Milnor Conjecture on Quadratic Forms

Toni Annala

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Abstract

In these notes, we explain the proof of the Milnor conjecture on quadratic forms by Orlov–Vishik–Voevodsky. This is the last talk of the IAS Milnor Conjecture reading seminar during the 22/23 academic year.

Contents

1 Introduction

The purpose of these notes is to prove the *Milnor conjecture on quadratic forms* [OVV07]: for a field k of characteristic 0

$$\mathrm{Gr}^I_* W(k) \cong K^M_*(k)/2, \tag{1}$$

where $\mathrm{Gr}^I_* W(k)$ denotes the associated graded ring of the *Witt ring* with a filtration given by the powers of the ideal I of even dimensional quadratic forms. By the Milnor conjecture [Voe03b, Voe03a] we may identify $K^M_*(k)/2$ with étale cohomology with $\mathbb{Z}/2$-coefficients, and thus

$$\mathrm{Gr}^I_* W(k) \cong H^*_{\text{ét}}(k; \mathbb{Z}/2). \tag{2}$$

Hence, the graded Witt ring is identified with étale cohomology groups that are often easy to compute in practice.

In order to prove the conjecture, we establish the exactness of the fundamental sequence

$$K^M_{*-n}(k) \xrightarrow{\underline{a}\cdot} K^M_*(k) \to K^M_*(K(Q_{\underline{a}})), \tag{3}$$

where $\underline{a} = (a_1, \ldots, a_n) \in (k\backslash\{0,1\})^{\times n}$, and $Q_{\underline{a}}$ is the *norm quadric*, i.e., the vanishing locus of the form $\langle\langle a_1, \ldots, a_{n-1}\rangle\rangle - \langle a_n\rangle$ on $\mathbb{P}^{2^{n-1}}$. Above, the first summand is the Pfister form, and $\langle a\rangle$ is the form ax^2. In other words, Eq. 3 characterizes the kernel of the pullback map from the Milnor K-theory of k to that of the function field of the norm quadric $Q_{\underline{a}}$: the kernel is principal, and generated by the pure symbol $\underline{a} \in K^M_n(k)$.

In fact, Orlov–Vishik–Voevodsky [OVV07] also characterize the kernel of $\underline{a}\cdot$. However, since this is not necessary for our purposes, we do not provide the details of the more refined result here.

Background and conventions

We will use the theory of motivic cohomology, see [Voe03b, Voe03a, VSF00, MVW06] for reference. Motivic cohomology gives a bigraded cohomology theory of schemes. In particular we have the following identifications

$$H^{n,n}(k;\mathbb{Z}) \cong K^M_n(k)$$
$$H^{2n,n}(X;\mathbb{Z}) \cong \mathrm{CH}^n(X)$$

first of which we will use extensively later.

Moreover, the following basic fact, which is essentially just Grothendieck vanishing applied to motivic cohomology (see e.g. [MVW06]), will be useful throughout the paper.

Lemma 1 (Vanishing lemma)**.** *Let X be a smooth variety of dimension d, and let A be an Abelian group. Then*

$$H^{m+i,m}(X, A) \cong 0$$

if $i > d$.

2 Witt ring and the I-filtration

Here we recall some basic facts from the theory of quadratic forms. Our main reference is [EL72]. We will work over a fixed base field k (of characteristic 0, or at least not 2).

A *quadratic space* is a finite dimensional k vector space V equipped with a nondegenerate quadratic form $q : V \to k$. All quadratic forms are diagonalizable. We will denote by $\langle a_1, \dots, a_r \rangle$ the diagonal quadratic form $a_1 x_1^2 + \cdots a_r x_r^2$. There are obvious notions of direct sums and tensor products of quadratic spaces. The *Pfister form* is denoted/defined by

$$\langle\langle a_1, \dots, a_r \rangle\rangle := \langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_r \rangle. \tag{4}$$

The *Grothendieck–Witt ring* $GW(k)$ is defined as the group completion of the Abelian monoid that is formed by the isomorphism classes of quadratic spaces. It is a ring, with multiplication given by the tensor product. The *hyperbolic form* $h = \langle 1, -1 \rangle$ is isomorphic to $\langle a, -a \rangle$ for all $a \in k^\times$, and therefore it has the curious property that

$$h \otimes q \cong h^{\oplus \dim(q)}. \tag{5}$$

Hence $\mathbb{Z}h \subset GW(k)$ is an ideal. The quotient ring

$$W(k) := \mathrm{GW}(k)/\mathbb{Z}h \tag{6}$$

is called the *Witt ring.*

If there exists $0 \neq v \in V$ such that $q(v) = 0$, q is called *isotropic.* A quadratic form that is not isotropic is *anisotropic.* Any quadratic form is of form $q \cong q_a \oplus h^{\oplus m}$, where q_a is the *anisotropic part of* q. By Witt's cancellation theorem, the isomorphism class of q_a is well defined. As every quadratic forms admits an embedding to a large enough hyperbolic space (e.g. $h \otimes q$), the elements of $W(k)$ correspond to isomorphism classes of anisotropic quadratic forms.

The kernel $I \subset W(k)$ of the rank map

$$\mathrm{rank} : W(k) \to \mathbb{Z}/2 \tag{7}$$

is called the *fundamental ideal.* The I-adic filtration will play a prominent role in this talk. The associated graded ring $\mathrm{Gr}^I_* W(k)$ is by definition

$$\mathrm{Gr}^I_n W(k) := I^n / I^{n+1}. \tag{8}$$

The Hauptsatz of Arason–Pfister [AP71] gives strong restrictions to quadratic forms that belong to powers of fundamental ideal.

Theorem 2 (Hauptsatz [AP71])**.** *If q is anisotropic form of positive dimension that belongs to I^n, then*

$$\dim(q) \geq 2^n. \tag{9}$$

In particular, the I-adic filtration of $W(k)$ is Hausdorff. The Pfister forms provide a generating system for the I-graded Witt ring.

Lemma 3. *The group $\mathrm{Gr}^I_n W(k)$ is generated as an abelian group by the classes of n-fold Pfister forms $\langle\langle a_1, \ldots, a_n \rangle\rangle$.*

Proof. It is enough to show that I/I^2 is generated by quadratic forms of form $\langle 1, -a\rangle$. But this is easy: as I/I^2 is 2-torsion, we have that

$$\langle 1, b\rangle + \langle 1, c\rangle = \langle b, c\rangle \in \mathrm{Gr}^I_1 W(k)$$

and as quadratic forms of the form $\langle b, c\rangle$ generate the fundamental ideal I, the claim follows. □

Hence, one may define a surjective ring homomorphism from the tensor algebra $T^*(k^\times) \twoheadrightarrow \mathrm{Gr}^I_* W(k)$ by the formula

$$(a_1, \ldots, a_r) \mapsto \langle\langle a_1, \ldots a_r \rangle\rangle \in \mathrm{Gr}^I_r W(k). \tag{10}$$

By the classification of twofold Pfister forms, this descends into a surjective ring homomorphism

$$s : K^M_*(k)/2 \twoheadrightarrow \mathrm{Gr}^I_* W(k) \tag{11}$$

The main result of this talk is that s is an isomorphism. In order to prove that, we need to prove the injectivity of s. The following partial result was known already in 1972.

Theorem 4. *The map*

$$s : K^M_*(k)/2 \to \mathrm{Gr}^I_* W(k)$$

is injective on pure symbols $\underline{a} = (a_1, \ldots, a_r)$.

Proof. This follows from [EL72, Theorem 3.2]. □

3 Fundamental exact sequence

Here we prove that the sequence of Eq. 3 is exact. We begin by stating a very general fact. Recall that if X is a smooth variety over k, then $\check{C}(X)$ is the simplicial scheme with $\check{C}(X)_n \cong X \times_k \cdots \times_k X$ ($n+1$ copies). It may be regarded as a subobject of $\mathrm{Spec}(k)$ in the sense that for a commutative ring R

$$\check{C}(X)(R) \simeq \begin{cases} * & \text{if } X \text{ has an } R\text{-point;} \\ \emptyset & \text{otherwise.} \end{cases} \tag{12}$$

Lemma 5. *For all $n \geq 0$, the sequence*

$$0 \to H^{n,n-1}(\check{C}(X);\mathbb{Z}/2) \xrightarrow{\tau} K_n^M(k)/2 \to K_n^M(K(X))/2 \tag{13}$$

is exact. The map τ corresponds to multiplication by the nontrivial element in $H^{0,1}(k;\mathbb{Z}/2) = \mu_2(k)$.

Proof. This exact sequence will come (essentially) from a long exact sequence induced by a cofibre sequence of motivic complexes. Note that there is a map of motives over k

$$\tau : \mathbb{Z}/2 \to \mathbb{Z}/2(1) \tag{14}$$

that corresponds to the nontrivial element of

$$\begin{aligned} \mathrm{Hom}(\mathbb{Z}/2, \mathbb{Z}/2(1)) &= H^{0,1}(k;\mathbb{Z}/2) \\ &= \{\pm 1\} \subset k^{\times}. \qquad (\mathbb{Z}/2(1) \simeq \mathbb{G}_m[1], \text{ see [MVW06]}) \end{aligned} \tag{15}$$

By the Beilinson–Lichtenbaum conjectures [Voe03a, §6], the canonical map

$$R\Gamma_{\mathrm{Zar}}(-;\mathbb{Z}/2(n)) \to \tau^{\leq n}\left(R\Gamma_{\text{ét}}(-;\mathbb{Z}/2)\right) \tag{16}$$

is an equivalence. Moreover, multiplication by τ induces the canonical maps between truncations[1]

$$\tau : \tau^{\leq n}\left(R\Gamma_{\text{ét}}(-;\mathbb{Z}/2)\right) \to \tau^{\leq n+1}\left(R\Gamma_{\text{ét}}(-;\mathbb{Z}/2)\right). \tag{17}$$

Hence, we obtain cofibre sequences

$$R\Gamma_{\mathrm{Zar}}(-;\mathbb{Z}/2(n-1)) \xrightarrow{\tau} R\Gamma_{\mathrm{Zar}}(-;\mathbb{Z}/2(n)) \to R\Gamma_{\mathrm{Zar}}(-;\underline{H}^{n,n}(\mathbb{Z}/2)[n]), \tag{18}$$

where $\underline{H}^{n,n}(\mathbb{Z}/s)$ is sheafification of the bidegree (n,n) motivic cohomology with $\mathbb{Z}/2$ coefficients.

From Eq. 18, we obtain the exact sequence

$$0 \to R^n\Gamma(-;\mathbb{Z}/2(n-1)) \to R^n\Gamma(-;\mathbb{Z}/2(n)) \to \Gamma(-;\underline{H}^{n,n}(\mathbb{Z}/2)). \tag{19}$$

Evaluating the above on $\check{C}(X)$, and using Beilinson–Lichtembaum conjecture and the fact that the canonical map $\check{C}(X) \to \mathrm{Spec}(k)$ is an étale-local equivalence, Eq. 19 transforms into

$$0 \to H^{n,n-1}(\check{C}(X);\mathbb{Z}/2) \to H^{n,n}(k,\mathbb{Z}/2) = K_*^M(k)/2 \to \Gamma(\check{C}(X);\underline{H}^{n,n}(\mathbb{Z}/2)). \tag{20}$$

[1] I didn't check this, but it seems reasonable. The case $n = 0$ is immediate from the definitions, and the other cases should not be much more difficult. Most likely the general case is a formal consequence of the case $n = 0$.

As $\underline{H}^{n,n}(\mathbb{Z}/2)$ is a homotopy invariant sheaf with transfers, pulling back to a Zariski dense open subscheme induces an injective homomorphism [MVW06, Lecture 11]. Checking compatibility with the simplicial structure, we deduce that the pullback

$$\Gamma(\check{C}(X);\underline{H}^{n,n}(\mathbb{Z}/2)) \hookrightarrow \Gamma(\check{C}(X);\underline{H}^{n,n}(\mathbb{Z}/2)) \tag{21}$$

is injective. The result follows by combining Eq. 20 and Eq. 21. □

Consider then a pure symbol $\underline{a} = (a_1, ..., a_n)$, and for simplicity, denote

$$\mathfrak{X}_{\underline{a}} := \check{C}(Q_{\underline{a}}) \tag{22}$$

As $\underline{a} = 0 \in K^M_*(K(Q_{\underline{a}}))$, we obtain a factorization

$$\begin{array}{ccccccc} & & K^M_{*-n}(k) & & & & \\ & & \downarrow \alpha & \searrow^{\underline{a}\cdot} & & & \\ 0 & \longrightarrow & H^{*,*-1}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2) & \xrightarrow{\tau} & K^M_*(k) & \longrightarrow & K^M_*(K(Q_{\underline{a}})) \end{array} \tag{23}$$

from the exactness of Eq. 13. It remains to be show that α is surjective.

Let us denote by $M_{\underline{a}}$ the *Rost motive*, which is a direct summand of the motive $M(Q_{\underline{a}})$ of $Q_{\underline{a}}$ [Ros98]. By [Voe03a, §4] we have a distinguished triangle of motives

$$M(\mathfrak{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathfrak{X}_{\underline{a}}) \xrightarrow{\mu'} M(\mathfrak{X}_{\underline{a}})(2^{n-1}-1)[2^n-1]. \tag{24}$$

As μ' is $K^M_*(k)/2$-linear, it coincides with multiplication by the element

$$\mu \in H^{2^n-1,2^{n-1}-1}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2) \tag{25}$$

corresponding to the composition of μ' with the projection $M(\mathfrak{X}_{\underline{a}}) \to M(k)/2$.

For convenience, let us define $K^M_*(k)$-modules

$$\mathbb{H}^i := \bigoplus_{n\in\mathbb{Z}} H^{n+i,n}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2). \tag{26}$$

Note that $\mathbb{H}^0 = K^M_*(k)/2$.

Lemma 6. *The multiplication by μ induces a surjection*

$$\mu\cdot : K^M_*(k)/2 \twoheadrightarrow \mathbb{H}^{2^{n-1}}. \tag{27}$$

In particular,

$$H^{2^n-1,2^{n-1}-1}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2) \cong \mathbb{Z}/2 \tag{28}$$

is generated by the nontrivial element μ.

Proof. As $M_{\underline{a}}$ is a summand of the motive of a smooth projective variety of dimension $2^{n-1}-1$, the vanishing lemma result of motivic cohomology implies that $H^{m+i,m}(M_{\underline{a}};\mathbb{Z}/2)\cong 0$ when $i>2^{n-1}-1$. Hence, the cofibre sequence of motives of Eq. 24 induces the exact sequence

$$K_m^M(k)/2 = H^{m,m}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2) \xrightarrow{\mu'} H^{m+2^n-1,m+2^{n-1}-1}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2) \to 0, \tag{29}$$

from which the claim follows. □

Next, we deduce the surjectivity of α from that of $\mu\cdot$. Composition of *Milnor operators* $Q_i : H^{*,*}(-;\mathbb{Z}/2)\to H^{*+2^{i+1}-1,*+2^i-1}(-;\mathbb{Z}/2)$ (see [Voe03b, Voe03a]) defines a map

$$d := Q_{n-2}\cdots Q_0 : \mathbb{H}^1 \to \mathbb{H}^{2^{n-1}} \tag{30}$$

of graded Abelian groups. In fact, d is a map of $K_*^M(k)/2$-modules.

Lemma 7. *The map $d:\mathbb{H}^1\to\mathbb{H}^{2^{n-1}}$ is $K_*^M(k)/2$-linear.*

Proof. By [Voe03b, Proposition 13.4], we have that

$$Q_i(xy) = Q_i(x)y + xQ_i(y) + \rho\sum_{E,F} c_{E,F}Q^E(x)Q^F(y), \tag{31}$$

where E,F range over subsets of $\{0,\dots,n-1\}$, Q^E is the cohomology operation $\prod_{e\in E}Q_e$, $\rho=[-1]\in H^{1,1}(k;\mathbb{Z}/2)$, and $c_{E,F}\in H^{*,*}(k;\mathbb{Z}/2)$. By the vanishing lemma, the Milnor operators Q_i kill elements of K_*^M. Hence, Eq. 31 implies that d is $K_*^M(k)/2$-linear. □

We will show that d is an isomorphism below.

Lemma 8. *The map $d:\mathbb{H}^1\to\mathbb{H}^{2^{n-1}}$ is an injection.*

Proof. Denote by $\tilde{\mathfrak{X}}_{\underline{a}}$ the cone of $\mathfrak{X}_{\underline{a}}\to\operatorname{Spec}(k)$, and $\tilde{\mathbb{H}}^i := \bigoplus_{m\in\mathbb{Z}} H^{m+i,m}(\tilde{\mathfrak{X}}_{\underline{a}};\mathbb{Z}/2)$. The vanishing lemma implies that we obtain natural isomorphisms

$$\mathbb{H}^i \to \tilde{\mathbb{H}}^{i+1}$$

for $i\geq 1$, which are compatible with cohomology operations. Hence, it suffices to show the injectivity of

$$\tilde{d} : \tilde{\mathbb{H}}^2 \to \tilde{\mathbb{H}}^{2^{n-1}+1}.$$

By [Voe03a, Corollary 3.8], the the sequence

$$\tilde{\mathbb{H}}^{m-2^i} \xrightarrow{Q_i} \tilde{\mathbb{H}}^m \xrightarrow{Q_i} \tilde{\mathbb{H}}^{m+2^i}$$

is exact for all m. The numerics aligns itself in such a fashion that the injectivity of $\tilde{d}$ follows from the triviality of $\tilde{\mathbb{H}}^1$, which in turn follows from the isomorphism $H^{p,q}(\mathfrak{X}_{\underline{a}};\mathbb{Z}/2)\to H^{p,q}(k;\mathbb{Z}/2)$ for $p\leq q$ and the vanishing lemma. □

Let $\underline{\tilde{a}} \in H^{n,n-1}(\mathfrak{X}_a; \mathbb{Z}/2)$ be the lift of $\underline{a}$ along τ (Eq. 23).

Lemma 9. *The composition*

$$K_*^M(k)/2 \xrightarrow{\underline{\tilde{a}}\cdot} \mathbb{H}^1 \xrightarrow{d} \mathbb{H}^{2^{n-1}} \tag{32}$$

coincides with multiplication by μ. In particular, d is a surjection.

Proof in the case $\underline{a} \neq 0 \in K_n^M(k)$. As both maps are $K_*^M(k)/2$-linear, it suffices that both maps obtain the same value on 1. Thus, we want to show that $d(\underline{\tilde{a}}) = \mu$. As $\underline{a} \neq 0$, then neither is $d(\underline{\tilde{a}})$, so it must coincide with μ by Lem. 8. □

In particular $\alpha = \underline{\tilde{a}}\cdot$ is surjective, as desired. Hence, we have proven the following result.

Theorem 10. *The sequence*

$$K_{*-n}^M(k) \xrightarrow{\underline{a}\cdot} K_*^M(k) \to K_*^M(K(Q_{\underline{a}})), \tag{33}$$

is exact (well, at least if $\underline{a} \neq 0$). □

4 Proof of the Milnor conjecture on quadratic forms

Here, we prove the Milnor conjecture on quadratic forms. In other words, we have to show that the map $s : K_*^M(k)/2 \to \mathrm{Gr}_*^I(k)$ (Eq. 11) is injective. We already know that s is an injection when restricted to pure symbols. Using the following Lemma, we can restrict to that case.

Lemma 11. *Let $0 \neq \alpha \in K_n^M(k)/2$. Then there exists a field extension L/k (not necessarily algebraic) such that*

$$\alpha = \underline{a} \in K_n^M(L)/2 \tag{34}$$

for some non-zero pure symbol $\underline{a} \in K_n^M(L)/2$.

Proof. Write

$$\alpha = n_1\underline{a_1} + \cdots + n_r\underline{a_r} \tag{35}$$

for some non-zero pure symbols $\underline{a_i} \in K_n^M(k)/2$ and non-zero integers n_i. Define

$$K_i := K(Q_{\underline{a_1}} \times_k \cdots \times_k Q_{\underline{a_i}}) \tag{36}$$

(this makes sense because quadrics of positive dimension are geometrically connected). By the fundamental exact sequence (Eq. 3)

$$\alpha = 0 \in K_n^M(K_r)/2. \tag{37}$$

Let j be the smallest integer such that $\alpha = 0 \in K_n^M(K_{j+1})$. By the exactness of

$$\mathbb{Z}/2 \cong K_0^M(K_j)/2 \xrightarrow{\underline{a_{j+1}}} K_n^M(K_j)/2 \to K_n^M(K_{j+1})/2, \tag{38}$$

we observe that

$$\alpha = \underline{a_{j+1}} \in K_n^M(K_j)/2. \tag{39}$$

Hence, we may set $L = K_j$. □

Proof of the Milnor conjecture on quadratic forms. Let $L, \underline{a}$ be as in the statement of Lem. 11. Then the claim follows from the commutativity of

$$\begin{array}{ccc} K_*^M(k)/2 & \xrightarrow{s} & K_*^M(L)/2 \\ \downarrow & & \downarrow \\ \mathrm{Gr}_*^I W(k) & \xrightarrow{s} & \mathrm{Gr}_*^I W(L) \end{array} \tag{40}$$

and the fact that $s(\underline{a}) \neq 0$. □

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