On Transitive Actions and Fibre Bundles

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1 Introduction

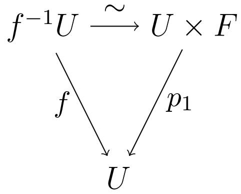
It is elementary to show that for a topological group G and a transitive continuous action $\mu : G \times X \to X$ the function $\pi : G \to X$, defined by sending $g \in G$ to gx_0 for some $x_0 \in X$, has homeomorphic fibres, i.e., for any $x, x' \in X$ the spaces $\pi^{-1}\{x\}$ and $\pi^{-1}\{x'\}$ are homeomorphic. The following question then naturally arises: when does π give G the structure of a fibre bundle over X? We show that the above question is essentially the same as whether or not π has local sections at x_0. Finally, we show that the existence of local sections is fairly easy to characterize, at least when G is a Lie group smoothly acting on a smooth manifold X. This gives a complete answer for our question in the case of smooth actions of Lie groups and smooth vector bundles.

2 Prerequisities

2.1 Local Sections and Fibre Bundles

A *local section* of a continuous map $f : X \to Y$ is a continuous map $s : U \to X$, where U is an open set of Y, such that $f \circ g$ is the identity function of U. A local section at $y \in Y$ is a local section $s : U \to X$ where $y \in U$.

Let $f : X \to Y$ be a continuous map of topological spaces and F any nonempty topological space. The map f is called an *F-fibre bundle*, if for every $y \in Y$, we have an open neighbourhood U of y and a homeomorphism $f^{-1}U$ making the diagram (*)



commute, where the map p_1 is just the canonical projection. The map f (or even the space X itself, when the *structure morphism* f is clear from the context) is called a *fibre bundle* if f is a F-fibre bundle for some topological space F.

The following lemma relates the two concepts. Denote by F the fibre of π over x_0.

Lemma 2.1. *The map π as defined in the introduction makes G an F-fibre bundle over X exactly when π has a local section at x_0.*

Proof. Assume first that π is a fibre bundle. Let U be an open neighbourhood of x_0, and $\psi : f^{-1}U \xrightarrow{\sim} U \times F$ a homeomorphism making the diagram (*) commute. Now for $F_0 \in F$, we may define a continuous map $U \to U \times F$ by $u \mapsto (u, F_0)$ and composing it with ψ^{-1} gives a continuous map $s : U \to X$. From the fact that the diagram (*) commutes it then follows that $f \circ s$ is the identity function, so s is a local section at x_0.

Assume then that there is a local section at x_0. The lemma following this one shows that π has local sections at every point of X. Let $x \in X$ be arbitrary and $s : U \to G$ a local section at x. Define a map $\psi : f^{-1}U \to U \times F$ by sending g to $(f(g), (s \circ f)(g)^{-1}g)$. The map is well-defined as $(s \circ f)(g)^{-1}g$ is a member of F:

$$f((s \circ f)(g)^{-1}g) = s(gx_0)^{-1}gx_0 = x_0$$

where the latter equation follows from the fact that $gx_0 = (f \circ s)(gx_0) = s(gx_0)x_0$. The map ψ is a continuous, and it does make the diagram (*) commute, so the only thing left to show is that ψ is a homeomorphism.

Define a continuous map $\phi : U \times F \to f^{-1}U$ by sending (u, g) to $s(u)g$. This map is well defined: as π is clearly a map of G-sets, it follows that $f(s(u)g) = s(u)f(g) = s(u)x_0 = u$. The map ϕ is also the inverse of ψ as

$$\phi(\psi(g)) = \phi(f(g), s(f(g))^{-1}g) = s(f(g))s(f(g))^{-1}g = g$$

and

$$\begin{aligned}\psi(\phi(u, g)) &= \psi(s(u)g) \\ &= (f(s(u)g), s(f(s(u)g))^{-1}s(u)g) \\ &= (u, s(u)^{-1}s(u)g) = (u, g).\end{aligned}$$

Thus ψ is a homeomorphism, which concludes our proof. □

The above proof used the following lemma.

Lemma 2.2. *If π has local section at x_0, then it has local sections at every point of X.*

Proof. Denote by s the local section $s : U \to X$ at x_0 whose existence was assumed, and let $x \in X$ be arbitrary. Because the action of G on X is transitive, we can find an element g of G which sends x_0 to x. Define a map $s' : gU \to X$ with formula $s'(u) = gs(g^{-1}u)$. The map s' is clearly continuous. Furthermore, it is a local section of π as

$$\pi(s'(u)) = \pi(gs(g^{-1}u)) = g\pi(s(g^{-1}u)) = gg^{-1}u = u.$$

This concludes the proof. □

2.2 Some Vector Analysis

It is a well known fact from elementary analysis that a smooth function $f : U \to V$ between open sets U and V of $\mathbb{R}^n$ has a local inverse at $x \in U$ if and only if the derivative matrix

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots \\ \partial_1 f_n(x) & \dots & \partial_n f_n(x) \end{pmatrix}$$

is invertible. This is called the inverse function theorem. As an application of this, we obtain the following lemma:

Lemma 2.3. *Let U and V be open sets of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, and $f : U \to V$ a smooth function. Now there exists a smooth local section s of f at $y_0 \in V$ if and only if the rank of $Df(x_0)$ is maximal for some x_0 in the fibre of y_0, i.e., $Df(x_0)$ defines a surjective linear map.*

Proof. If there is a local section s at y_0, then as $f \circ s = \mathrm{id}$, we get that

$$I = D(f \circ s)(y_0) = Df(s(y_0))Ds(y_0),$$

and hence $Df(s(y_0))$ must have maximal rank.

Assume then that $Df(x_0)$ has maximal rank. Now from elementary linear algebra it follows that there must be m columns of $Df(x_0)$ that are linearly independent. By permuting the coordinates of $\mathbb{R}^n$ if necessary, we may assume that these are the first m columns. Define a function $\bar{f} : U \to V \times \mathbb{R}^{n-m}$ by sending $x = (x_1, \dots, x_n)$ to $(f(x), x_{m+1}, \dots, x_n)$. The derivative matrix

$$Df(x) = \begin{pmatrix} \partial_1 f_1(x) & \cdots & \partial_m f_1(x) & \partial_{m+1} f_1(x) & \cdots & \partial_n f_1(x) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial_1 f_m(x) & \cdots & \partial_m f_m(x) & \partial_{m+1} f_m(x) & \cdots & \partial_n f_m(x) \\ 0 & \cdots & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 1 \end{pmatrix}$$

is clearly invertible, so there exists a local smooth inverse function $h : V' \times W \to U'$ of $\bar{f}$, where U' and V' are open subsets of U and V respectively, and W is an open set of $\mathbb{R}^{n-m}$. Let $w \in W$. Now we can define a continuous map $s : V \to U'$ by sending v to $h(v, w)$. This is a local section of f. In order to see this, recall that $f = p_1 \circ \bar{f}$, and thus

$$(f \circ s)(v) = p_1(\bar{f}(h(v, w))) = p_1(v, w) = v.$$

As h is smooth, it is clear that also s is. □

3 G as a Smooth Fibre Bundle

We are ready to prove the main theorem. Assume that G is a Lie group and that the transitive action on X is smooth. Defining π as before makes it a smooth map.

Theorem 3.1. *The map π is a smooth fibre bundle if and only if the induced map on tangent spaces $T_e G \to T_{x_0} X$ is surjective.*

Proof. It is easy to show that π is a smooth fibre bundle if and only if there is a smooth local section of π at x_0. Essentially the same proof holds as in the continuous case, just replace the word "continuous" with the word "smooth" everywhere.

We already know that π has local smooth section at x_0 if and only if the tangent map $T_{g_0} G \to T_{x_0} X$ is surjective for some g_0 in the fibre of x_0. The problem is that g_0 is not necessarily e. In any case, if $T_{g_0} G \to T_{x_0} X$ is surjective for some g_0, then $\pi : G \to X$ is a fibre bundle, so finding a local section of π at x_0 sending x_0 to e is trivial. Hence the tangent map $T_e G \to T_{x_0} X$ is surjective as well, and the theorem follows. □

This condition is fairly straightforward to verify, at least in the cases where a closed subgroup of $GL(n, \mathbb{R}^n)$ acts on a submanifold of $\mathbb{R}^n$ with its natural action.