Y = X \times \mathbb{A}_{\mathbb{C}}^n$ be a variety with a \mathbb{C}^* -action which fixes X and which acts with positive weights on $\mathbb{A}_{\mathbb{C}}^n$. Denote by $\pi : Y \rightarrow X$ the projection. Let $g : Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$ be a \mathbb{C}^* equivariant map, where \mathbb{C}^* acts with weight 1 on $\mathbb{A}_{\mathbb{C}}^1$. Denote by $\kappa : Y_0 \rightarrow Y$ the closed immersion of the zero locus of g , and by $j : Y \setminus Y_0 \rightarrow Y$ the open immersion of its complement. Let \mathcal{F} be a sheaf on $Y \setminus Y_0$ locally constant on \mathbb{C}^* -orbits. Then*

$$\pi_* j_! \mathcal{F} = 0.$$

Proof. Consider the blow-up $r : \text{Bl}_X Y \rightarrow Y$. There is an isomorphism $\text{Bl}_X Y \cong X \times \text{Tot}_{\mathbb{P}^{n-1}} \mathcal{O}(-1)$, and we denote by q the natural projection $\text{Bl}_X Y \rightarrow X \times \mathbb{P}^{n-1}$ onto the exceptional fiber. Consider the commutative diagram

$$\begin{array}{ccc}
& & Y \setminus Y_0 \\
& \swarrow \tilde{j} & \downarrow j \\
\text{Bl}_X Y & \xrightarrow{r} & Y \\
\downarrow q & & \downarrow \pi \\
W := X \times \mathbb{P}^{n-1} & \xrightarrow{\pi'} & X
\end{array}$$

Where \tilde{j} is the unique morphism through which j factors. We denote by $W_0 = (Y_0 - X)/\mathbb{C}^* \subset W$. There are natural isomorphisms

$$\pi_* j_! \simeq \pi_* r_* \tilde{j}_! \simeq \pi'_* q_* \tilde{j}_!$$

and so it is sufficient to prove that $q_* \tilde{j}_! \mathcal{F} = 0$. For this, let $w \in W$ and consider the cartesian diagram:

$$\begin{array}{ccc}
\mathbb{A}_{\mathbb{C}}^1 & \xrightarrow{a} & \text{Tot}_W(\mathcal{O}(-1)) \\
\downarrow q_w & & \downarrow q \\
w & \xrightarrow{a_0} & W.
\end{array}$$

The maps a and a_0 are the natural inclusions. Using smooth base change, we have that $a_* q_* \mathcal{F} = q_{w*} a_* \mathcal{F}$. We thus need to argue that $q_{w*} a_* \tilde{j}_! \mathcal{F} = 0$. When w is not in $W \setminus W_0$, we have that $a_* \tilde{j}_! = 0$. When $w \in W \setminus W_0$, we need to show $q_{w*} j_1^\circ a^\circ \mathcal{F} = 0$, where the maps a° and j_1° are defined via the diagram:

$$\begin{array}{ccc}
\mathbb{C}^* & \xrightarrow{j_1^\circ} & \mathbb{A}_{\mathbb{C}}^1 \\
\downarrow a^\circ & & \downarrow a \\
Y - Y_0 & \xrightarrow{\tilde{j}} & \text{Bl}_X Y = \text{Tot}_W \mathcal{O}(-1).
\end{array}$$

We thus need to show that $H(j_1^\circ a^{\circ*} \mathcal{F}) = 0$. This is true because $a^{\circ*} \mathcal{F}$ is a locally constant sheaf on $\mathbb{A}_{\mathbb{C}}^1 - 0$, and the result is a special case of [5, Lemma A.3, page 58]. \square

The above proposition allows us to compare the cohomology of sheaves locally constant on \mathbb{C}^* -orbits on $X \times \mathbb{A}_{\mathbb{C}}^n$ or $Z \times \mathbb{A}_{\mathbb{C}}^n$ with their restrictions onto the zero fiber.

Corollary 7.2. *Recall the framework introduced at the beginning of the section. The vertical arrows in the following diagrams are isomorphisms:*

$$\begin{array}{ccc} \pi_* \pi^* \mathcal{E} & \longrightarrow & \pi_* p_* p^* \pi^* \mathcal{E} \\ \downarrow \text{iso.} & & \downarrow \text{iso.} \\ \pi_* \kappa_{0*} \kappa_0^* \pi^* \mathcal{E} & \longrightarrow & \pi_* \kappa_{0*} \kappa_0^* p_* p^* \pi^* \mathcal{E}. \end{array}$$

and

$$\begin{array}{ccc} \pi_* i_* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* \tilde{p}_* \tilde{p}^* i^! \pi^* \mathcal{E} \\ \downarrow \text{iso.} & & \downarrow \text{iso.} \\ \pi_* i_* \iota_{0*} \iota_0^* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* \iota_{0*} \iota_0^* \tilde{p}_* \tilde{p}^* i^! \pi^* \mathcal{E}. \end{array}$$

Proof. The complexes of sheaves $\pi^* \mathcal{E}$, $p_* p^* \pi^* \mathcal{E}$, $i_0^! \pi^* \mathcal{E}$ and $\tilde{p}_* \tilde{p}^* i_0^! \pi^* \mathcal{E}$ are constant on \mathbb{C}^* orbits in $Y - Y_0$ and $(Z \times \mathbb{A}^1) \setminus (Z \times \mathbb{A}^1)_0$ respectively. Let

$$j: (X' \times \mathbb{A}^1) \setminus (X' \times \mathbb{A}^1)_0 \hookrightarrow (X' \times \mathbb{A}^1)$$

be the inclusion. Then by Proposition 7.1 there is an isomorphism $\pi_* j_! j^* \pi^* \mathcal{E} \cong 0$ and in the distinguished triangle

$$\pi_* j_! j^* \pi^* \mathcal{E} \rightarrow \pi_* \pi^* \mathcal{E} \rightarrow \pi_* \kappa_{0!} \kappa_0^* \pi^* \mathcal{E}$$

the second morphism is an isomorphism. The other three claims follow similarly. \square

The next proposition compares the nearby cohomology and the restriction to a nonzero fiber for a sheaf locally constant on \mathbb{C}^* -orbits.

Proposition 7.3. *Let $Y = X \times \mathbb{A}_{\mathbb{C}}^n$ be a smooth variety with a \mathbb{C}^* action which fixes X and acts with positive weights on $\mathbb{A}_{\mathbb{C}}^n$. Denote by $\pi: Y \rightarrow X$ the projection. Consider a \mathbb{C}^* -equivariant function $g: Y \rightarrow \mathbb{A}_{\mathbb{C}}^1$, where the \mathbb{C}^* action on $\mathbb{A}_{\mathbb{C}}^1$ has weight 1. Denote by $\kappa_1: Y_1 := g^{-1}(1) \rightarrow Y$ the inclusion of the fiber over 1. Let \mathcal{F} be a sheaf on $Y \setminus Y_0$ locally constant on \mathbb{C}^* -orbits. There exists a natural map $p_* p^* \mathcal{F} \rightarrow \kappa_{1*} \kappa_1^* \mathcal{F}$ which induces an isomorphism:*

$$\pi_* p_* p^* \mathcal{F} \rightarrow \pi_* \kappa_{1*} \kappa_1^* \mathcal{F}.$$

Proof. Consider the diagram

$$\begin{array}{ccccc} \tilde{Y}' & & & & \\ & \searrow \tilde{i}' & & & \\ & & \tilde{Y}_1 & \xrightarrow{\tilde{\kappa}_1} & \tilde{Y} & \xrightarrow{a} & \mathbb{A}^1 \\ & \searrow k & \downarrow \tilde{p}_1 & & \downarrow p & & \downarrow \text{exp} \\ & & Y_1 & \xrightarrow{\kappa_1} & Y & \xrightarrow{g} & \mathbb{A}^1 \\ & \searrow \tilde{p}' & & & & & \\ & & & & & & \end{array}$$

Here \tilde{Y} and \tilde{Y}_1 are defined such that the two squares are cartesian. Further, \tilde{Y}' is the fiber over zero of the map $a: \tilde{Y} \rightarrow \mathbb{C}$, so the map $\tilde{p}': \tilde{Y}' \rightarrow Y_1$ is an isomorphism.

The squares are cartesian, uniquely determining k . Note that $\tilde{p}_1^! = \tilde{p}_1^*$. We consider the composition α of morphisms

$$p_* p^* \mathcal{F} \rightarrow p_* \tilde{i}'_* \tilde{i}'^* p^* \mathcal{F} \cong \kappa_{1*} \tilde{p}'_* \tilde{p}'^* \kappa_1^* \mathcal{F} \cong \kappa_{1*} \kappa_1^* \mathcal{F}$$

The map

$$m : \tilde{Y}' \times \mathbb{A}^1 \rightarrow \tilde{Y}$$

defined by $m(y, z) = (e^{z/d}y, z) \in Y \times_{\mathbb{A}^1} \mathbb{A}^1$ is an isomorphism. In the following diagram, for which the sub-diagram of uncurved arrows is commutative, we use m to identify $\tilde{Y}' \times \mathbb{A}^1$ and \tilde{Y} . Then \tilde{i}' is the inclusion of the zero fiber of the trivial \mathbb{A}^1 bundle, while we define ϖ to be the projection onto the \tilde{Y}' factor:

$$\begin{array}{ccc} \tilde{Y}' & \xleftarrow{\varpi} & \tilde{Y}' \times \mathbb{A}^1 \\ \downarrow t & \tilde{i}' & \downarrow p \\ X & \xleftarrow{\pi} & Y \end{array}$$

Here t is defined by $t = \pi \kappa_1 \tilde{p}'$. Since ϖ is a projection with contractible fibers, the natural transformation

$$\varpi_*(\text{id} \rightarrow \tilde{i}'_* \tilde{i}'^*) \varpi^*$$

is an isomorphism η . Using this notation, we need to show that the following natural map is an isomorphism:

$$t_* \varpi_* p^* \mathcal{F} \rightarrow t_* \varpi_* \tilde{i}'_* \tilde{i}'^* p^* \mathcal{F}.$$

Since we assume that \mathcal{F} is locally constant on \mathbb{C}^* orbits, there is a sheaf \mathcal{E} on \tilde{Y}' such that $p^* \mathcal{F} \cong \varpi^* \mathcal{E}$. In the commutative diagram

$$\begin{array}{ccc} t_* \varpi_* p^* \mathcal{F} & \longrightarrow & t_* \varpi_* \tilde{i}'_* \tilde{i}'^* p^* \mathcal{F} \\ \downarrow \cong & & \downarrow \cong \\ t_* \varpi_* \varpi^* \mathcal{E} & \xrightarrow{\cong} & t_* \varpi_* \tilde{i}'_* \tilde{i}'^* \varpi^* \mathcal{E} \end{array}$$

the top horizontal morphism is an isomorphism, as the other three are. \square

Corollary 7.4. *The following diagram commutes, where the horizontal arrows are isomorphisms:*

$$\begin{array}{ccccc} \pi_* i_* \iota_{1*} \iota_1^! i^! \pi^* \mathcal{E} & \xleftarrow{\text{iso.}} & \pi_* i_* \tilde{p}'_* \tilde{p}'^* i^! \pi^* \mathcal{E} & \xrightarrow{\text{iso.}} & \pi_* i_* \psi_h i^! \pi^* \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow \\ \pi_* \kappa_{1*} \kappa_1^* \pi^* \mathcal{E} & \xleftarrow{\text{iso.}} & \pi_* p_* p^* \pi^* \mathcal{E} & \xrightarrow{\text{iso.}} & \pi_* \psi_g \pi^* \mathcal{E}. \end{array}$$

Proof. The left square clearly commutes. Its horizontal arrows are isomorphism by Proposition 7.3 because the sheaves $\pi^* \mathcal{E}$ on $X \times \mathbb{A}_{\mathbb{C}}^n$ and $i^! \pi^* \mathcal{E}$ on $Z \times \mathbb{A}_{\mathbb{C}}^n$ are constant on \mathbb{C}^* -orbits.

For the right square, observe that, by the definition of nearby cycles:

$$\begin{aligned} \pi_* i_* \psi_h i^! \pi^* \mathcal{E} &= \pi_* i_* \iota_{0*} \iota_0^* \tilde{p}'_* \tilde{p}'^* i^! \pi^* \mathcal{E} \\ \pi_* \psi_g \pi^* \mathcal{E} &= \pi_* \kappa_{0*} \kappa_0^* p_* p^* \pi^* \mathcal{E}. \end{aligned}$$

We can thus rewrite the right square as follows:

$$\begin{array}{ccc} \pi_* i_* \tilde{p}_* \tilde{p}^* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* \iota_{0*} \iota_0^* \tilde{p}_* \tilde{p}^* i^! \pi^* \mathcal{E} \\ \downarrow & & \downarrow \\ \pi_* p_* p^* \pi^* \mathcal{E} & \longrightarrow & \pi_* \kappa_{0*} \kappa_0^* p_* p^* \pi^* \mathcal{E}. \end{array}$$

The square clearly commutes and its horizontal maps are isomorphisms by Corollary 7.2. \square

Lemma 7.5. *Let Y be a variety with a \mathbb{C}^* action, and let $g : Y \rightarrow \mathbb{A}^1$ be a homogeneous regular function. Denote by $\kappa_1 : Y_1 \rightarrow Y$ be the inclusion of the fiber over 1. Consider a sheaf \mathcal{F} locally constant on \mathbb{C}^* -orbits. Then*

$$\kappa_1^* \mathcal{F} = \kappa_1^! \mathcal{F}[2].$$

Proof. We use the commutative diagram from the proof of Proposition 7.3. Since \tilde{p}' is an isomorphism, the lemma follows from the claim that

$$(i_1 \tilde{p}')^* \mathcal{F} \cong (i_1 \tilde{p}')^! \mathcal{F}[2].$$

Since p is locally a homeomorphism, we have $p^! \mathcal{F} \cong p^* \mathcal{F}$. Recall the maps

$$\begin{array}{ccc} \tilde{Y}' & \xleftarrow{\varpi} & \tilde{Y}' \times \mathbb{A}^1 \\ & \xrightarrow{\tilde{i}'} & \end{array}$$

from Proposition 7.3. The sheaf \mathcal{F} is locally constant on \mathbb{C}^* -orbits, so there exists a sheaf \mathcal{E} on \tilde{Y}' such that $\mathcal{F} = \varpi^* \mathcal{E}$. The desired isomorphism follows now from $\tilde{i}'^* \varpi^* = \tilde{i}'^! \varpi^*[2]$. \square

PROOF OF THEOREM 1.2. Using Corollaries 7.2 and 7.4, the vanishing cycles fit in distinguished triangles:

$$(32) \quad \begin{array}{ccccc} \pi_* i_* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* \iota_{1*} \iota_1^* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* \varphi_h i^! \pi^* \mathcal{E} \\ \downarrow & & \downarrow & & \downarrow (A) \\ \pi_* \pi^* \mathcal{E} & \longrightarrow & \pi_* \kappa_{1*} \kappa_1^* \pi^* \mathcal{E} & \longrightarrow & \pi_* \varphi_g \pi^* \mathcal{E}. \end{array}$$

We need to show that the map (A) is an isomorphism. Consider the diagram:

$$\begin{array}{ccccc} (Z \times \mathbb{A}^1)_1 & \xrightarrow{i_1} & (X' \times \mathbb{A}^1)_1 & \xleftarrow{j_1} & (U \times \mathbb{A}^1)_1 \\ \downarrow \iota_1 & & \downarrow \kappa_1 & & \downarrow l_1 \\ Z \times \mathbb{A}^1 & \xrightarrow{i} & X' \times \mathbb{A}^1 & \xleftarrow{j} & U \times \mathbb{A}^1. \end{array}$$

Consider the distinguished triangle

$$i_{1*} i_1^! \rightarrow \text{id} \rightarrow j_{1*} j_1^* \xrightarrow{[1]}$$

applied to the sheaf $\kappa_1^* \pi^* \mathcal{E}[2]$ on $X' \times \mathbb{A}^1$:

$$(33) \quad i_{1*} i_1^! \kappa_1^* \pi^* \mathcal{E} \rightarrow \kappa_1^* \pi^* \mathcal{E} \rightarrow j_{1*} j_1^* \kappa_1^* \pi^* \mathcal{E} \xrightarrow{[1]} .$$

Lemma 7.5 implies that:

$$i_{1*} i_1^! \kappa_1^* \pi^* \mathcal{E} = i_{1*} \iota_1^* \kappa_1^* \pi^* \mathcal{E}[2] = i_{1*} \iota_1^* i^* \pi^* \mathcal{E}[2] = i_{1*} \iota_1^* i^! \pi^* \mathcal{E}.$$

The triangle (33) thus becomes:

$$(34) \quad i_{1*} \iota_1^* i^! \pi^* \mathcal{E} \rightarrow \kappa_1^* \pi^* \mathcal{E} \rightarrow j_{1*} j_1^* \kappa_1^* \pi^* \mathcal{E} \xrightarrow{[1]} .$$

Putting together the diagram (32), the triangle (34), and the equality $j_* \iota_{1*} \iota_1^* j^* = \kappa_{1*} j_{1*} j_1^* \kappa_1^*$, we obtain the commutative diagram:

$$\begin{array}{ccccc}
\pi_* i_* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* l_{1*} l_1^* i^! \pi^* \mathcal{E} & \longrightarrow & \pi_* i_* \varphi_h i^! \pi^* \mathcal{E} \\
\downarrow & & \downarrow & & \downarrow (A) \\
\pi_* \pi^* \mathcal{E} & \longrightarrow & \pi_* \kappa_{1*} \kappa_1^* \pi^* \mathcal{E} & \longrightarrow & \pi_* \varphi_g \pi^* \mathcal{E} \\
\downarrow & & \downarrow & & \\
\pi_* j_* j^* \pi^* E & \xrightarrow{(B)} & \pi_* j_* l_{1*} l_1^* j^* \pi^* E & &
\end{array}$$

The map (A) is an isomorphism if and only if the map (B) is an isomorphism, so it is enough to show that (B) is an isomorphism. Let $\tilde{\pi} : U \times \mathbb{A}^1 \rightarrow U$. It is enough to show the isomorphism:

$$\tilde{\pi}_* j^* \pi^* \mathcal{E} \cong \tilde{\pi}_* l_{1*} l_1^* j^* \pi^* \mathcal{E}.$$

For a point $u \in U$, let $i_u : \mathbb{A}^1 \rightarrow U \times \mathbb{A}^1$ be the inclusion of the fiber over u . We need to verify:

$$H(\mathbb{A}^1, i_u^* j^* \pi^* \mathcal{E}) \cong H(\text{pt}, i_{\text{pt}}^* j^* \pi^* \mathcal{E}),$$

which is true because the sheaf $j^* \pi^* \mathcal{E}$ is constant along the fiber \mathbb{A}^1 . \square

As in the case of undeformed cohomological dimensional reduction, we can easily generalize to stacks, i.e. the following corollary is a generalization of [5, Cor.A.9]

Corollary 7.6. *Let G be an algebraic group, and let X be a G -equivariant variety, with $\tilde{X} = X \times \mathbb{A}_{\mathbb{C}}^n$ the total space of a G -equivariant bundle over X . Let $\bar{g} \in \Gamma(\tilde{X}/G)$, and let $g \in \Gamma(\tilde{X}/G)$ be the induced function on the stack. Assume in addition the \mathbb{G}_m -equivariance assumptions of Theorem 1.2. Let $X' \subset X$ be a G -invariant subvariety, then there is a natural isomorphism of cohomologically graded monodromic mixed Hodge structures*

$$\mathrm{H}_c(\tilde{X}'/G, \phi_g^{\mathrm{mon}} \mathbb{Q}_{\tilde{X}/G}) \cong \mathrm{H}_c((\tilde{Z} \cap \tilde{X}')/G, \phi_{g^{\mathrm{red}}}^{\mathrm{mon}} \mathbb{Q}_{\tilde{Z}/G}).$$

Proof. Using the notation in Section 5 and formula (25), the above statement follows from the natural isomorphism:

$$\mathrm{H}_c^j(\tilde{X}' \times_G \mathrm{Fr}(n, N), \phi_g^{\mathrm{mon}} \mathbb{Q} \otimes \mathfrak{L}^{-nN}) \cong \mathrm{H}_c^j((\tilde{Z} \cap \tilde{X}') \times_G \mathrm{Fr}(n, N), \phi_{g^{\mathrm{red}}}^{\mathrm{mon}} \mathbb{Q} \otimes \mathfrak{L}^{-nN}),$$

which is the claim (7) of Theorem 1.2 applied to the regular function induced by g :

$$\bar{g} : \tilde{X} \times_G \mathrm{Fr}(n, N) = X \times_G \mathrm{Fr}(n, N) \rightarrow \mathbb{A}_{\mathbb{C}}^1.$$

\square

Set Q to be the three loop quiver, with loops labelled a, b, c , and let Q' be the quiver obtained by removing the loop a . We define W_d as in (18).

As a corollary of the deformed dimensional reduction theorem, we compute the cohomological BPS invariants of $\mathbb{C}(Q, W_d)$, giving an alternative proof of the existence of the isomorphism (28).

Corollary 7.7. *The cohomological DT invariants for (Q, W_d) and dimension $n \geq 1$ are*

$$\mathrm{BPS}_{Q, W_d, n} \cong \mathrm{H}(\mathbb{A}_{\mathbb{C}}^1, \phi_{x^d}^{\mathrm{mon}} \mathbb{Q}) \otimes \mathfrak{L}^{1/2}.$$

Proof. Consider the affinization maps

$$\begin{aligned}
p_n : \mathfrak{M}_n(Q) &\rightarrow \mathcal{M}_n(Q) \\
p'_n : \mathfrak{M}_n(Q') &\rightarrow \mathcal{M}_n(Q')
\end{aligned}$$

from the moduli stacks to the coarse moduli spaces. Let $h = \mathfrak{Tr}(c^d) \in \Gamma(\mathfrak{C}_n)$. Corollary 7.6 gives an isomorphism

$$\mathrm{H}_c \left(\mathfrak{M}_n(Q), \phi_{\mathfrak{Tr}(W_d)}^{\mathrm{mon}} \mathbb{Q} \right) \otimes \mathfrak{L}^{-n^2} \cong \mathrm{H}_c(\mathfrak{C}_n, \phi_h^{\mathrm{mon}} \mathbb{Q}).$$

We fix a cohomological degree i , and a number $N \gg 0$ depending on i . Let $\mathrm{Fr}(n, N)$ be the space of n -tuples of linearly independent vectors in \mathbb{C}^N . Let

$$\begin{aligned} \mathcal{M} &:= \mathbb{X}_n(Q) \times_{\mathrm{GL}_n} \mathrm{Fr}(n, N) \\ \mathcal{M}' &:= \mathbb{X}_n(Q') \times_{\mathrm{GL}_n} \mathrm{Fr}(n, N) \\ \mathcal{C} &:= \mathbf{C}_n \times_{\mathrm{GL}_n} \mathrm{Fr}(n, N) \end{aligned}$$

and let $\bar{h} \in \Gamma(\mathcal{M}')$ be the function induced by $\mathrm{Tr}(c^d)$. We have natural maps $\bar{\pi}$, \bar{p} , \bar{p}' , and ϖ as follows:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\bar{\pi}} & \mathcal{M}' \\ \downarrow \bar{p} & & \downarrow \bar{p}' \\ \mathcal{M}_n(Q) & \xrightarrow{\varpi} & \mathcal{M}_n(Q'). \end{array}$$

Then there are isomorphisms:

$$\begin{aligned} \mathrm{H}_c^i(\mathfrak{C}_n, \phi_h^{\mathrm{mon}} \mathbb{Q}) &\stackrel{(0)}{\cong} \mathrm{H}_c^i(\mathcal{C}, \phi_{\bar{h}}^{\mathrm{mon}} \mathbb{Q}_{\mathcal{C}} \otimes \mathfrak{L}^{-nN}) \\ &\cong \mathrm{H}_c^i(\mathcal{M}', \phi_{\bar{h}}^{\mathrm{mon}} \mathbb{Q}_{\mathcal{C}} \otimes \mathfrak{L}^{-nN}) \\ &\stackrel{(1)}{\cong} \mathrm{H}_c^i(\mathcal{M}_n(Q'), \phi_{\mathfrak{Tr}(c^d)}^{\mathrm{mon}} \bar{p}'! \mathbb{Q}_{\mathcal{C}} \otimes \mathfrak{L}^{-nN}) \\ &\stackrel{(2)}{\cong} \mathrm{H}_c^i(\mathcal{M}_n(Q'), \phi_{\mathfrak{Tr}(c^d)}^{\mathrm{mon}} \bar{p}'! \bar{\pi}_! \phi_{\alpha}^{\mathrm{mon}} \mathbb{Q}_{\mathcal{M}} \otimes \mathfrak{L}^{-nN-n^2}) \\ &\cong \mathrm{H}_c^i(\mathcal{M}_n(Q'), \phi_{\mathfrak{Tr}(c^d)}^{\mathrm{mon}} \varpi! \bar{p}_! \phi_{\alpha}^{\mathrm{mon}} \mathbb{Q}_{\mathcal{M}} \otimes \mathfrak{L}^{-nN-n^2}) \\ &\stackrel{(3)}{\cong} \mathrm{H}_c^i(\mathcal{M}_n(Q'), \phi_{\mathfrak{Tr}(c^d)}^{\mathrm{mon}} \varpi! \mathrm{Sym}(\Delta_{n,*} \mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2} \otimes \mathfrak{L}^{-3/2} \otimes \mathrm{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}) |_{\mathcal{M}_n(Q')}) \\ &\cong \mathrm{H}_c^i(\mathcal{M}_n(Q'), \phi_{\mathfrak{Tr}(c^d)}^{\mathrm{mon}} \mathrm{Sym}(\Delta'_{n,*} \mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2} \otimes \mathfrak{L}^{-1/2} \otimes \mathrm{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}) |_{\mathcal{M}_n(Q')}) \\ &\stackrel{(4)}{\cong} \mathrm{H}_c^i(\mathcal{M}_n(Q'), \mathrm{Sym}(\Delta'_{n,*} \phi_{ny^d}^{\mathrm{mon}} \mathbb{Q}_{\mathbb{A}_{\mathbb{C}}^2} \otimes \mathfrak{L}^{-1/2} \otimes \mathrm{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}}) |_{\mathcal{M}_n(Q')}) \end{aligned}$$

Isomorphism (0) follows as in [13, Sec.2.2] from the fact that, up to removing a very high codimension substack, \mathcal{C} is a nN -dimensional affine fibration over \mathfrak{C}_n . Isomorphism (1) follows from the fact that p_n is approximated by proper maps (and so commutes with vanishing cycles functors [13, Prop.4.3]). Isomorphism (2) follows from usual cohomological dimensional reduction [5, Thm.A.1]. Isomorphism (3) follows from the calculation of the BPS sheaves for the quiver with potential (Q, W) in [6], while Isomorphism (4) comes from commutativity of vanishing cycles functors with Sym [13, Prop.3.11].

We deduce that

$$\begin{aligned} \bigoplus_{n \geq 0} \mathrm{H}_c \left(\mathfrak{M}_n(Q), \phi_{\mathfrak{Tr}(W_d)}^{\mathrm{mon}} \mathbb{Q} \right) \otimes \mathfrak{L}^{-n^2} &\cong \bigoplus_{n \geq 0} \mathrm{H}_c(\mathfrak{C}_n, \phi_h^{\mathrm{mon}} \mathbb{Q}) \\ &\cong \mathrm{Sym} \left(\bigoplus_{n \geq 1} \mathrm{H}_c(\mathbb{A}_{\mathbb{C}}^2, \phi_{y^d}^{\mathrm{mon}} \mathbb{Q}) \otimes \mathfrak{L}^{1/2} \otimes \mathrm{H}_c(\mathrm{pt}/\mathbb{C}^*)_{\mathrm{vir}} \right) \end{aligned}$$

as required. \square

8. K-THEORETIC DEFORMED DIMENSIONAL REDUCTION

For a stack \mathcal{X} , denote by $D^b(\mathcal{X})$ the derived category of bounded complexes of sheaves on \mathcal{X} and by $\text{Perf}(\mathcal{X}) \subset D^b(\mathcal{X})$ its full subcategory of perfect complexes. Denote by $G_0(\mathcal{X})$ and $K_0(\mathcal{X})$ the Grothendieck groups of $D^b(\mathcal{X})$ and $\text{Perf}(\mathcal{X})$, respectively.

8.1. K-theoretic DT invariants. We recall the construction of the K-theoretic Hall algebra and the PBW theorem for KHAs from [29]. Let (Q, W) be a quiver with potential. For a dimension vector $d \in \mathbb{N}^I$, let $\mathcal{X}(d)$ be the stack of dimension d representations of Q . The potential W determines a regular function:

$$\text{Tr}(W) : \mathcal{X}(d) \rightarrow \mathbb{A}^1.$$

We assume that the only singular value is 0, and denote by $\mathcal{X}(d)_0$ the fiber over 0. The category of singularities $D_{sg}(\mathcal{X}(d)_0)$ is defined as the triangulated quotient:

$$D_{sg}(\mathcal{X}(d)_0) = D^b(\mathcal{X}(d)_0)/\text{Perf}(\mathcal{X}(d)_0).$$

The K-theoretic Hall algebra of (Q, W) has underlying \mathbb{N}^I -graded vector space

$$\text{KHA}(Q, W) = \bigoplus_{d \in \mathbb{N}^I} K_0(D_{sg}(\mathcal{X}(d)_0)),$$

and the multiplication $m = p_*q^*$ is defined via the maps of attracting loci:

$$\mathcal{X}(d) \times \mathcal{X}(e) \xleftarrow{q_{d,e}} \mathcal{X}(d, e) \xrightarrow{p_{d,e}} \mathcal{X}(d+e).$$

The PBW theorem in [29] says that one can define subcategories $\mathbb{M}(d) \subset D_{sg}(\mathcal{X}(d)_0)$ and a filtration F^\cdot on $\text{KHA}(Q, W)$ such that there is an isomorphism of algebras:

$$\text{gr}^F \text{KHA}(Q, W) = \text{dSym} \left(\bigoplus_{d \in \mathbb{N}^I} K_0(\mathbb{M}(d)) \right),$$

where the right hand side is a deformation of the symmetric algebra. In analogy with the cohomological BPS invariants, we call the vector spaces $K_0(\mathbb{M}(d))$ the *K-theoretic BPS invariants of (Q, W)* .

One can define a smaller version of KHA which can be compared with CoHA via the Chern character. There is a Chern character, see [29, Section 5]:

$$(35) \quad \text{ch} : K_0(D_{sg}(\mathcal{X}(d)_0)) \rightarrow H(\mathcal{X}(d)_0, \varphi_{\text{Tr } W} \mathbb{Q})$$

with image in the monodromy fixed part $H(\mathcal{X}(d)_0, \varphi_{\text{Tr } W} \mathbb{Q})^{T-1}$. The algebra $\text{gr KHA}(Q, W)$ is the associated graded of KHA with respect to the cohomological filtration on CoHA. The PBW theorem [29, Section 8] for gr KHA says that there are categories $\mathbb{M}(d)_0 \subset \mathbb{M}(d)$ and a filtration E^\cdot on gr KHA such that there is an isomorphism of algebras:

$$\text{gr}^E \text{gr KHA}(Q, W) \cong \text{Sym} \left(\bigoplus \text{gr } K_0(\mathbb{M}(d)_0)[[h]] \right).$$

For (Q, W_d) as in Subsection 1.2, the Chern character map (35) is zero, and thus $\text{gr KHA}(Q, W) = 0$.

8.2. K-theoretic dimensional reduction. The cohomological/ motivic dimensional reduction theorem has a categorical analogue due to Isik [22]. We recall its K-theoretic version [29, Theorem 2.9]:

Theorem 8.1. *Let $V \times \mathbb{A}^n$ be a representation of a reductive group G , and consider the quotient stack $\mathcal{X} = X \times \mathbb{A}^n/G$. Consider a regular function:*

$$g = t_1 f_1 + \cdots + t_n f_n : \mathcal{X} \rightarrow \mathbb{A}^1,$$

where t_1, \dots, t_n are the coordinates on \mathbb{A}^n and $f_1, \dots, f_n : X \rightarrow \mathbb{A}^1$ are G -equivariant regular functions. Denote by $\mathcal{X}_0 \subset \mathcal{X}$ be the zero locus of g , by $i : \mathcal{Z} \hookrightarrow \mathcal{X}_0$ the zero locus of f_1, \dots, f_n , and by \mathcal{K} the Koszul complex for f_1, \dots, f_n . The pushforward map i_* induces an isomorphism:

$$G_0(\mathcal{Z}) \cong G_0(\mathcal{K}) \xrightarrow{i_*} K_0(D_{sg}(\mathcal{X}_0)).$$

The Koszul complex of f_1, \dots, f_n is defined as follows. Consider the vector bundle $\mathcal{E} = \mathcal{O}_{\mathcal{X}}^n$ with the section $\sigma : \mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{E}$ given by (f_1, \dots, f_n) . It induces a map $\mathcal{E}^\vee = \mathcal{O}_{\mathcal{X}}^n \xrightarrow{d} \mathcal{O}_{\mathcal{X}}$ defined on sections by $d(s_1, \dots, s_n) = s_1 f_1 + \dots + s_n f_n$. The Koszul complex of f_1, \dots, f_n is a stack with ring of regular functions:

$$\Lambda \cdot \mathcal{E}^\vee[1] = \left(\dots \rightarrow \Lambda^2 \mathcal{E}^\vee \rightarrow \mathcal{E}^\vee \xrightarrow{d} \mathcal{O}_{\mathcal{X}} \right).$$

The zeroth cohomology of $\Lambda \cdot \mathcal{E}^\vee[1]$ is $\mathcal{O}_{\mathcal{Z}}$, where $\mathcal{Z} \subset \mathcal{X}$ is the zero locus of $f_i = 0$ for $1 \leq i \leq n$.

The above dimensional reduction implies a weaker version of a deformed dimensional reduction. Before we state it, we recall the definition of relative categories of singularities. Consider a closed immersion $i : \mathcal{Z} \rightarrow \mathcal{X}$ of quotient stacks, where $\mathcal{O}_{\mathcal{Z}}$ has a finite resolution by flat $\mathcal{O}_{\mathcal{X}}$ -modules. Then the *relative category of singularities* [17] is defined as the quotient:

$$D_{sg}(\mathcal{Z}/\mathcal{X}) := D^b(\mathcal{Z}) / \langle \text{Im}(Li^* : D^b(\mathcal{X}) \rightarrow D^b(\mathcal{Z})) \rangle,$$

where $\langle - \rangle$ denotes the smallest thick subcategory containing the objects in $(-)$.

Proposition 8.2. *Let $V \times \mathbb{A}^n$ be a representation of a reductive group G , and let $\mathcal{X} = X \times \mathbb{A}^n / G$. Consider a regular function:*

$$g = f_0 + t_1 f_1 + \dots + t_n f_n : \mathcal{X} \rightarrow \mathbb{A}^1,$$

where t_1, \dots, t_n are the coordinates on \mathbb{A}^n and $f_0, \dots, f_n : X \rightarrow \mathbb{A}^1$ are G -equivariant regular functions. Denote by $\mathcal{X}_0 \subset \mathcal{X}$ be the zero locus of g , by $\mathcal{Z} \hookrightarrow \mathcal{X}$ be the zero locus of f_1, \dots, f_n , by \mathcal{K} the Koszul stack $\mathcal{O}_{\mathcal{Z}} \xrightarrow{f_0} \mathcal{O}_{\mathcal{Z}}$, and by $\iota : \mathcal{K} \rightarrow \mathcal{X}_0$ the natural closed immersion. Pushforward along ι induces a surjective map:

$$\iota_* : K_0(D_{sg}(\mathcal{K}/\mathcal{Z})) \rightarrow K_0(D_{sg}(\mathcal{X}_0)).$$

When f_1, \dots, f_n is a regular sequence and $f_0 : \mathcal{Z} \rightarrow \mathbb{A}^1$ is flat, Hirano [21] proved that the above map actually induces an equivalence of categories:

$$\iota_* : D_{sg}(\mathcal{K}/\mathcal{Z}) \cong D_{sg}(\mathcal{X}_0).$$

Proof. Let $\mathcal{Y} = V \times \mathbb{A}^1 / G$ and consider the regular function

$$h = t_0 f_0 + \dots + t_n f_n : \mathcal{Y} \rightarrow \mathbb{A}^1.$$

For a substack $\mathcal{W} \subset \mathcal{Y}$, denote by \mathcal{W}_0 the fiber over 0 of h and by \mathcal{W}' the open subset of \mathcal{W} with $t_0 \neq 0$. We abuse notation and use \mathcal{Z} and \mathcal{K} for substacks of \mathcal{Y} introduced in Proposition 8.2; they differ by a product with \mathbb{A}^1 from the ones above. Let $\iota : \mathcal{K} \rightarrow \mathcal{Y}_0$. Observe that $\mathcal{Y}'_0 \cong \mathcal{X}_0 \times \mathbb{G}_m$. Let $j : \mathcal{Y}'_0 \subset \mathcal{Y}_0$. It is enough to show that the following map is surjective:

$$(36) \quad j^* \iota_* : K_0(D_{sg}(\mathcal{K}/\mathcal{Z})) \rightarrow K_0(D_{sg}(\mathcal{Y}'_0)).$$

We first explain that the map is well-defined. Let $i : \mathcal{K} \rightarrow \mathcal{Z}$. We need to show that

$$j^* \iota_* K_0(\langle \text{Im}(Li^* : D^b(\mathcal{Z}) \rightarrow D^b(\mathcal{K})) \rangle) = 0.$$

It suffices to show that:

$$(37) \quad j^* \iota_* i^* G_0(\mathcal{Z}) = 0.$$

Consider the commutative diagram, where the left square commutes by proper base change:

$$\begin{array}{ccccccc} G_0(\mathcal{K}) & \xrightarrow{\iota_*} & G_0(\mathcal{Y}_0) & \xrightarrow{j^*} & G_0(\mathcal{Y}'_0) & \twoheadrightarrow & K_0(D_{sg}(\mathcal{Y}'_0)) \\ i^* \uparrow & & \uparrow & & \uparrow & & \\ G_0(\mathcal{Z}) & \longrightarrow & K_0(\mathcal{Y}) & \xrightarrow{j^*} & K_0(\mathcal{Y}') & & \end{array}$$

The restriction map $K_0(\mathcal{Y}') \rightarrow G_0(\mathcal{Y}'_0)$ factors through $K_0(\mathcal{Y}'_0) \rightarrow G_0(\mathcal{Y}'_0)$, so the composition (37) is indeed zero.

The map in Equation (36) is well-defined, and it is enough to show that the following map is surjective:

$$(38) \quad j^* \iota_* : G_0(\mathcal{K}) \rightarrow K_0(D_{sg}(\mathcal{Y}'_0)).$$

The map $\iota_* : G_0(\mathcal{K}) \rightarrow K_0(D_{sg}(\mathcal{Y}_0))$ is an isomorphism by Theorem 8.1. The restriction $j^* : K_0(D_{sg}(\mathcal{Y}_0)) \rightarrow K_0(D_{sg}(\mathcal{Y}'_0))$ is surjective. Indeed, the restrictions $G_0(\mathcal{Y}_0) \rightarrow G_0(\mathcal{Y}'_0)$ and $K_0(\mathcal{Y}_0) \rightarrow K_0(\mathcal{Y}'_0)$ are surjective, and the claim follows from the diagram:

$$\begin{array}{ccccc} K_0(\mathcal{Y}_0) & \longrightarrow & G_0(\mathcal{Y}_0) & \twoheadrightarrow & K_0(D_{sg}(\mathcal{Y}_0)) \\ \downarrow & & \downarrow & & \downarrow \\ K_0(\mathcal{Y}'_0) & \longrightarrow & G_0(\mathcal{Y}'_0) & \twoheadrightarrow & K_0(D_{sg}(\mathcal{Y}'_0)). \end{array}$$

This means that indeed the map in equation (38) is surjective, and thus the claim of the proposition follows. \square

Theorem 8.3. *Consider the quiver with potential (Q, W_d) introduced in Subsection 1.2. Let $e \in \mathbb{N}$, and consider the category $\mathbb{M}(e) \subset D_{sg}(\mathcal{X}(e)_0)$ mentioned in Subsection 8.1. Then*

$$K_0(\mathbb{M}(e)) = 0.$$

Proof. We will show that $K_0(D_{sg}(\mathcal{X}(e)_0)) = 0$.

Let $\mathcal{C}(e) \subset \mathfrak{gl}(e) \times \mathfrak{gl}(e) / GL(e)$ be the stack of commuting matrices (y, z) , and let $\mathcal{K}(e)$ be the Koszul stack with regular functions $\mathcal{O}_{\mathcal{C}(e)} \xrightarrow{\text{Tr}(z^d)} \mathcal{O}_{\mathcal{C}(e)}$. By Proposition 8.2, it is enough to show that the following map is surjective:

$$(39) \quad G_0(\mathcal{C}(e)) \rightarrow G_0(\mathcal{K}(e)).$$

Before we begin the proof, we introduce some notations:

- $\mathcal{A}(e)$ is the closed substack of $\mathcal{C}(e)$ where z has one eigenvalue,
- $\mathcal{A}^0(e) \subset \mathcal{A}(e)$ is the closed substack where z is nilpotent,
- denote by \underline{e} a partition $e_1 + \cdots + e_k = e$,
- for a partition \underline{e} , define

$$\mathcal{B}(\underline{e}) := (\mathcal{A}(e_1) \times \cdots \times \mathcal{A}(e_k)) \times_{\mathcal{C}(e)} \mathcal{K}(e),$$

- for a partition as above, let $Z \subset \mathbb{A}^k$ be the zero locus of $\sum_{i=1}^k e_i z_i^d = 0$,
- for a partition \underline{e} , let s be the number of distinct terms and m_1, \dots, m_k their multiplicities; consider the group $\mathfrak{S}(\underline{e}) := \mathfrak{S}_{m_1} \times \cdots \times \mathfrak{S}_{m_k}$.

Observe that:

$$(40) \quad \mathcal{A}(e) \cong \mathbb{A}^1 \times \mathcal{A}^0(e) \text{ and } \mathcal{B}(\underline{e}) \cong Z \times (\mathcal{A}^0(e_1) \times \cdots \times \mathcal{A}^0(e_k)).$$

Let $\mathfrak{S}(\underline{e})$ act on $\mathcal{A}(e_1) \times \cdots \times \mathcal{A}(e_k)$ by permuting the factors of the same size. The stack $\mathcal{C}(e)$ has a stratification by locally closed substacks

$$(\mathcal{A}(e_1) \times \cdots \times \mathcal{A}(e_k)) / \mathfrak{S}(\underline{e})$$

where z has k different eigenvalues with multiplicities e_1, \dots, e_k . By an excision argument, it is enough to show that the following maps are surjective:

$$G_0^{\mathfrak{S}(\underline{e})}(\mathcal{A}(e_1) \times \dots \times \mathcal{A}(e_k)) \rightarrow G_0^{\mathfrak{S}(\underline{e})}(\mathcal{B}(\underline{e}))$$

for all partitions \underline{e} . By Equation (40), it is enough to show that the following map is surjective:

$$G_0^{\mathfrak{S}(\underline{e})}(\mathbb{A}^k \times \mathcal{A}^0(e_1) \times \dots \times \mathcal{A}^0(e_k)) \rightarrow G_0^{\mathfrak{S}(\underline{e})}(Z \times \mathcal{A}^0(e_1) \times \dots \times \mathcal{A}^0(e_k)),$$

and thus that the following map is surjective:

$$(41) \quad K_0^{\mathfrak{S}(\underline{e})}(\mathbb{A}^k) \rightarrow G_0^{\mathfrak{S}(\underline{e})}(Z).$$

For a subset $S \subset \{1, \dots, k\}$, let $\mathbb{A}_S^k \subset \mathbb{A}^k$ be the locally closed subset with $z_i = 0$ for i not in S and $z_i \neq 0$ for i in S . The locally closed sets \mathbb{A}_S^k stratify \mathbb{A}^k . Observe that:

$$(42) \quad \mathbb{A}_S^k \cong \mathbb{G}_m^{|S|}.$$

Let $Z_S \subset Z$ be the intersection of \mathbb{A}_S^k and Z . Using Equation (42), we also use the notation $Z_S \subset \mathbb{G}_m^{|S|}$ for the zero locus $\sum_{i \in S} e_i z_i^d = 0$. The group $\mathfrak{S}(\underline{e})$ acts naturally on subsets S of $\{1, \dots, k\}$ and permutes the corresponding strata \mathbb{A}_S^k . Every set S is a unordered partition for its sum of elements and it thus has a corresponding permutation group \mathfrak{S} . By an excision argument, it is enough to show that for each set $S \subset \{1, \dots, k\}$, the following map is surjective:

$$(43) \quad K_0^{\mathfrak{S}}(\mathbb{A}_S^k) \cong K_0^{\mathfrak{S}}(\mathbb{G}_m^{|S|}) \rightarrow G_0^{\mathfrak{S}}(Z_S) \cong K_0^{\mathfrak{S}}(Z_S).$$

Consider the étale map $\mathbb{G}_m^{|S|} \rightarrow \mathbb{G}_m^{|S|}$ which sends z_i to z_i^d . Étale morphisms induce isomorphisms in rational K -theory. The preimage of $Z_S \subset \mathbb{G}_m^{|S|}$ is $T_S \subset \mathbb{G}_m^{|S|}$, the zero locus of $\sum_{i \in S} e_i z_i = 0$. The restriction to a hyperplane map is surjective:

$$K_0^{\mathfrak{S}}(\mathbb{G}_m^{|S|}) \rightarrow K_0^{\mathfrak{S}}(T_S),$$

so the map in Equation (43) is indeed surjective. \square

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