CAPACITIES FROM S^1 EQUIVARIAN SH

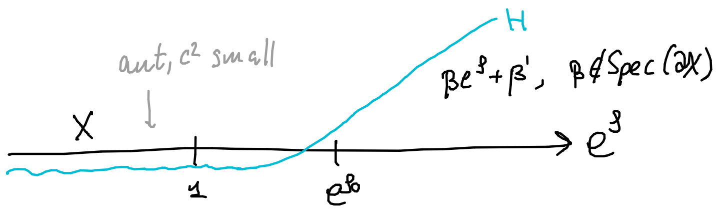
following (mostly) Gutt-Hutchings

PLAN:
- Recap of SH
- Def of S^1 equiv SH
- Properties (action filtration, U & δ maps)
- Capacities (def + properties)
- compute c_k "the hard way" → Compute EVERYTHING for $\mathbb{D}^2$
- compute c_k "the easy way" → Compute c_k for ellipsoids
- Gromov width of concave toric → Application to symp embeddings.

Setting: (X,λ) Liouville domain (e.g. ☆-shaped $\subset \mathbb{R}^{2n}$)

$$\hat{X} := X \cup [0,\infty)\times\partial X \,, \quad \hat{\lambda} := \begin{cases} \lambda & \text{on } X \\ e^{\rho}\lambda|_{\partial X} & \text{on } \underset{\rho}{[0,\infty)} \times \partial X \end{cases}$$

Recap of SH: SH is defined as a limit of FH over properly chosen Hamiltonians, called Admissible Hamiltonians: $H : S^1 \times \hat{X} \to \mathbb{R}$



(1) On $S^1\times X$: H is <0, aut, c^2 small.

(2) On $S^1\times[\rho_0,\infty)\times\partial X$: $H(\theta,\rho,y) = \beta e^{\rho} + \beta'$, for $\beta\notin Spec(\partial X,\lambda)$ "slope" of H, $\beta'\in\mathbb{R}$.

(3) On $S^1\times[0,\rho_0]\times\partial X$: H c^2 close to $h(e^{\rho})$ where h strictly convex.

A.c.s:
- J^{θ} compatible with $\hat{\omega} = d\hat{\lambda}$
- cylindrical end: On $[\rho_1,\infty)$ J^{θ} indep of θ, inv under ρ-trans, preserves ξ, $J(\partial_\rho) = R_0$

! 1-per orbits of H are either const, i.e. crit pts, or contained in $[0,\rho_0]$ and corr to Reeb orbits. Choosing "nice" a.c.s,

$$\Longrightarrow \quad SH(X,\lambda) := \varinjlim_{H,J \text{ adm}} HF(H,J).$$

(maps are continuation wrt monotone homotopies)

S^1-equiv SH: We have a group, S^1, acting on the loop space: $S^1 \circlearrowright \mathcal{L}\hat{X}$ by $\gamma \mapsto \gamma(\cdot+\varphi)$ and want a homology that "sees" this action. If the action were free, could take homology of the quotient. Not free (const loops). The idea is to look at the product of the loop space with a contractible space (so that the product is h.e. to the original space) on which the action is free; → S^∞: $\mathcal{L}\hat{X}\times S^\infty$, diag action is free. We will "approximate" S^∞ by f.d. spheres and then take the limit. Consider $S^{2N+1}\subset\mathbb{C}^{N+1}$ unit sphere. S^1 acts by rotation on each coordinate. $S^{2N+1}/S^1 = \mathbb{CP}^N$.

Consider $\mathcal{L}\hat{X} \times S^{2N+1}$ with $(\gamma, z) \mapsto (\gamma(\cdot + \varphi), \varphi \cdot z)$.

Want an action functional on the product which is S^1-inv.

For that consider Parametrized Hamiltonians:

$H : S^1 \times \hat{X} \times S^{2N+1} \to \mathbb{R}$, S^1-inv: $H(\theta + \varphi, x, \varphi \cdot z) = H(\theta, x, z)$.

The action functional is: $\mathcal{A}_H(\gamma, z) := -\int_\gamma \hat{\lambda} - \int_{S^1} H(\theta, \gamma(\theta), z)\, d\theta$.

One approach to S^1-equiv SH is to do Morse theory for this functional: Take the complex gen by its crit pts and define the differential by counting negative gradient flow lines. A drawback for this approach is that the gradient equations are coupled PDEs, which makes computations difficult.

Another approach is to fix a nice function on the sphere and define the differential by counting solutions of a parametrized FE (like continuation) where the param is by negative grad flow lines of the previously fixed nice fn.

$f_N : \mathbb{C}P^N \to \mathbb{R}^N$, $\quad f_N([z_0 : \dots : z_N]) := \dfrac{\sum_{j=0}^N j|z_j|^2}{\sum_{j=0}^N |z_j|^2}$

f_N is a perfect Morse function.

<u>Exc</u> $\text{Crit}(f_N) = \{ z_i := [0 : \dots : \underset{i}{1} : 0 \dots : 0] \}$, ind $z_i = 2i$.

$\tilde{f}_N : S^{2N+1} \to \mathbb{R}$ the lift (S^1-inv).

We'll consider param S^1-inv Hamiltonians satisfying:

(i) $\forall z \in S^{2N+1}$, $H_z := H(\cdot, \cdot, z)$ is SH-adm with β, β' indep of z (possibly deg)

(ii) $\forall z \in \text{Crit}(\tilde{f}_N)$, H_z is non-deg.

(iii) H is nondecreasing along flow of $-\nabla \tilde{f}_N$.

↑ so that differential will be action decreasing.

Trivial thought: What happens for $H = H' - \tilde{f}_N$ where H' is SH-adm? • if H' aut $\to$ H_z deg $\forall z$. • If H' time-dep then H not S^1-inv.

The complex

instead of taking the generators to be $\mathrm{Crit}\,\mathcal{A}_H$ we consider $\mathcal{P}^{S^1}(H) = \{(\gamma, z) \mid z \in \mathrm{Crit}(\tilde{f}_N),\ \gamma \in \mathcal{P}_1(H_z)\}$.

for $p \in \mathcal{P}^{S^1}(H)$, denote S_p = orbit of p under S^1 action.

$$CF^{S^1,N}(H) := \mathrm{span}_{\mathbb{Q}}\{S_p \mid p \in \mathcal{P}^{S^1}(H)\}$$

on ☆ shaped: Graded by $|(\gamma,z)| = \mathrm{ind}_{f_N} z - CZ(\gamma)$

If we demand in addition that, over crit pts of $\tilde{f}_N$, H coincides with a fixed Hamiltonian, the complex takes a simpler form:

"Simple" Hams: Assume in addition $\exists\, H' : S^1 \times \hat{X} \to \mathbb{R}$ SH-adm s.t. for $z \in \mathrm{Crit}(\tilde{f}_N)$, $H(\theta, x, z) = H'(\theta - \phi(z), x) + c(z)$ where $\phi : \mathrm{Crit}(\tilde{f}_N) \to S^1$ given by $\phi(0,\dots,0,e^{2\pi i\varphi},0,\dots,0) = \varphi$.

$$\Longrightarrow \quad CF^{S^1,N}(H) = \mathbb{Q}\{1, u, \dots, u^N\} \otimes CF(H') \quad (\text{here } u^i \text{ rep } z_i)$$

The differential

$J = J^\theta_z = J^{\theta+\varphi}_{\varphi\cdot z}$ + SH adm.

for $p_\pm \in \mathcal{P}^{S^1}(H)$, $\hat{\mathcal{M}}(S_{p_-}, S_{p_+}; J)$ is the set of pairs (η, u)

$\eta : \mathbb{R} \to S^{2N+1}$

$u : \mathbb{R} \times S^1 \to \hat{X}$

$$(\mathrm{PFE}) \quad \begin{cases} \dot{\eta} + \nabla \tilde{f}_N(\eta) = 0 \\ \partial_s u + J^\theta_{\eta(s)}\left(\partial_\theta u - X_{H^\theta_{\eta(s)}} \circ u\right) = 0 \\ \lim\limits_{s \to \pm\infty} (u(s), \eta(s)) \in S_{p_\pm} \end{cases}$$

$\mathbb{R}, S^1$ act on $\hat{\mathcal{M}} \to \mathcal{M}(S_{p-}, S_{p+}; J)$ the quotient.

$$\partial^{S^1} S_{p-} := \sum_{p+} \# \mathcal{M}(S_{p-}, S_{p+}, J) \cdot S_{p+}$$

s.t. $\dim \mathcal{M} = 0$ $\to$ $p+$ $\quad$ signed count

$|p_-| - |p_+| = 1$.

For "simple" Ham: recall $H = H'(\theta - \phi(z), x) + c(z)$ for $z \in Crit(\hat{f}_N)$

Intuitive explanation: Want to show $\mathcal{M}$ dep only on the difference of powers of u. The flowlines $z_i \to z_{i-k}$ are $\mathbb{CP}^k$ embedded in $\mathbb{CP}^N$. If H, J look the same on these $\mathbb{CP}^k$ for all i $\mathcal{M}$ does not dep on i.

$i_0, i_1 : S^{2N+1} \hookrightarrow S^{2N+3}$ given by $\begin{cases} i_0(z) = (z_0, .., z_N, 0) \\ i_1(z) = (0, z_0, .., z_N) \end{cases}$

suppose $\forall\ 0 \le k \le N-1$:

- $i_j^* H_{k+1} = H_k + \text{const}$ (could dep on z).
- $i_j^* J_{k+1} = J_k$

Under additional assumptions on (H, J), the differential splits into a sum $\partial^{S^1} = \sum_{i=0}^{N} u^{-i} \otimes \varphi_i$ where $\varphi_i : CF(H') \circlearrowleft$, $u^{-i}(u^k) = \begin{cases} u^{k-i} & \text{if } k \ge i \\ 0 & \text{ow}. \end{cases}$

The S^1 equiv SH: $SH^{S^1}(X, \lambda) = \varinjlim_{N, H, J} HF^{S^1, N}(H, J)$ $\leftarrow$ homology of $(CF^{S^1, N}(H), \partial^{S^1})$

The maps in this direct limit are continuations wrt monotone homotopies.

Partial order: $(N_1, H_1) \le (N_2, H_2) \leftrightarrow N_1 \le N_2,\ H_1 \le H_2|_{S^{2N_1+1}}$ Take hom K on $S^1 \times \hat{X} \times S^{2N_1+1}$ between that is monotone, $\partial_s K \ge 0$

The map $CF^{S^1 N_1}(H_1) \longrightarrow CF^{S^1 N_2}(H_2)$ is the composition of continuation and inclusion.

Proposition: enough to consider "simple" H, J.

Properties: ① Action filtration and positive part.

As in FH, we can consider the subcomplex generated by orbits of action smaller than L and its homology:

for $L \in \mathbb{R}$, $CF^{S^1, N, \le L} = \text{span}_{\mathbb{Q}} \{ S_p \mid \mathcal{A}(p) \le L \} \rightsquigarrow HF^{S^1, N, \le L}$

Then, $SH^{S^1, \le L}(X, \lambda) := \varinjlim HF^{S^1, N, \le L}$.

We can also consider the quotient complex corresponding to orbits with action $\geq \ell$: $\dfrac{CF^{S^1,N,\leq L}}{CF^{S^1,N,\leq \ell}} \rightsquigarrow HF^{S^1,N,\geq \ell,\leq L}$

$SH^{S^1,\leq L,\geq \ell} = \varinjlim HF^{S^1,N,\leq L,\geq \ell}$.

Positive part: $SH^{S^1,+}(X,\lambda) = SH^{S^1,\geq \varepsilon}$ for $0<\varepsilon<\min \mathrm{Spec}(\partial X)$

Similarly, have action filtration on $SH^{S^1,+}$:

$SH^{S^1,+,\leq L} \xrightarrow{i_L} SH^{S^1,+}$ $\quad$ i_L = limit of maps induced by inclusion.

② U map : $SH^{S^1} \circlearrowleft$. for "simple" H, $CF^{S^1,N} = \mathbb{Q}\{1,..,u^N\} \otimes CF(H')$

U induced by $u^i \otimes \gamma \mapsto \begin{cases} u^{i-1} \otimes \gamma & \text{if } i \geq 1 \\ 0 & \text{ow} \end{cases}$ $\quad$ - chain map
- action pres.

! The fact that this is a chain map follows easily from the "decomposed" form of ∂^{S^1}. $\quad$ • U restricts to $SH^{S^1,+}$

[LES for U: $\cdots \longrightarrow SH^+ \xrightarrow{i_*} SH^{S^1,+} \xrightarrow{U[-2]} SH^+ \xrightarrow{B[+1]} \cdots$]
induced by inclusion $\quad$ induced by $u^i \otimes \gamma \mapsto \varphi_{i+1}(\gamma)$

③ δ map :

$$\begin{tikzcd} SH^{S^1,+}(X,\lambda) \arrow[rr, "{\delta[-1]}"] & & H_*(X,\partial X) \otimes H_*(BS^1) \arrow[dl, "i_* + \text{iso}"] \\ & SH^{S^1}(X,\lambda) \arrow[ul, "\pi_*"] & \end{tikzcd}$$

$\quad\quad\quad \cong SH^{S^1,<\varepsilon}$

Given rep β of a class in $SH^{S^1,+}$, $\beta \in CF^{S^1,N,\geq \varepsilon}$ a cycle.
Then $\partial^{S^1}\beta \in CF^{S^1,N,\leq \varepsilon} \longrightarrow$ a class in $SH^{S^1,<\varepsilon} \cong H_*(X,\partial X) \otimes H_*(BS^1)$

Capacities: $c_k(X,\lambda) := \inf\{L : \exists \beta \in SH^{S^1,+,\leq L} \text{ s.t } \delta U^{k-1} i_L \beta = [X] \otimes [pt]\}$

c_k is the smallest action of a rep of a class in $SH^{S^1,+}$ whose image under δU^{k-1} is $[X] \otimes [pt]$.

"Exc 0" If $SH^{S^1}(X) = 0$ (eg $X = \text{✯}$ shaped) then $c_1 < \infty$.

Sol. $c_1 = \inf\{L : \exists \beta \;\; \delta i_L \beta = [X] \otimes [pt]\} \longleftrightarrow [X] \otimes [pt] \in \mathrm{im}\,\delta$.

By exact triangle, $\mathrm{im}\,\delta = \ker i_* = H_*(X,\partial X) \otimes H_*(BS^1)$.

The "intuitive" solution: Suppose $\alpha \in CF^{S^1,N,<\varepsilon}$ rep $[X]\otimes[pt]$ (actually, its image under the iso) then $\alpha \in CF^{S^1,N}$ (assuming N large enough) is boundary: $\partial^{S^1}\beta = \alpha$. Then $\pi\beta \in CF^{S^1,\geq\varepsilon}$ rep a class in $SH^{S^1,+}$ and $\delta[\pi\beta] = [\alpha] = [X]\otimes[pt]$.

Properties: Theorem (GH): When $(X,\lambda) \subset (\mathbb{R}^{2n},\lambda_0)$ nice ☆-shaped domain, the capacities c_k sat:

CONFORMALITY: $c_k(r\cdot X) = r^2 c_k(X)$ — gen Liouville: c_k 1-hom wrt rescaling at λ.

INCREASING: $c_1(X) \leq c_2(X) \leq \cdots < \infty$ — holds for gen Liouville wo $<\infty$.

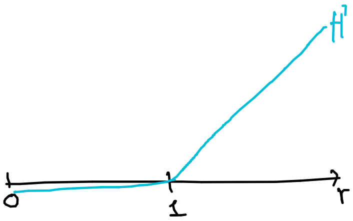
MONOTONICITY: if $X' \overset{S}{\hookrightarrow} X$ then $c_k(X') \leq c_k(X)$ — for gen Liouville only wrt gen Liouville emb $[\varphi^*\lambda - \lambda'] = 0 \in H$

SPECTRALITY: if $\lambda_0|_{\partial X}$ non-deg then

$c_k(X) = \int_\gamma \lambda_0$ for Reeb orbit γ of index $n-1+2k$. — holds for Liouville wo ind.

Computing EVERYTHING for $\mathbb{D}^2 \subset \mathbb{C}^2$.

Use "simple" Hams: $CF^{S^1,N} = \mathbb{Q}\{1,..,u^N\}\otimes CF(H')$



$CF(H') = \mathbb{Q}\{m, \check{\gamma}_1, \hat{\gamma}_1, \check{\gamma}_2, \hat{\gamma}_2, \ldots\}$

min $H' = 0$ — non-deg pert of orbits going k times around $\partial\mathbb{D}$.

Action & indices: $*\ \mathcal{A}(m) \approx 0$, $\mathcal{A}(\gamma_k) \approx \pi k$

$*\ |m| = \mathrm{ind}_{-H'}(m) - n = 2-1 = 1$ $\quad *\ |\check{\gamma}_k| = 2k$, $|\hat{\gamma}_k| = 2k+1$.

degree: 1 2 3 4 5 ...

$$m \xleftarrow{\partial} \check{\gamma}_1 \xrightarrow{\varphi_1} \hat{\gamma}_1 \xleftarrow{\partial} \check{\gamma}_2 \xrightarrow{\varphi_1} \hat{\gamma}_2 \xleftarrow{\partial} \cdots$$

Differential: $\partial^{S^1} = \sum u^{-i}\otimes\varphi_i$, $\deg(\varphi_i) = 2i-1$, $\varphi_0 = \partial$.

- for $i > 1$, $\varphi_i = 0$ since increase deg by >1, and are action non-increasing.

- $\varphi_0 = \partial$. Since $SH_*(\mathbb{D}) = 0$ every orbit is either bdry or not closed

m closed $\to \partial \check{\gamma}_1 = m$. $\partial \hat{\gamma}_1 \neq \check{\gamma}_1 \leftarrow$ not closed $\Rightarrow \partial \hat{\gamma}_1 = 0 \to \partial \check{\gamma}_2 = \hat{\gamma}_1$.

- φ_1 increases deg by 1, and action non-increasing $\Rightarrow$

$\varphi_1(\hat{\gamma}_k) = 0$, $\varphi_1(\check{\gamma}_k) = a_k \cdot \hat{\gamma}_k$. claim: $a_k = k$ (φ_1 coincides with the BV operator which counts sols to FE wrt $H^\theta_{s,\phi} : \hat{X} \to \mathbb{R}$ sat. $H^\theta_{s,\phi} = H^\theta$ for $s \ll 0$, $H^\theta_{s,\phi} = H^{\theta+\phi}$ for $s \gg 0$. (and linear dep only on s out of a compact). When the orbit is $\frac{1}{k}$ periodic have sol for $\phi_i = \frac{i}{k}$.

Overall, $\partial^{S^1}(u^i \otimes \check{\gamma}_j) = u^i \otimes \hat{\gamma}_{j-1} + j \cdot u^{i-1} \otimes \hat{\gamma}_j$, $\partial^{S^1}(u^i \otimes \hat{\gamma}_j) = 0$.

<u>δ map:</u> δ induced by $\ker \partial^{S^1,+} \subset CF^{S^1,N,+} \xrightarrow{\partial^{S^1}} CF^{S^1,N,<\varepsilon}$

Let us find domain and target. Notice $CF^{S^1,N,<\varepsilon}$ gen by $u^i \otimes m$.

Let's find $\ker \partial^{S^1,+}$. If $\partial^{S^1}\alpha \neq 0$ then α must contain rep $[X]\otimes[u^i]$

$u^i \otimes \check{\gamma}_1 \longrightarrow \partial^{S^1} u^i \otimes \check{\gamma}_1 = u^i \otimes m + u^{i-1} \otimes \hat{\gamma}_1$

$\Longrightarrow \partial^{S^1,+} \neq 0$ (if $i \neq 0$) $\to$ let's "fix" it: substract $u^{i-1} \otimes \check{\gamma}_2 \ldots$

$\alpha_i := u^i \otimes \check{\gamma}_1 - u^{i-1} \otimes \check{\gamma}_2 + \cdots + (-1)^i (i-1)!\, 1 \otimes \check{\gamma}_{i+1}$, $\partial^{S^1,+}\alpha_i = 0$ and

$\partial^{S^1}\alpha_i = u^i \otimes m \Longrightarrow \delta$ maps $\alpha_i \mapsto u^i \otimes m$.

<u>Capacities:</u> c_k = smallest action of a cycle mapped to $[X]\otimes[pt]$ by $\delta \cdot U^{k-1}$!

$$[X]\otimes[pt] \overset{??}{=} [1 \otimes m] \Rightarrow \text{Want } \alpha \text{ st } 1 \otimes m = \partial^{S^1} U^{k-1} \alpha \overset{\substack{U \text{ chain map} \\ \downarrow}}{=} U^{k-1} \partial^{S^1} \alpha$$

$\Longleftrightarrow \partial^{S^1}\alpha = u^{k-1} \otimes m \,(+\, a_1 \cdot u^{k-2} \otimes m + a_2 \cdot u^{k-3} \otimes m \ldots)$ ← Cannot have $\check{\gamma}_i$ here since $\partial^{S^1,+}\alpha = 0$!

already computed! $\Longrightarrow \alpha = \alpha_{k-1} = u^{k-1} \otimes \check{\gamma}_1 + \cdots + (-1)^{k-1}(k-1)!\, 1 \otimes \check{\gamma}_k$

$c_k(\mathbb{D}) = \mathcal{A}(\alpha_{k-1}) = \pi k$.

This was actually computing the capacities the "hard" way. In many cases (including this) a much easier way to compute is only using the properties:

Computing c_k for ellipsoids / computing c_k the "easy" way

Let $E = E(a_1, \dots, a_n) = \left\{ \sum_{i=1}^{n} \frac{\pi |z_i|^2}{a_i} \le 1 \right\}$, s.t. $\frac{a_i}{a_j} \notin \mathbb{Q}$ for $i \neq j$.

There are exctly n simple Reeb orbits on ∂E:

$$\gamma_i(t) = \left(0, \dots, \sqrt{\frac{a_i}{\pi}} e^{2\pi i t}, 0, \dots, 0\right) \quad \text{and} \quad \mathcal{A}(\gamma_i^m) = m a_i$$

and $CZ(\gamma_i^m) = n - 1 + 2 \sum_{j=1}^{n} \left\lfloor \frac{m a_i}{a_j} \right\rfloor$. ← the action of γ_i^m

By Spectrality property, $c_k(E)$ is the action of γ of index $2k+n-1$.

$$CZ(\gamma_i^m) = n-1+2k \longleftrightarrow \sum_{j=1}^{n} \left\lfloor \frac{m a_i}{a_j} \right\rfloor = k \overset{\text{Exc}}{\longleftrightarrow} \mathcal{A}(\gamma_i^m) = M_k(a_1, \dots, a_n)$$

$\Longrightarrow c_k(E) = M_k(a_1, \dots, a_n)$

Here $(M_k(a_1, \dots, a_n))_{k \in \mathbb{N}}$ is the seq of natural mult. of a_j arranged in non-dec order.

Similarly one can compute c_k for what is called convex and concave toric domains (I'll define in a moment concave) only using the properties, only the result and the pf are more complicated.

Application to symplectic embeddings:

Can use the capacities c_k to show that the "best" embedding of the ball into a concave toric domain is inclusion.

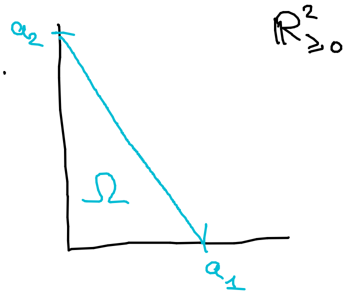
Def • A toric domain in $\mathbb{C}^n$ is a domain of the form

$X_\Omega = \mu^{-1}(\Omega)$ for some $\Omega \subset \mathbb{R}^n_{\geq 0}$, where $\mu: \mathbb{C}^n \to \mathbb{R}^n_{\geq 0}$ given by

$\mu(z_1, \dots, z_n) = \pi(|z_1|^2, \dots, |z_n|^2)$

• X_Ω is a concave toric domain if Ω is compact and $\mathbb{R}^n_{\geq 0} \setminus \Omega$ is convex.

Convex toric domains are convex sets X_Ω for convex Ω.



Example: Every ellipsoid is a concave toric domain.

Theorem (GH): For every concave toric domain

$$X_\Omega \subset \mathbb{C}^n, \quad c_G(X_\Omega) = \max\{a \mid B(a) \subset X_\Omega\}$$

The Gromov width is the size (πr^2) of the largest ball that can be symp embedded into X_Ω.

The main ingredient in the proof of this theorem is a formula for c_k (actually c_1) for convex toric domains:

Lemma: X_Ω concave toric $\Rightarrow$ $c_k(X_\Omega) = \max\{[v]_\Omega \mid v \in \mathbb{N}^n_{>0}, \sum_{i=1}^n v_i = k+n-1\}$

where $[v]_\Omega := \min\{\langle v, w\rangle \mid w \in \overline{\partial\Omega \cap \mathbb{R}^n_{>0}}\}$

This function is the analog of "dual norm" for concave instead of convex domains.

Pf that Lemma $\Rightarrow$ Thm: Let $a_{max} := \max\{a : B(a) \subset X_\Omega\}$, then $c_G(X_\Omega) \geq a_{max}$.

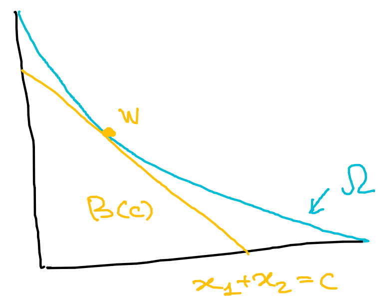
Let's show $c_G(X_\Omega) \leq a_{max}$ in 2 steps:

(1) for any ball $B(a) \overset{s}{\hookrightarrow} X_\Omega$, $a \leq c_1(X_\Omega)$: Indeed, $a = c_1(B(a))$ which, by monotonicity of c_1, $\leq c_1(X_\Omega)$.

(2) $c_1(X_\Omega) - a_{max}$: Indeed, by the Lemma:

$$c_1(X_\Omega) = [(1,\dots,1)]_\Omega = \min\{\sum_{i=1}^n w_i \mid w \in \overline{\partial\Omega \cap \mathbb{R}^n_{>0}}\} = a_{max}:$$

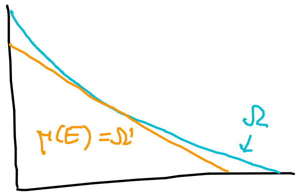
Set $c := \sum w_i$ for w minimizer then $\mu^{-1}(\{\sum x_i \leq c\})$ is a ball of capacity c that is contained in Ω.



"Proof" of the Lemma: Fix concave toric X_Ω.

$\boxed{\leq}$ Construct a "nice" non-deg perturbation of $X_{\Omega'}$ for which every periodic orbit of index $2k+n-1$ corr to $v \in \mathbb{N}^n_{>0}$ with $\sum v_i \leq k+n-1$, and its action is $[v]_\Omega$. Then, $c_k \leq \max \mathrm{Spec}_{2k+n-1} \leq$ RHS.

$\boxed{\geq}$ For v on which max is attained, def $\Omega' := \{x \in \mathbb{R}^n_{\geq 0} \mid \langle x, v\rangle \leq [v]_\Omega\}$



Then $\Omega' \subset \Omega$ and $\mu^{-1}(\Omega')$ is ellipsoid.

Monotonicity $\Rightarrow$ $c_k(\Omega) \geq c_k(E = X_{\Omega'}) \geq [v]_\Omega$

We saw for ellipsoids: $CZ(\gamma_i^m) = n-1+2\sum_{j=1}^{n} \left\lfloor \frac{ma_i}{a_j} \right\rfloor = n-1+2k$

$\Leftrightarrow$ $\sum_{j=1}^{n} \left\lfloor \frac{ma_i}{a_j} \right\rfloor = k$ and ma_i is the action. Moreover,

Noticing $E = E\left(\frac{[v]_\Omega}{v_1}, \dots, \frac{[v]_\Omega}{v_j}\right)$ $\left(\text{since } 1 \geq \sum \frac{x_i v_i}{[v]_\Omega} = \sum \frac{x_i}{[v]_\Omega / v_i}\right)$

we find $c_k(E) = \left\{A : \sum_{j=1}^{n} \left\lfloor \frac{A}{[v]_\Omega / v_j} \right\rfloor = k \right\}$

Notice that if $A < [v]_\Omega$ then $\sum_{j=1}^{n} \left\lfloor \frac{A}{[v]_\Omega} \cdot v_j \right\rfloor \leq \sum_{j=1}^{n} (v_j - 1)$
$= \sum_{j=1}^{n} v_j - n = k+n-1-n = k-1.$

Therefore, $c_k(X_\Omega) \geq c_k(E) = A \geq [v]_\Omega$.