# Floer Theory of Disjointly Supported Hamiltonians

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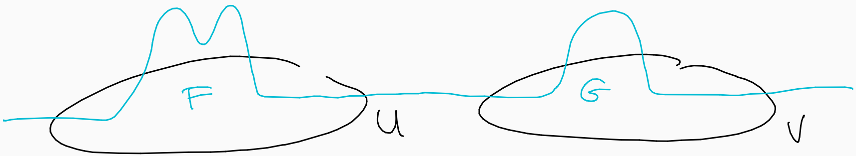
Joint with Yaniv Ganor

## PLAN:

- Background (the motivating question and related works)
- Setting
- Results (or, applications of the main tool)
- The main tool: Barricades.

# Background

$CF(H)$ for $H \approx F + G$ disj supp




as a v.s.: $CF(H) = C_U \oplus C_V \oplus \langle \text{crit. pts outside} \rangle$

($C_U$) gen by orbits in $U$

Differential $= ?$

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**Theorem (HLS, ”Max-formula”)**
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**Remark:** By Poincaré duality for spectral invariants:

$$c(F+G;[pt]) = \min\{c(F;[pt]), c(G;[pt])\}$$

# Setting

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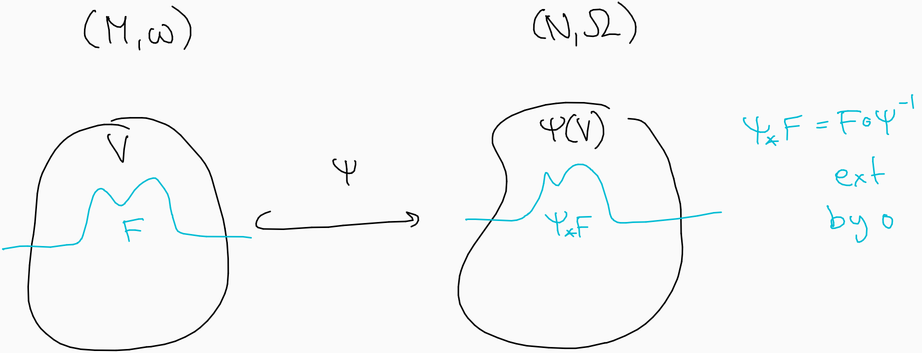
  More generally, consider domains with contact-type, incompressible boundaries. Call these CIB domains.

  $i_* : \pi_1(\partial V) \to \pi_1(M)$ injective

# Results




$$c_M(F : [M]) = c_N(\Psi_* F : [N])$$

**Theorem 1:**
Let $(M, \omega)$ and $(N, \Omega)$ be symplectically aspherical manifolds and assume that $V \subset M$ is a CIB domain, embedded into $N$, $\psi : V \hookrightarrow N$, such that the image is again a CIB domain. For every $F$ supported in $V$,

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**Remark:**
The asphericity and incompressibility assumptions are necessary.

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**Definition:**
For a non degenerate Hamiltonian $F$, consider homotopies of Hamiltonians $H$ and a.c.s. $J$ such that $H_- = F$, and denote by $\mathcal{M}(H,J)$ is the set of solutions of Floer equation with respect to $(H,J)$. Then,

$$c_{AHS}(F) = \sup_{(H,J)} \min_{u \in \mathcal{M}(H,J)} \mathcal{A}_F(u(-\infty)).$$

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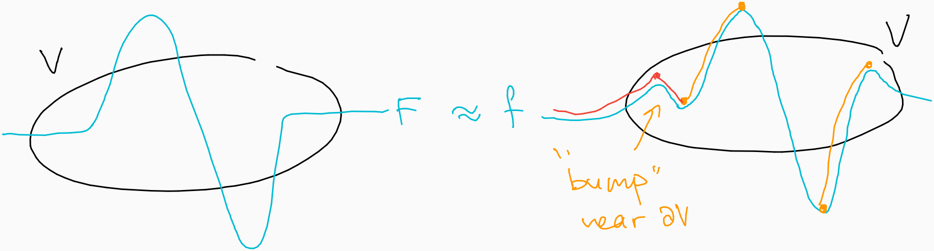
**Theorem (HLS):**
On surfaces (other than $S^2$) and for autonomous Hamiltonians, every action selector satisfying the min-formula coincides with $c(\cdot; [pt])$.

# The main tool: Barricades

## Morse Theory




For $f$:

1. $u$ starts in $V$, away from $\partial V$

   $\Rightarrow u \subset V \setminus N(\partial V)$

2. $u$ ends in $V \Rightarrow u \subset V$.

# The main tool: Barricades

**Theorem 0:**
Suppose $F$ is supported in a CIB domain $V$. Then, there exists a perturbation $f$ of $F$, and an almost complex structure $J$, such that for every solution $u$ of the Floer equation with respect to $(f, J)$:

1. If $u$ starts in $V \setminus \mathcal{N}(\partial V)$, then $\mathrm{im}(u) \subset V \setminus \mathcal{N}(\partial V)$.
2. If $u$ ends in $V$, then $\mathrm{im}(u) \subset V$.

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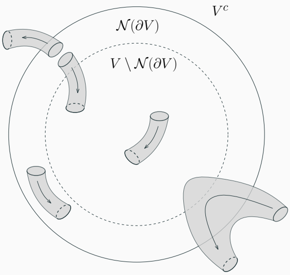
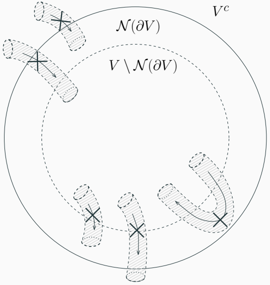
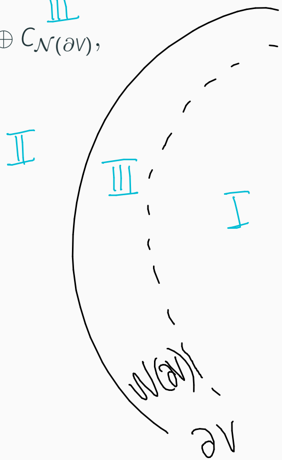



Allowed trajectories




Forbidden trajectories

Under the decomposition $CF(f) = \overset{\text{I}}{C_{V \setminus \mathcal{N}(\partial V)}} \oplus \overset{\text{II}}{C_{V^c}} \oplus \overset{\text{III}}{C_{\mathcal{N}(\partial V)}}$,

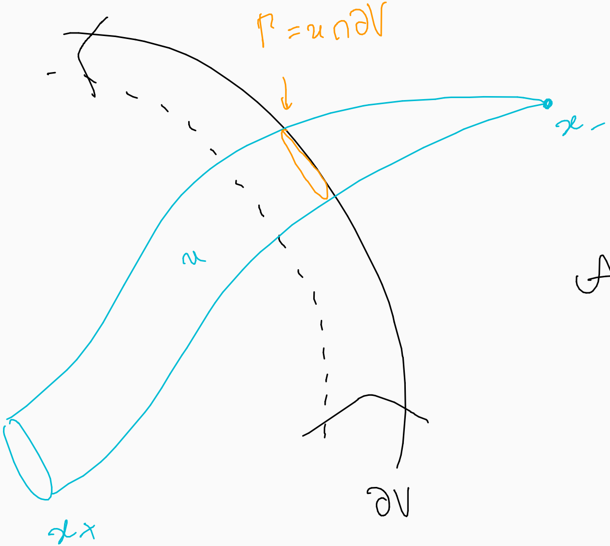



the differential takes a diagonal block form:

$$\partial_{f,J} = \begin{pmatrix} \partial|_{V \setminus \mathcal{N}(\partial V)} & 0 & \partial|_V \\ 0 & * & * \\ 0 & 0 & \partial|_V \end{pmatrix}$$

counts traj ⊂ V ↗

# A word about the proof




**Claim:**

$$\mathcal{A}(\Gamma) > \max_{V^c} f$$

# Thank you!