Derived algebraic cobordism

Towards bivariant Conner–Floyd theorem

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Motives and What Not, June 17 2020

Goals

- Recall classical algebraic bordism.

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- Introduce derived algebraic cobordism.
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- Give examples of open questions.
- Focus on one in particular (trying to prove bivariant Conner–Floyd).

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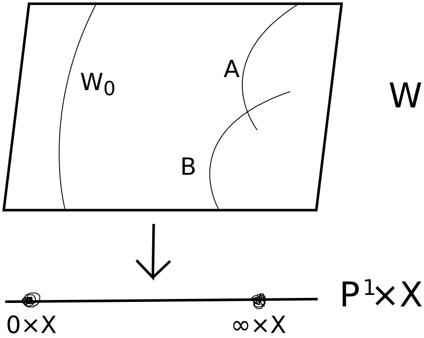
then

$$[W_0 \to X] = [A \to X] + [B \to X] - [\mathbb{P}_{A\cap B}(\mathcal{N}_{A\cap B/A} \oplus \mathcal{O}) \to X]$$

in $\Omega_i(X)$.

Double point cobordism pictorially



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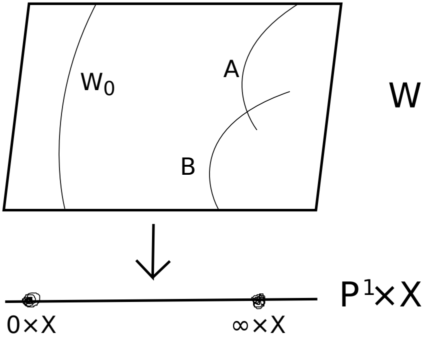

$$[W_0 \to X] = [A \to X] + [B \to X] - [\mathbb{P}_{A\cap B}(\mathcal{N}_{A\cap B/A} \oplus \mathcal{O}) \to X] \in \Omega_i(X)$$

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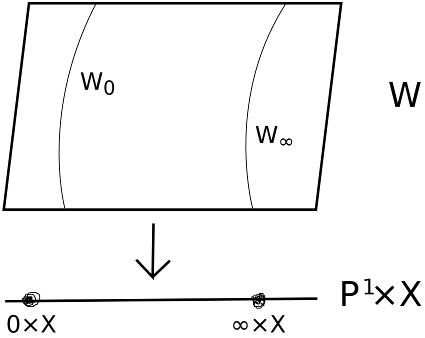

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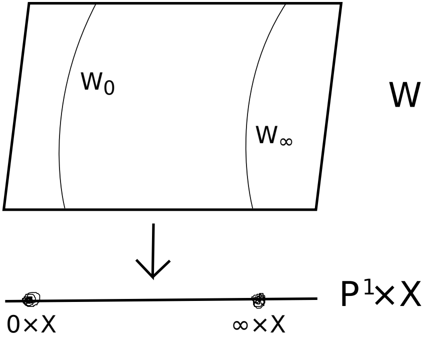

$$[W_0 \to X] = [W_\infty \to X] \in \Omega_i(X)$$

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- More generally, if $f : X \to Y$ is l.c.i. of pure relative dimension d get pullback

$$f^! : \Omega_*(Y) \to \Omega_{*+d}(X),$$

but no nice formula in general.

Basic properties of Ω_* cont.

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- $\Omega_*(X)$ is a module over $\Omega_*(\mathrm{Spec}(k))$, action given by

$$[V \to \mathrm{Spec}(k)] \cdot [W \to X] := [V \times W \to X]$$

Chern classes in Ω_*

Given a line bundle $\mathscr{L}/X$, can define *Chern class operator*

$$c_1(\mathscr{L}) := s^! \circ s_* : \Omega_*(X) \to \Omega_{*-1}(X),$$

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Then

$$\begin{aligned} c_1(\mathscr{L}_1 \otimes \mathscr{L}_2) &= c_1(\mathscr{L}_1) + c_1(\mathscr{L}_2) \\ &\quad + \sum_{i,j \geq 1} a_{ij} \cdot c_1(\mathscr{L}_1)^i \circ c_1(\mathscr{L}_1)^j, \end{aligned}$$

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- $\mathbb{Z}_a$ = integers with formal group law $x + y \Rightarrow$ then

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- $\mathbb{Z}_m$ = integers with formal group law $x + y - xy \Rightarrow$ then

$$\mathbb{Z}_m \otimes_{\mathbb{L}} \Omega_*(X) \cong K_0(X)$$

$$[V \xrightarrow{f} X] \mapsto [Rf_*(\mathcal{O}_V)].$$

(We call this Conner–Floyd.)

Other properties

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- *Homotopy invariance*: E/X rank r vector bundle $\Rightarrow$

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$$\mathbb{Z}_a \otimes_{\mathbb{L}} \Omega^*(X) \cong \mathrm{CH}^*(X)$$

and

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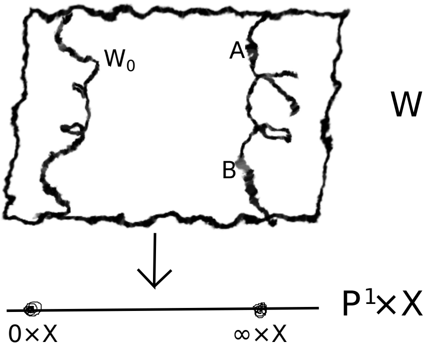

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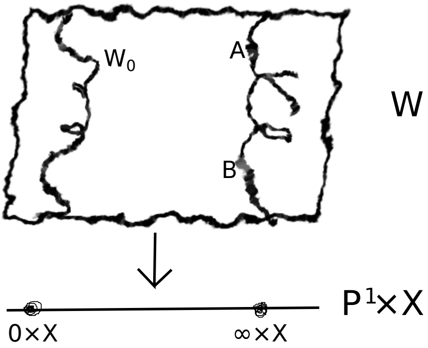

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- If $f : X \to Y$ is projective and quasi-smooth of relative dimension $-d$, get *pushforward* $f_! : \underline{\Omega}^*(X) \to \underline{\Omega}^{*+d}(Y)$

$$f_![V \xrightarrow{g} X] \to [V \xrightarrow{f \circ g} Y].$$

Cohomological Conner–Floyd

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Comparable with the classical result if $A = k$, $\mathrm{char}(k) = 0$ and X is smooth. $\underline{\Omega}^*$ is not homotopy invariant!

Derived Grothendieck–Riemann–Roch

We can define *pre Chow-ring*

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commuting with pullbacks, commuting with pushforwards twisted by a Todd class, and $\mathbb{Q} \otimes ch$ *is an isomorphism.*

Projective bundle formula

Given a vector bundle E of rank r on X, have *Chern classes*

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Theorem (A.-Yokura, A.)

If E is a rank r vector bundle on X, then

$$\underline{\Omega}^*(\mathbb{P}(E)) \cong \underline{\Omega}^*(X)[t]/\big(t^r - c_1(E^\vee)t^{r-1} + \cdots + (-1)^r c_r(E^\vee)\big),$$

with $t \sim c_1(\mathcal{O}(1))$.

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- $\underline{\Omega}^*$ has a natural extension to a bivariant theory $(\underline{\Omega}^*(X) = \underline{\Omega}^*(X \to X))$. Is there a bivariant version of Conner–Floyd

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- $\underline{\Omega}^*$ homotopy invariant for smooth schemes over a field (regular base)? Would follow from being able to find quasi-smooth compactifications of quasi-smooth schemes + little more.

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- What about localization? Is there a short exact sequence

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  for $Z \hookrightarrow X$ closed and U the open complement. What is $\underline{\Omega}^*(X)_Z$?
- Higher $\underline{\Omega}^*$? Is $\underline{\Omega}^*$ representable in some sort of motivic homotopy theory without $\mathbb{A}^1$-invariance?

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However, such a statement is likely false. Need to add more relations to $\underline{\Omega}^*$ to get *Algebraic Cobordism* Ω^*. Being able to prove bivariant Conner–Floyd would be a good indicator that we have found the right definition of Ω^*.

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Motivation for bivariant Conner–Floyd 2

This problem is also easier than the other open problems: it is much easier to understand quasi-coherent sheaves on derived schemes, than it is to understand, for example, finding quasi-smooth compactifications.

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on equivalence classes of projective morphisms to X, so that the composition $V \to Y$ is quasi-smooth of relative dimension $-i$. The relations are the analogues of the derived double point cobordism relations.

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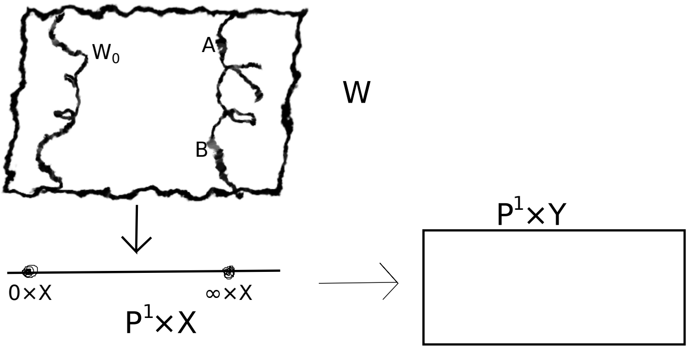
then

$$[W_0 \to X] = [A \to X] + [B \to X] \\ - [\mathbb{P}_{A \cap^h B}(\mathcal{N}_{A \cap^h B/A} \oplus \mathcal{O}) \to X]$$

in $\underline{\Omega}^i(X \hookrightarrow Y)$.

Bivariant double point cobordism pictorially



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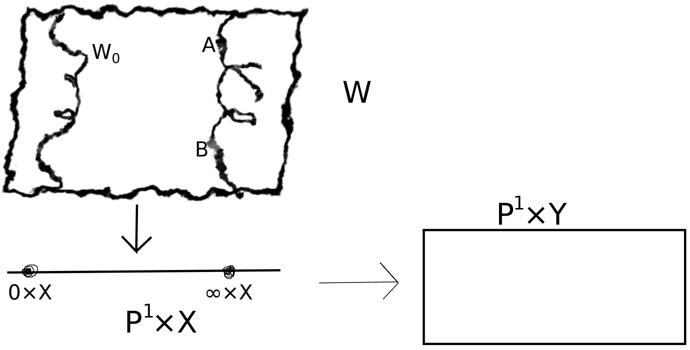

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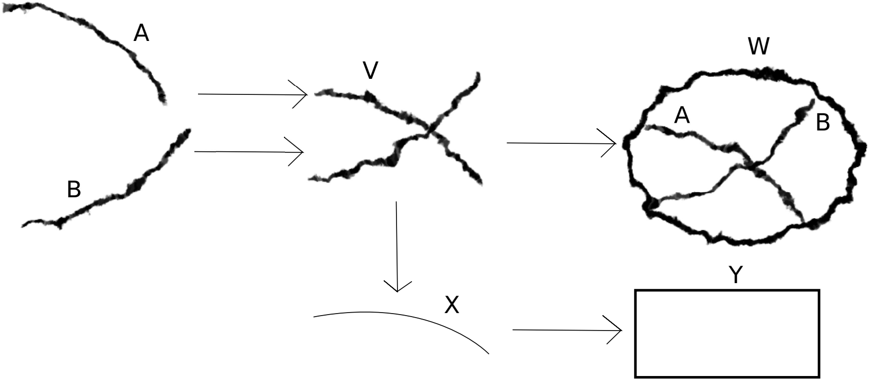
Enforcing the relation

$$[V \to X] = [A \to X] + [B \to X] - [\mathbb{P}_{A \cap^h B}(\mathcal{N}_{A \cap^h B/A} \oplus \mathcal{O}) \to X]$$

for all such, we obtain $\Omega^*(X \hookrightarrow Y)$.

Extra relations pictorially



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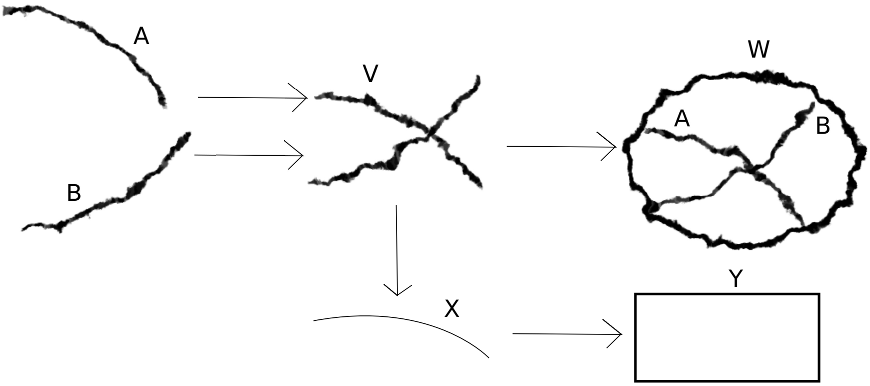

We are now able to break more cycles (e.g. $[V \to X]$) into simpler pieces.

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(Addition $\sim$ cofibre sequences).

Strategy of the proof

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Need to find an inverse.

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$$c_1(\mathcal{F}) = c_1(\mathcal{F}') + c_1(\mathcal{F}'')$$

for

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

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However, $c_1(E_j)$ do not lift to $\Omega^*(X \hookrightarrow Y)$.

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Above, CK^* is the *connective K-theory*.

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Constructing localized Chern classes for CK^* is enough for the most important applications (bivariant CF, GRR): CK^* "interpolates" between CH^* and K^0.

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Theorem (Hudson–Ikeda–Matsumura–Naruse)

$$\pi_{f!}\left(c_1(\mathcal{O}(1))^{k-1}\right) = \sum_{j=k}^{\infty} (-\beta)^{j-k} \binom{j-1}{j-k} c_i(\mathrm{Cone}(f))$$

in $\mathrm{CK}^k(Y)$.

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Corollary

$$c_k\left(\mathrm{Cone}(f)\right) = \pi_{f!}\left(\frac{c_1(\mathcal{O}(1))^{k-1}}{\left(1-\beta c_1(\mathcal{O}(1))\right)^k}\right)$$

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In particular,

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in $\mathrm{CK}^*\big(V(\mathrm{det}(f)) \hookrightarrow Y\big)$.

π'_{f*} is a bivariant pushforward

$$\mathrm{CK}^*(\mathrm{Deg}_f(Y) \to \mathrm{Deg}_f(Y)) \to \mathrm{CK}^*(V(\mathrm{det}(f)) \hookrightarrow Y)$$

$$[W \to \mathrm{Deg}_f(Y)] \mapsto [W \to V(\mathrm{det}(f))].$$

Splitting principle

We want to show that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E' & \longrightarrow & E & \longrightarrow & E'' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\ 0 & \longrightarrow & F' & \longrightarrow & F & \longrightarrow & F'' & \longrightarrow & 0 \end{array}$$

implies

$$c_1^{\mathrm{loc}}\big(\mathrm{Cone}(f)\big) := c_1^{\mathrm{loc}}\big(\mathrm{Cone}(f')\big) + c_1^{\mathrm{loc}}\big(\mathrm{Cone}(f'')\big)$$

in $\mathrm{CK}^1\big(V(\det(f)) \hookrightarrow Y\big)$.

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Let's sketch a proof for the case where $r = 2$.

Special case of splitting principle

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{L}_1 & \longrightarrow & \mathscr{L}_1 \oplus \mathscr{L}_2 & \longrightarrow & \mathscr{L}_2 & \longrightarrow & 0 \\
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Want:

$$c_1^{\mathrm{loc}}\left(\mathrm{Cone}(\psi_1 \oplus \psi_2)\right) := c_1^{\mathrm{loc}}\left(\mathrm{Cone}(\psi_1)\right) + c_1^{\mathrm{loc}}\left(\mathrm{Cone}(\psi_2)\right)$$

in $\mathrm{CK}^1\big(V(\det(\psi_1 \oplus \psi_2)) \hookrightarrow Y\big)$.

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Then

$$\begin{array}{ccc} \mathrm{Deg}_f(Y) = V(s_1) \cap^h V(s_2) & \hookrightarrow & \mathbb{P}(\mathscr{L}_1 \oplus \mathscr{L}_2) \\ & \searrow & \downarrow \\ & & Y \end{array}$$

$V(s_1)$ and $V(s_2)$ pictorially

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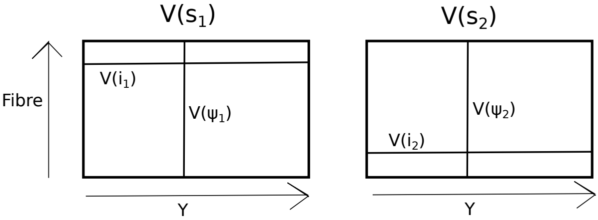
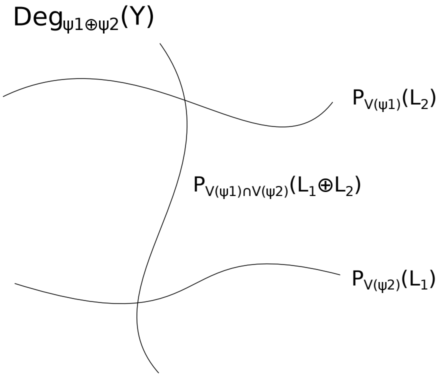


Picture of the degeneracy scheme



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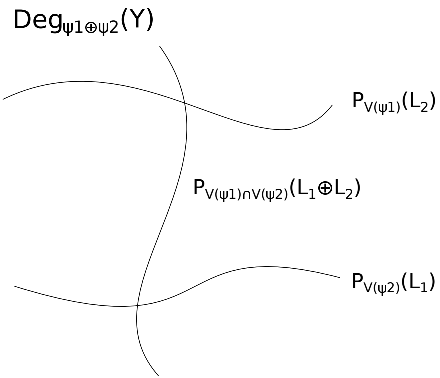

SNC-relations $\Rightarrow$
$[\mathrm{Deg}_{\psi_1 \oplus \psi_2}(Y) \to \mathrm{Deg}_{\psi_1 \oplus \psi_2}(Y)] \in \mathrm{CK}^1\left(\mathrm{Deg}_{\psi_1 \oplus \psi_2}(Y) \to Y\right)$
breaks into pieces.

Splitting principle 4

Apply this to

$$\frac{[\mathrm{Deg}_{\psi_1 \oplus \psi_2}(Y) \to \mathrm{Deg}_{\psi_1 \oplus \psi_2}(Y)]}{1 - \beta c_1(\mathcal{O}(1))} \in \mathrm{CK}^1\big(\mathrm{Deg}_{\psi_1 \oplus \psi_2}(Y) \to Y\big)$$

and push forward to $\mathrm{CK}^1\big(V(\det(\psi_1 \oplus \psi_2)) \hookrightarrow Y\big)$.

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in $\mathrm{CK}^1\big(V(\det(\psi_1 \oplus \psi_2)) \hookrightarrow Y\big)$. Generalizing to higher ranks is easy.

Longer complexes

Not clear how to generalize this to longer complexes.

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Not clear how to generalize this to longer complexes.

- Not clear if a longer complex could somehow be factored into shorter ones

$$\begin{array}{ccccccccc}
 & & Q_3 & \longrightarrow & \mathcal{E}_2 & & Q_1 & & \\
 & & \uparrow & & \downarrow & & \uparrow & \searrow & \\
\cdots & \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 \\
 & & \uparrow & & \downarrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathcal{E}_3 & & Q_2 & \longrightarrow & \mathcal{E}_1 & &
\end{array}$$

Longer complexes

Not clear how to generalize this to longer complexes.

- Not clear if a longer complex could somehow be factored into shorter ones

$$\begin{array}{ccccccccc}
 & & Q_3 & \longrightarrow & \mathcal{E}_2 & & Q_1 & & \\
 & & \uparrow & & \downarrow & & \uparrow & \searrow & \\
\cdots & \longrightarrow & E_3 & \longrightarrow & E_2 & \longrightarrow & E_1 & \longrightarrow & E_0 \\
 & & \uparrow & & \downarrow & & \uparrow & & \\
\cdots & \longrightarrow & \mathcal{E}_3 & & Q_2 & \longrightarrow & \mathcal{E}_1 & &
\end{array}$$

- What should replace degeneracy schemes for longer complexes?

Thank you!

- Annala, T.: Chern classes in precobordism theories, *submitted*, 2019.
- Annala, T.: Precobordism and cobordism, *preprint*, 2020.
- Annala, T.; Yokura, S.: Bivariant algebraic cobordism with vector bundles, *submitted*, 2019.
- Fulton, W.; MacPherson, R.: Categorical framework for the study of singular spaces, *Memoirs of the American Mathematical Society*, 31(243), 1981.
- Hudson, T.; Ikeda, T.; Matsumura, T.; Naruse, H.: Degeneracy loci classes in K-theory — determinantal and Pfaffian formula, *Advances in Mathematics* 320:115–156, 2017.
- Hudson, T.; Matsumura, T.: Segre classes and Damon–Kempf–Laksov formula in algebraic cobordism, *Mathematische Annalen*, 374:1439–1457, 2019.

- Levine, M.; Morel, F.: Algebraic Cobordism, *Springer Monographs in Mathematics*, 2007.
- Levine, M.; Pandharipande, R.: Algebraic cobordism revisited. *Inventiones Mathematicae*, 176(1):63–130, 2009.
- Lowrey, P.; Schürg, T.: Derived algebraic cobordism, *Journal of the Institute of Mathematics of Jussieu*, 15(2):407–443, 2016.