Diagram Chasing in Abelian Categories

Toni Annala

Contents

Chapter 1

Overview

This is a short note, intended only for personal use, where I fix diagram chasing in general abelian categories. I didn't want to take the Freyd-Mitchell embedding theorem for granted, and I didn't like the style of the Freyd's book on the topic. Therefore I had to do something else. As this was intended only for personal use, and as I decided to include this to the application quite late, I haven't touched anything in chapter 2. Some vague references to "Freyd's book" are made in the passing, they mean the book Abelian Categories by Peter Freyd.

How diagram chasing is fixed then? The main idea is to chase subobjects instead of elements. The sections 2.1 and 2.2 contain many standard statements about abelian categories, proved perhaps in a nonstandard way. In section 2.3 we define the image and inverse image functors, which let us transfer subobjects via a morphism of objects. The most important theorem in this section is probably 2.3.11, which states that for a subobject U of X, and a morphism $f : X \to Y$, we have $ff^{-1}U = U \cap \mathrm{im} f$. Some other results are useful as well, for example 2.3.2, which says that the image functor associated to a monic morphism is "injective".

Finally, the section 2.4 contains proofs of five lemma, nine lemma, snake lemma and zigzag lemma. As snake lemma requires showing the existence of a morphism satisfying certain assumptions, we need prove some additional things before being able to prove it. The most important of these is 2.4.11.

Chapter 2

Abelian Categories

2.1 Definition and basic properties

Definition 2.1.1. A category $\mathcal{C}$ is *additive* if

1. $Hom_{\mathcal{C}}(X,Y)$ is an abelian group for all X and Y.
2. The composition of morphisms is biadditive.
3. $\mathcal{C}$ has a zero object.
4. $\mathcal{C}$ has finite products.
5. $\mathcal{C}$ has finite coproducts.

It is well known, that if the assumptions 1-3 above are satisfied then 4 and 5 are equivalent, and moreover they are equivalent to

6. Given objects X and Y, we have an object $X \oplus Y$ together with maps $i_1 : X \to X \oplus Y$, $i_2 : Y \to X \oplus Y$, $p_1 : X \oplus Y \to X$ and $p_2 : X \oplus Y \to Y$ that satisfy $p_a \circ i_b = \delta_{ab}$ and $i_1 \circ p_1 + i_2 \circ p_2 = 1$.

The object in 6, together with the four maps satisfying the above conditions, is called the *direct sum* of X and Y. The "projections" p_1, p_2 make it a direct product and the "injections" i_1, i_2 make it a direct sum.

The notion of "zero map" could now have two different meanings, namely a map that factors trough a zero object and a map that is the the zero element of the abelian group of morphisms. These two notions however coincide, which is seen from the biadditivity of composition of morphisms.

It is clear that if $\mathcal{C}$ is additive then the opposite category $\mathcal{C}^{op}$ is additive as well. Therefore we can prove pairs of dual statements in additive categories by proving either one of them.

Definition 2.1.2. An additive category $\mathcal{C}$ is *abelian* if

1. $\mathcal{C}$ has kernels and cokernels.

2. Any monomorphism is the kernel of its cokernel.

3. Any epimorphism is the cokernel of its kernel.

Again it is clear that the opposite category of an abelian category is still abelian, so we may use duality arguments. One does not need to assume that an abelian category comes with additive structure, if we drop the assumptions 1 and 2 in 2.1.1 then one can define an additive structure for the category using the other assumptions. (See for example <insert Freyd's book here>). From now on, all the categories are assumed to be abelian unless otherwise stated.

Using the fact that limits (colimits) can be constructed using only products and kernels (coproducts and cokernels), we obtain the following.

Proposition 2.1.3. *Abelian categories have finite limits and finite colimits.*

Especially, fibre products exist in abelian categories.

In abelian categories monomorphisms and epimorphisms have the following alternative descriptions

Proposition 2.1.4. *Let $f : X \to Y$ be a map in an abelian category. The following are equivalent:*

1. *f is a monomorphism.*

2. *Composition by f sends nonzero maps to nonzero maps.*

3. *The kernel of f is* 0.

Proof. 2 follows from 1 trivially. If 2 holds, then for $U \to X \to Y = 0$ we have that $U \to X = 0$, so it factors uniquely trough 0, giving us 3. If we have two different maps g and g' such that $f \circ g = f \circ g'$, then $g - g'$ is nonzero but $f \circ (g - g') = 0$. As nonzero maps cannot factor trough 0, we see that 0 cannot be the kernel of f. Therefore 1 follows from 3. □

Dually we have:

Proposition 2.1.5. *Let $f : X \to Y$ be a map in an abelian category. The following are equivalent:*

1. *f is an epimorphism.*

2. *Precomposition by f sends nonzero maps to nonzero maps.*

3. *The cokernel of f is* 0*.*

For abelian groups, modules over a ring or essentielly any other type such algebraic structure, a bijective morphism is always an isomorphism. This generalizes to abelian categories.

Proposition 2.1.6. *A map $f : X \to Y$ is isomorphism if and only if it is both monic and epic.*

Proof. Clearly an isomorphism is both epic and monic. Let $f : X \to Y$ be both an epimorphism and a monomorphism. Now $Y \to 0$ is its cokernel. Now by definition of abelian categories, $X \to Y$ is its kernel. On the other hand, clearly also $Y \xrightarrow{1} Y$ is its kernel. By essential uniqueness of kernel we have an isomorphism $g : X \xrightarrow{\sim} Y$ s.t.

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle 1} \\ & & Y \end{array}$$

commutes, i.e., $f = g$ and f is an isomorphism. $\square$

Proposition 2.1.7. *If f is monic, then g and $f \circ g$ have the same kernel. If g is epic, then f and $f \circ g$ have the same cokernel.*

Proof. By monicness of f, we have that $f \circ g \circ a = 0$ if and only id $g \circ a = 0$, so the kernels must be the same. The second follows from duality. $\square$

2.2 Subobjects and quotient objects

We begin with the following observation, which follows directly from the definition of monomorphism.

Proposition 2.2.1. *Let us have a monomorphism $U \to X$ and two monomorphisms $V \to U$ and $V' \to U$. Let $V \to V'$ be a map. Now*

$$\begin{array}{ccc} V & \longrightarrow & V' \\ \downarrow & \swarrow & \\ U & & U \\ \downarrow & \swarrow & \\ X & & \end{array}$$

commutes if and only if

$$\begin{array}{ccc} V & \longrightarrow & V' \\ \downarrow & \swarrow & \\ U & & \end{array}$$

commutes.

Definition 2.2.2. A *map* of objects *over X* is a map of objects U_1 and U_2 with *structure maps* $U_i \to X$ s.t.

$$\begin{array}{ccc} U_1 & \longrightarrow & U_2 \\ & \searrow & \downarrow \\ & & X \end{array}$$

commutes. We define the category of *subobjects* of X to be subcategory of objects over X with monic structure maps modulo isomorphisms. As the structure maps are monic, there exists at most one map between any two subobjects. The map must be a monomorphism as well.

The subobjects have the following order relation: we say that U is contained in U' if there is a map of subobjects $U \to U'$. This is sometimes denoted by $U \leq U'$. From the opening proposition of this section, we see that the subobjects of a subobject U can be naturally regarded as subobjects of X contained in U and vice versa.

Definition 2.2.3. Dually to the definition of subobjects one can define the *quotient objects* of X as equivalence classes of epimorphisms $X \to U$. If there exists a map of quotient objects $U \to U'$, then we say that U *covers* U' ($U \geq U'$).

By the definition of abelian category, we have a bijection between the subobjects of X and the quotient objects of X. This bijection is given by assigning each subobject its cokernel, and its inverse by assigning each quotient object its kernel. It is easy to see the following:

Proposition 2.2.4. *The bijection between subobjects and quotient objects is order reversing.*

To simplify notation, and to show clearer the similarity of the theory of abelian categories and the theory of, for example, abelian groups, given a subobject $K \to X$, we denote its cokernel by X/K.

Definition 2.2.5. Let U and U' be subobjects of X. We define the *intersection* of U and U', denoted by $U \cap U'$, to be the largest subobject of X contained both in U and in U'.

We define the *sum* of U and U', denoted by $U + U'$, to be the smallest subobject of X containing both U and U'.

The intersection is very easy to describe.

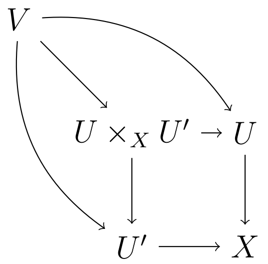
Proposition 2.2.6. *The intersection of two subobjects U and U' of X exists, and is isomorphic to $U \times_X U'$, where the fibre product is taken inside the original abelian category.*

Proof. It is known that now $U \times_X U' \to U$ is monic, and thus is $U \times_X U' \to U \to X$, i.e., $U \times_X U'$ defines a subobject of X. Let us have a subobject V of X contained both in U and in U'. Now the following diagram commutes

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ U' & \longrightarrow & X \end{array}$$

by the definition of maps of subobjects of X.

Now we have an unique map $V \to U \times_X U'$ s.t.



commutes, and thus $V \to U \times_X U'$ is a map of subobjects of X. This proves that $U \times_X U' = U \cap U'$. □

Definition 2.2.7. Given a map $f : X \to Y$, we define its *image* to be the smallest subobject of Y trough which f factors. Dually one has the notion of *coimage* of f, namely the smallest quotient object of X trough which f factors.

The following propositions are direct consequences of the definitions.

Proposition 2.2.8. *Given a monomorphism $f : X \to Y$ the subobject defined by the equivalence class of f is the image of f.*

Given an epimorphism $f : X \to Y$ the quotient object defined by the equivalence class of f is the coimage of f.

Proposition 2.2.9. *The image of $X \to Y$ is 0 iff the map is the zero map.*

At this point, the existence of images is not clear. But as it turns out, the image does not only exist, but also has a nice description by kernels and cokernels. Before proving this, we will need a couple of lemmas.

Lemma 2.2.10. *Let U and U' be subobjects of X. If U is not contained in U', then $U \cap U'$ is properly contained in U.*

Proof. The intersection must be contained in U, but by assumptions it cannot be U, so it is properly contained in U. □

Lemma 2.2.11. *Let U and U' be subobjects of Y. If $f : X \to Y$ factors trough U and U', then it factors trough $U \cap U'$.*

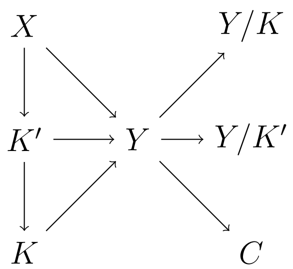
Proof. This is just the universal property of fibered product. □

Theorem 2.2.12. *Let $f : X \to Y$ be a map. Its image is given by the kernel of $Y \to$* $\operatorname{coker}(f)$.

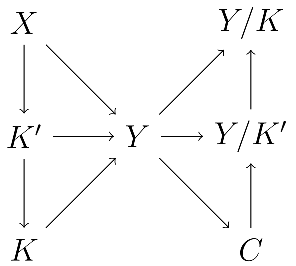
Dually, the coimage of f is given by the cokernel of $\ker(f) \to X$*, i.e., it is* $X/\ker(f)$.

Proof. Denote by C the cokernel of f and by K the kernel of $Y \to C$. Now, as $X \to Y \to C = 0$, f will factor trough K. Assume that K is not the image of f, i.e., there is a subobject K' of Y not containing K s.t. f factors trough it. By previous lemmas, we may assume that K' is properly contained in K.

Now the situation may be illuminated using the following commutative diagram:



Using the universal property of cokernels and the commutativity of the diagram, we see that we can complete the previous diagram to obtain the following:



By the definition of abelian category, we have that $Y/K = C$, so $C \leq Y/K' \leq C$, i.e., $Y/K' = C = Y/K$, which then proves that $K' = K$, a contradiction. Thus K is the image of f. □

The following alternative description of image is sometimes useful.

Proposition 2.2.13. *Let us have a map $f : X \to Y$. A subobject U of Y trough which f factors is the image of f if and only if the factored map $X \to U$ is epic.*

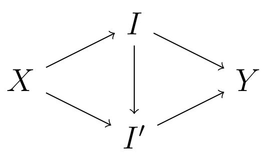
Dually, a quotient object U of X is the coimage if and only if the factored map $U \to Y$ is monic.

Proof. If $X \to \mathrm{im} f$ wouldn't be epic, then there would be a proper subobject of $\mathrm{im} f$, trough which it would factor, which would give a subobject of Y properly contained in $\mathrm{im} f$, trough which f would then factor. This is, of course, a contradiction.

Let U be a subobject of Y s.t. $X \to U$ is epic. Let K be a subobject of Y strictly contained in U. Now $U \to U/K$ is a nonzero map, so f cannot factor trough K, lest $f \to U \to U/K = 0$ contradicting the epicness. □

This gives us the following factorization theorem for maps in abelian categories.

Theorem 2.2.14. *Let $X \to Y$ be a morphism in abelian category. Now it has factorization $X \to I \to Y$, where $X \to I$ is epic and $I \to Y$ is monic. Moreover, if $X \to I' \to Y$ is another such factorization, then we have an isomorphism $I \xrightarrow{\sim} I'$ s.t.*



commutes.

Proof. The existence follows from the previous proposition. Both I and I' are now the image of $X \to Y$, so we have an isomorphism s.t. the right triangle in the above diagram commutes. By monicness of $I' \to Y$ also the left triangle commutes, which proves the claim. □

The next proposition is just compilation of various facts about images that are now immediate.

Proposition 2.2.15. *Let us have a map $f : X \to Y$.*

1. *The coimage of f is isomorphic to the image of f.*
2. *The map f is epic if and only if the image of f is Y.*
3. *The image of f is 0 if and only if f is the zero map.*
4. *If g is epic, then f and $f \circ g$ have the same image.*

Proof.

1. 2.2.13
2. 2.1.5 and the fact that $f : X \to Y$ is zero if and only if X is its kernel.
3. A nonzero map cannot factor trough a zero object.
4. By 2.1.7 they have the same cokernel, which proves the claim. □

Using the existence of images, we may show that the sum of subobjects exists.

Proposition 2.2.16. *Let U and U' be subobjects of X. Now the sum $U + U'$ exists and is the image of $U \oplus U' \to X$.*

Proof. Precomposing with the canonical injections, one sees that the image of $U \oplus U' \to X$ must contain both U and U'. On the other hand, if a subobject V contains both U and U', then the map $U \oplus U' \to X$ factors trough it, so it contains the image of $U \oplus U' \to X$. □

2.3 The image and inverse image functors

Definition 2.3.1. Let $f : X \to Y$ be a map. We define the *image functor associated to* f from the category of subobjects of X to the category of subobjects of Y by sending each subobject U to the image of $U \to X \to Y$. If we have a map of subobjects $U \to U'$, then fU' is a subobject trough which $U \to X \to Y$ factors, so we have a map $fU \to fU'$. Functorality is satisfied by the uniqueness of maps of subobjects.

Using the image functor, we obtain yet another characterization for monicness of a map.

Proposition 2.3.2. *A map $f : X \to Y$ is monomorphism if and only if the image functor associated to f is injective.*

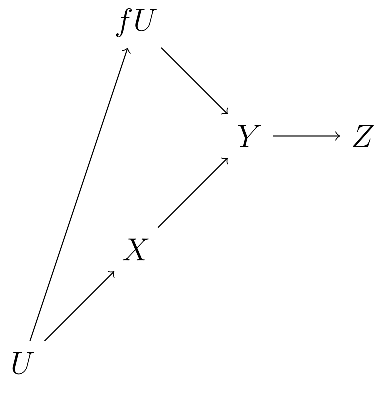
Proof. If f is not monic, then it has a nonzero kernel $K \to X$. As $K \to X \to Y = 0$, we have that $fK = 0 = f0$, i.e., the image functor is not injective.

If f is monic, then so are $U \to X \to Y$ and $U' \to X \to Y$. Therefore they are their own images (2.2.8). The rest follows from the fact that subobjects of subobjects of Y can be naturally regarded as subobjects of Y (2.2.1). □

Proposition 2.3.3. *Let $f : X \to Y$ and $g : Y \to Z$ be maps. Now $g(fU)$ and $(g \circ f)U$ are naturally isomorphic.*

Proof. By the nature of category of subobjects of any object, it is enough to show that $g(fU) = (g \circ f)U$, i.e., the maps $fU \to Y \to Z$ and $U \to X \to Y \to Z$ have the same image.

Our situation can be illuminated by the following commutative diagram:



We know that $U \to fU$ must be epic, so $fU \to Y \to Z$ and $U \to fU \to Y \to Z = U \to X \to Y \to Z$ have the same images, which proves the claim. □

The image functor satisfies the following special properties for any map f.

Proposition 2.3.4. *Let $f : X \to Y$ be a map.*

a. $f0 = 0$.

b. $f(U + U') = fU + fU'$.

Proof.

a. This one is already clear.

b. $U + U'$ is the image of $U \oplus U' \to X$ so $f(U + U')$ is the image of $U \oplus U' \to X \to Y$. This then factors to $U \oplus U' \to fU \oplus fU' \to Y$. If we can show that $U \oplus U' \to fU \oplus fU'$ is epic, then we would be done, as the image of $fU \oplus fU' \to Y$ is $fU + fU'$.

   Let us have a nonzero map $fU \oplus fU' \to W$. This map is uniquely determined by maps $fU \to W$ and $fU' \to W$. Now at least one of them, say $fU \to W$, must be nonzero. As $U \to fU$ is epic, $U \to fU \to W$ is nonzero and thus $U \oplus U' \to fU \oplus fU' \to W$ is nonzero. Therefore $U \oplus U' \to fU \oplus fU'$ is epic, which concludes our proof. □

We have the following characterization of kernels using the image functor.

Proposition 2.3.5. *Let $f : X \to Y$ be a map. Now the kernel of f is the largest subobject K of X s.t. $fK = 0$.*

Proof. The kernel is a subobject of X. If $fK = 0$, then $K \to X \to Y = 0$ by 2.2.15, and $K \to X$ factors trough the kernel, i.e., K is contained in the kernel of f. □

Definition 2.3.6. Let $f : X \to Y$ be a map. We define the *inverse image functor*, f^{-1}, from the category of subobjects of Y to the category of subobjects of X by simply setting $f^{-1}U = U \times_Y X$. By the properties of fibre product $U \times_Y X \to X$ will always be monic. The induced mappings are just the ones given by the fibre product.

The following characterization may motivate the previous definition.

Proposition 2.3.7. *Let $f : X \to Y$ be a map and U a subobject of Y. Now $f^{-1}U$ is the largest subobject V of X s.t. fV is contained in U.*

Proof. If fV is contained in U, then $V \to X \to Y$ factors trough $U \to Y$. Therefore we have the following commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

which gives a map $V \to f^{-1}U$ of subobjects of X.

Finally, as

$$\begin{array}{ccc} f^{-1}U & \longrightarrow & U \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

commutes, we have that $f^{-1}U \to X \to Y$ factors trough U, which proves the claim. □

The preimage functor has the following easy basic properties.

Proposition 2.3.8. *Let $f : X \to Y$ be a map.*

1. *$f^{-1}0$ is the kernel of f.*
2. *U is contained in $f^{-1}fU$.*
3. *If $g : Y \to Z$ is a map, then $f^{-1}(g^{-1}U)$ is naturally isomorphic to $(g \circ f)^{-1}U$.*
4. *If f is monic, then $f^{-1}fU = U$.*
5. *If $i : K \to X$ is the inclusion of a subobject, then $i^{-1}U = K \cap U$.*

Proof. 1. This is clear as the kernel of f is the largest subobject K of X s.t. $fK = 0$.

2. Clear as $f^{-1}fU$ is the largest subobject V of X s.t. fV is contained in fU.
3. This follows from the fact that $(U \times_Z Y) \times_Y X$ is naturally isomorphic to $U \times_Z X$.
4. Clear as the image functor associated to f is injective.
5. $i^{-1}U$ is the largest subobject of X contained in K that is also contained in U, so it is $K \cap U$. □

Lemma 2.3.9. *Let $f : X \to Y$ be epic. Now the inverse image functor is injective.*

Proof. Let us have two subobjects K_1 and K_2 of Y. Now the kernel of $X \xrightarrow{f} Y \to Y/K_i$ is the inverse image of 0, i.e., $f^{-1}K_i$. If $f^{-1}K_1 = f^{-1}K_2$, then the kernels of $X \to Y \to Y/K_1$ and $X \to Y \to Y/K_2$ coincide. As $X \to Y$ is epic, we have that Y/K_1 and Y/K_2 define the same quotient object of X. By the dual of 2.2.1 they must then be the same quotient objects of Y. This is equivalent to $K_1 = K_2$ which concludes our proof. □

Corollary 2.3.10. *Let $f : X \to Y$ be epic. Now $ff^{-1}U = U$.*

Proof. By previous, it is enough to show that $f^{-1}ff^{-1}U = f^{-1}U$. We know that $ff^{-1}U$ is contained in U. The image of $f^{-1}U$ is contained in $ff^{-1}U$, and if V is a strictly larger subobject of X, then fV could not be contained in U, and therefore it would not be contained in $ff^{-1}U$ either. Therefore $f^{-1}U$ is the largest subobject which maps into $ff^{-1}U$, i.e., $f^{-1}ff^{-1}U = f^{-1}U$. □

In the general case, where f is not necessarily epic, we obtain the following.

Theorem 2.3.11. *Let $f : X \to Y$ be a map. Now $ff^{-1}U = U \cap \text{im}f$.*

Proof. Now f factors into $X \to \text{im}f \to Y$, where $X \to \text{im}f$ is epic. Using the previous corollary, and the fact that $U \cap \text{im}f$ is a subobject of $\text{im}f$ in a natural way, we have that $ff^{-1}(U \cap \text{im}f) = U \cap \text{im}f$. As $ff^{-1}U$ is contained in $U \cap \text{im}f$, we have $U \cap \text{im}f \leq ff^{-1}U \leq U \cap \text{im}f$, which proves the claim. □

This will be very useful in diagram chasing, if a subobject is contained in the image of some map, then the image of its preimage is the subobject itself! Moreover, using the previous theorem, we can say something about the image of an intersection.

Theorem 2.3.12. *Let $f : X \to Y$ be a map, U a subobject of X and V a subobject of f. Now $f(U \cap f^{-1}V) = fU \cap V$.*

Proof. Let $i : U \to X$ be inclusion of the subobject into X. Now the pullback of V via $U \to X \to Y$ is $U \cap f^{-1}V$, and its image along $U \to X \to Y$ is $fU \cap V$ by the previous theorem, which proves the claim. □

We already know that U is contained in $f^{-1}fU$, but a more complete description of $f^{-1}fU$ is necessary. In the case of modules it is easy to show that $f^{-1}fU = U + K$, where K is the kernel of f. In fact, this holds in any abelian category.

Lemma 2.3.13. *Let us have a map $f : X \to Y$, whose kernel is K. Now the image fU is $(K + U)/K$.*

Proof. By the properties of the image functor $f(U + K) = fU + fK = fU$. Now the kernel of $U + K \to X \to Y$ is K, so the coimage of that map is $(U + K)/K$. By the fact that images and coimages are the same (2.2.15) we are done. □

Theorem 2.3.14. *Let U and K be subobjects of X.*

1. *$(U + K)/K = U/(U \cap K)$ as subobjects of X/K.*

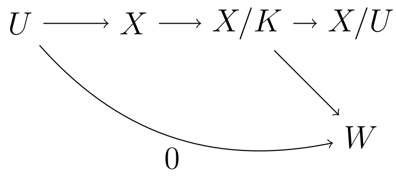
2. *If U contains K, then $(X/K)/(U/K) = X/K$ as quotient objects of X.*

Proof. These are the generalizations of the second and the third Noether isomorphism theorems.

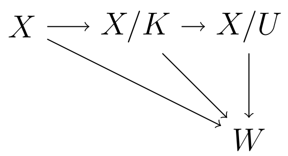
1. The kernel of $U \to X \to X/K$ is $U \cap K$, and therefore its coimage is $U/(U \cap K)$. By the above lemma this equals to $(U + K)/K$ as an subobject of X/K, which proves the claim.

2. As U contains K, we have that U/K is its image in $X \to X/K$. Now $(X/K)/(U/K)$ is the cokernel of $U \to X \to X/K$, so we want to show that X/U is the cokernel as well. As $K \leq U$, we have a map of quotient objects $X/K \to X/U$.

   If we have a map $X/K \to W$ such that

   commutes, then it gives a map $X \to W$ killing U, so by the universal property of cokernel, we obtain an unique map $X/U \to W$ such that the big triangle in

   commutes. By the epicness of $X \to X/K$, also the right triangle must commute, so $X/K \to X/U$ is indeed the cokernel of $U \to X \to X/K$. □

Theorem 2.3.15. *Let $f : X \to Y$ be a map with kernel K. Now $f^{-1}fU = U + K$.*

Proof. First we reduce to the situation where f is epic. We know by (2.2.13) that f may be factored to $X \xrightarrow{f'} I \xrightarrow{i} Y$, where f' is epic and i is monic. Now $f^{-1}fU = f'^{-1}i^{-1}if'U = f'^{-1}f'U$, so we may indeed assume that f is epic.

Denote by U' the subobject $U + K$. Recall that $fU' = fU$, so $f^{-1}fU = f^{-1}fU'$. Now $f^{-1}fU$ is the kernel of $X \to Y \to X/fU'$, which by the third Noether isomorphism theorem is the kernel of $X \to X/U'$. Therefore $f^{-1}fU' = U' = U + K$, which proves the claim. □

2.4 Exact sequences and diagram chasing

Definition 2.4.1. The sequence $X \xrightarrow{f} Y \xrightarrow{g} Z$ is *exact* if $\mathrm{im} f = \ker g$. A longer sequence is exact if any such shorter sequence in it is exact.

The next properties follow immediately from previous results.

Proposition 2.4.2. *Some basic properties of exactness.*

1. $0 \to X \to Y$ *is exact if and only if* $X \to Y$ *is monic.*
2. $X \to Y \to 0$ *is exact if and only if* $X \to Y$ *is epic.*
3. $0 \to X \to Y \to 0$ *is exact if and only if* $X \to Y$ *is an isomorphism.*
4. *If* $0 \to X \to Y \to Z \to 0$ *is exact, then* $Z = Y/X$.

It is known that exactness of a sequence is a self-dual notion, so if $X \to Y \to Z$ is exact, then $Z^{op} \to Y^{op} \to X^{op}$ is exact in the opposite category. This allows us sometimes to prove pairs of dual statements about exact sequences/diagrams.

As an application of the results in the previous section, we will now prove the five lemma for arbitary abelian categories.

Lemma 2.4.3. ***Four Lemma.*** *Let us have the following commutative diagram, with exact rows*

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \beta} & & \big\downarrow{\scriptstyle \gamma} & & \big\downarrow{\scriptstyle \delta} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \end{array}$$

If β and δ are monic and α is epic, then γ is monic.

Proof. It is enough to show that the only subobject of c of C s.t. $\gamma c = 0$ is 0. Let c be such an subobject. Now its image in D must be 0 by monicness of δ. Thus c is contained in the image of B in C. Let b be the pullback if c from C to B and $b' = \beta b$. By commutativity, and the fact that image of b in C is c, we have that the image of b' in C' is 0. Denote by a' the preimage of b' in A' and by a the preimage of a' in A. Now the image of a in $A \to B \to B'$ is b', so the image of a in B must be b by monicness of β. Thus the image of b in C must be 0, proving the monicness of γ. □

Using the four lemma and its dual, we obtain the five lemma as a corollary.

Corollary 2.4.4. ***Five Lemma.*** *Let us have the following commutative diagram, with exact rows*

$$\begin{array}{ccccccccc}
A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\
\downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \epsilon \\
A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E'
\end{array}$$

If β and δ are isomorphisms, α is epic, and ϵ is monic, then γ is an isomorphism.

As another application, we prove the nine lemma.

Theorem 2.4.5. ***Nine Lemma.*** *Let us have the following commutative diagram with exact columns and exact middle row.*

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_1 & \longrightarrow & B_1 & \longrightarrow & C_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_2 & \longrightarrow & B_2 & \longrightarrow & C_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_3 & \longrightarrow & B_3 & \longrightarrow & C_3 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Now the top row is exact if and only if the bottom row is exact.

Proof. Clearly $A_1 \to B_1$ is monic and $B_3 \to C_3$ is epic. Moreover, the facts that $A_1 \to B_1 \to C_1 = 0$ and $A_3 \to B_3 \to C_3 = 0$ follow without any assumptions on the exactness of top/bottom rows.

Let the top row be exact and a_3 a subobject of A_3 whose image in B_3 is 0. Let a_2 be its inverse image in A_2, and b_2 its image in B_2. As the image of b_2 in B_3 is 0, we can pull b_2 to a subobject b_1 of B_1. By monicness of $C_1 \to C_2$ the image of b_1 in C_1 is 0, so we may pull it back to $a_2 \subset A_1$. By the monicness of $A_2 \to B_2$, the image of a_1 in A_2 is exactly a_2, and hence $a_3 = 0$. Therefore, if the top row is exact, then $A_3 \to B_3$ is monic. Using duality, we see that if the bottom row is exact, then $B_1 \to C_1$ is epic.

Let the bottom row be exact and b_1 a subobject of B_1 that is sent to 0 in C_1. Now its image b_2 in B_2 is sent to 0 in C_2, so we can pull it back to $a_2 \subset A_2$ in a good way. As the image of b_2 in B_3 is 0 and $A_3 \to B_3$ is monic, we see that the image of a_2 in A_3 is 0, so we have a good preimage in A_1. The image of this object in B_1 must be exactly b_1 as

$B_1 \to B_2$ is monic. Therefore the image of $A_1 \to B_1$ contains the kernel of $B_1 \to C_1$, so it is exactly the kernel as $A_1 \to B_1 \to C_1 = 0$. We proved that from the exactness of the bottom row, one obtains the exactness of $A_1 \to B_1 \to C_1$. From duality, the exactness of the top row gives the exactness of $A_3 \to B_3 \to C_3$. This concludes our proof. $\square$

Sometimes we would to like to show the existence of some maps, whose existence is not known a priori. This will be needed in the proof of snake lemma or similar statements. Our next goal is to obtain the necessary tools for proving such claims.

We begin by giving an alternative characterization for square being a pullback and pushout. This characterization, and the two lemmas following that characterization, are taken from <Freyd's book [honestly, look up how to cite already...]>.

Lemma 2.4.6. *Let us have maps $f : X \to Z$ and $g : Y \to Z$. An object W together with maps $W \to X$ and $W \to Y$ is the pullback if and only if the sequence $0 \to W \to X \oplus Y \to Z$ is exact, where the map $X \oplus Y \to Z$ is given by f and $-g$.*

Proof. If $0 \to W \to X \oplus Y \to Z$ is exact then for any pair of maps $U \to X$ and $U \to Y$ s.t.

$$\begin{tikzcd} U \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

commutes, we have an unique map $U \to W$ s.t.

$$\begin{tikzcd} U \arrow[d] \arrow[dr] & \\ W \arrow[r] & X \oplus Y \end{tikzcd}$$

commutes. This is to say, that we have an unique map $U \to W$ s.t.

$$\begin{tikzcd} U \arrow[dr] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\ & W \arrow[r] \arrow[d] & X \arrow[d] \\ & Y \arrow[r] & Z \end{tikzcd}$$

commutes. The square

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

commutes as $W \to X \oplus Y \to Z = 0$, so we see that W is the fibre product of X and Y over Z. It is straightforward to verify that the fibre product satisfies these properties, so we are done. □

Lemma 2.4.7. *Let $X \to Z$ and $Y \to Z$ be maps. If $X \times_Z Y \to X$ is monic, then so is $Y \to Z$.*

Proof. Denote by W the fibre product $X \times_Z Y$. If $Y \to Z$ is not monic, then it has a nonzero kernel $K \to Y$. Together with the zero map $K \to X$ it gives a nonzero map $K \to W$ s.t. $K \to W \to Y = 0$, so $X \times_Z Y \to X$ cannot be monic. □

Lemma 2.4.8. *Let $X \to Z$ be a map and $Y \to Z$ be an epic map. Now $X \times_Z Y \to X$ is epic, so epicness is preserved under base change.*

Proof. We will prove the dual claim. Let

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

be pushout, $W \to X$ monic. We want to to show that $Y \to Z$ is a monomorphism. By the dual of 2.4.6 we have that $W \to X \oplus Y \to Z \to 0$ is exact. As $W \to X$ is monic, also $W \to X \oplus Y$ is monic, so actually $0 \to W \to X \oplus Y \to Z \to 0$ is exact. Therefore the square is pullback, and the monicness of $Y \to Z$ follows from the previous lemma. □

Next we prove that fibre products are "local". The following lemma has counterparts at least in the theory of topological spaces, schemes, covers and so on.

Lemma 2.4.9. *Let $X \to Z$ and $Y \to Z$ be maps, U a subobject of X. Now the preimage of U in $X \times_Z Y \to X$ is $U \times_Z Y$.*

Proof. Denote by W the fibre product and by W' the preimage of U. Let

$$\begin{array}{ccc} V & \longrightarrow & U \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

commute. This gives an unique map $V \to W$, s.t. $V \to W \to X$ factors trough U, so $V \to W$ factores uniquely to $V \to W'$, which proves the claim. □

Theorem 2.4.10. *Let $X \to Z$ and $Y \to Z$ be epic maps, $W = X \times_Z Y$. Now all the maps in*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

are epic. Moreover, for any subobject X' of X we get the same subobject of Y regardless of which route we take (using images and preimages).

Proof. Let Z' be the image of X' in Z and Y' the preimage of Z' in Y. Let W' be the preimage of X' in W. By epicness and commutativity, the image of W' in Z is Z', so the image of W' in Y' is at least contained in Y'. Now W' is the fibre product $X' \times_Z Y'$, and as the images of X' and Y' are exactly Z', $W' = X' \times'_Z Y'$. The maps $X' \to Z'$ and $Y' \to Z'$ are epic, and therefore $W' \to Y'$ is epic. This proves that the image of W' in Y is Y', which was exactly what we wanted to prove. □

This allows us to show existence of certain types of maps in certain situations, as the next theorem shows.

Theorem 2.4.11. *Let us have the following commutative diagram*

$$\begin{array}{ccccc} & & B & \longrightarrow & A \\ & & \downarrow & & \\ D & \longrightarrow & C & & \\ \downarrow & & & & \\ E & & & & \end{array}$$

and a subobject of U with the following property: for any subobject U' contained in U, as we go down from A to E using preimages and images, at each point the image of the preimage is the original object itself. If the subobject 0 of X gives us 0 in E, then we have a map $U \to E$, s.t., the image functor associated to it is exactly the one we get from preimages and images in the above diagram.

Proof. Replacing A with the subobject U, B with the preimage of U and so on, we may assume that the rows and columns of

$$
\begin{array}{ccccc}
 & & B & \longrightarrow & A & \longrightarrow & 0 \\
 & & \downarrow & & & & \\
D & \longrightarrow & C & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & & & \\
E & & 0 & & & & \\
\downarrow & & & & & & \\
0 & & & & & &
\end{array}
$$

are exact. Denote by W the fibre product of D and B over C. Now $W \to A$ and $W \to E$ are epic, so actually we are in the following situation

$$
\begin{array}{ccc}
W & \longrightarrow & W/K \\
\downarrow & & \\
W/K' & &
\end{array}
$$

where $W/K = A$ and $W/K' = B$. By assumptions and 2.4.10 K is contained in K', so we obtain a map $W/K \to W/K'$. The image functor of this map is clearly the functor obtained by composing the preimage functor of $W \to W/K$ with the image functor of $W \to W/K'$ because $W \to W/K'$ factors to $W \to W/K \to W/K'$, and because the composition of preimage and image functors of $W \to W/K$ is the identity functor. But this functor is by 2.4.10 the original functor, so we have proven the claim. □

Remark 2.4.12. One should note that the above theorem generalizes easily to longer such diagrams.

Now we are ready to prove the snake lemma.

Theorem 2.4.13. ***Snake Lemma.*** *Let us have the following commutative diagram with exact rows*

$$
\begin{array}{ccccccc}
 & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & &
\end{array}
$$

Now there exists a connecting morphism $\ker c \to \mathrm{coker} a$*, s.t. the sequence* $\ker a \to \ker b \to \ker c \to \mathrm{coker} a \to \mathrm{coker} b \to \mathrm{coker} c$ *is exact.*

Proof. This is the complete diagram of all objects of interest:

$$
\begin{array}{ccccccc}
 & & \ker a & \longrightarrow & \ker b & \longrightarrow & \ker c & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \mathrm{coker} a & \longrightarrow & \mathrm{coker} b & \longrightarrow & \mathrm{coker} c & &
\end{array}
$$

The proof for exactness of $\ker a \to \ker b \to \ker c$ is a standard diagram chasing, much like in the proofs of five lemma and nine lemma, so we are not going to prove them. The exactness of $\mathrm{coker} a \to \mathrm{coker} b \to \mathrm{coker} c$ follows from duality.

We may use 2.4.11 to give a map $\ker c \to \mathrm{coker} a$, it is straightforward to check that the diagram satisfies the assumptions.

First of all $\ker b \to \ker c \to \mathrm{coker} a = 0$. This because if we send $\ker b$ to B' along the path $\ker b \to kerc \to C \to B \to B'$, then it is exactly the same subobject as the image of the kernel of $B \to C$ in B'. Therefore the preimage of that subobject in A' lies in the image of $A \to A'$, so indeed, the image of $\ker b$ in $\ker c$ is sent to 0. By duality we also obtain that $\ker c \to \mathrm{coker} a \to \mathrm{coker} b = 0$.

Next we will show that the kernel of $\mathrm{coker} a \to \mathrm{coker} b$ is contained in the image of $\ker c \to \mathrm{coker} a$. Pull $\mathrm{coker} a$ to A' and push it to B'. This subobject has zero image in both C' and $\mathrm{coker} b$. Pull it back to B and push it to C. This map subobject has image 0 in C so it is contained in the kernel of c. This proves the claim and concludes our proof for exactness of $\ker c \to \mathrm{coker} a \to \mathrm{coker} b$. The exactness of $\ker b \to \ker c \to \mathrm{coker} a$ follows from duality. □

Remark 2.4.14. Both in the proof of nine lemma, and in the proof of snake lemma, we have chosen the subclaims to prove from dual pairs of claims in a very specific way. **This is because their duals may not be as easy to prove using subobjects!** The concept of subobject is not self-dual, so if we would like to prove these hard duals as easily as the claims that we now prove, we would have to create the theory of quotient objects that considers their images and pullback.

Definition 2.4.15. A sequence is *complex* if the composition of any consecutive morphisms is the zero morphism. Before we start talking about homology we will need a lemma.

Lemma 2.4.16. *Let us have the following commutative diagram*

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

where the left vertical maps are epic, and the right vertical maps are monic. Now there is an unique map $B \to B'$ making the diagram commute. Moreover, if $B \to B'$ makes either of the little squares commute, then it makes the whole diagram commute.

Proof. Let $K \to A$ be the kernel of $A \to B$. By the definition of abelian category $A \to B$ is the cokernel of $K \to A$. As our diagram commutes and as $B' \to C'$ is monic, we have that $K \to A \to A' \to B' = 0$. Therefore there is an unique map $B \to B'$ making the left little square in

$$\begin{array}{ccccc} A & \longrightarrow & B & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' \end{array}$$

commute. By epicness of $A \to B$ the right little square commutes as well, so we are done. The second claim is also clear. □

Definition 2.4.17. Let $A \xrightarrow{f} B \xrightarrow{g} C$ be a complex. Now we obtain the map $\ker g \to B \to \mathrm{coker} f$. We define the *homology* of the original sequence at B to be the image of the map $\ker g \to \mathrm{coker} f$.

This definition is functorial: if

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{g} & C \\ \downarrow & & \downarrow & & \downarrow \\ A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' \end{array}$$

commutes, then we have unique maps $\ker g \to \ker g'$ and $\mathrm{coker} f \to \mathrm{coker} f'$ s.t.

$$\begin{array}{ccccc} \ker g & \longrightarrow & B & \longrightarrow & \mathrm{coker} f \\ \downarrow & & \downarrow & & \downarrow \\ \ker g' & \longrightarrow & B' & \longrightarrow & \mathrm{coker} f' \end{array}$$

commutes. Therefore, using the previous lemma, we obtain an unique map between the homologies that makes the related diagram commute.

Lemma 2.4.18. *Let us have the following commutative diagram with complexes as rows*

$$
\begin{array}{ccccccc}
A & \xrightarrow{f} & B & \xrightarrow{g} & C & \xrightarrow{h} & D \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
A' & \xrightarrow{f'} & B' & \xrightarrow{g'} & C' & \xrightarrow{h'} & D'
\end{array}
$$

Now we have an unique commutative diagram of form

$$
\begin{array}{ccccccc}
B & \longrightarrow & \mathrm{coker} f & \longrightarrow & \ker h & \longrightarrow & C \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
B' & \longrightarrow & \mathrm{coker} f' & \longrightarrow & \ker h' & \longrightarrow & C'
\end{array}
$$

where $B \to \mathrm{coker} f \to \ker h \to D = g$ *and* $B' \to \mathrm{coker} f' \to \ker h' \to D' = g'$.

Proof. Factorization of g to $B \to \mathrm{coker} f \to \ker h \to D$ follows trivially from the fact that the rows are complexes and from universal properties of kernels and cokernels. We already know that we have an unique map $\mathrm{coker} f \to \mathrm{coker} f'$ ($\ker h \to \ker h'$) making the left (right) little square commute. Commutativity of the middle little square follows from the fact that maps to cokernels are epic and maps from kernels are monic, and from the fact that the big square commutes. $\square$

Lemma 2.4.19. *In the case of previous lemma, we have that the homology at B is naturally isomorphic to the kernel of* $\mathrm{coker} f \to \ker h$ *and that the homology at C is naturally isomorphic to the cokernel of* $\mathrm{coker} f \to \ker h$.

Proof. Now we have the following sequence $\ker f \to B \to \mathrm{coker} f \to \ker h \to C \to \mathrm{coker} h$. Denote the kernel and cokernel of $\mathrm{coker} f \to \ker h$ by H and H' respectively. Using the universal properties of kernels and cokernels, we obtain the diagram

$$
\begin{array}{ccccccccccc}
\ker g & \longrightarrow & B & \longrightarrow & \mathrm{coker} f & \longrightarrow & \ker h & \longrightarrow & C & \longrightarrow & \mathrm{coker} g \\
& \searrow & & \nearrow & & & & \searrow & & \nearrow & \\
& & H & & & & & & H' & &
\end{array}
$$

which commutes. Now $\ker g \to H$ is epic and $H' \to \mathrm{coker} g$ is monic. It is enough to show the first one, the second follows from duality. As $\ker h \to C$ is epic, $\ker g$ is the kernel of $B \to \mathrm{coker} f \to \ker h$. As the image of f is contained in $\ker g$, the preimage of the image of $\ker g$ to $\mathrm{coker} f$ is exactly $\ker g$. On the other hand, the inverse image functor associated to $B \to \mathrm{coker} f$ is injective and the pullback of the kernel of H to B is the kernel of $B \to \ker h = \ker g$, we have that the image of $\ker g$ in $\mathrm{coker} f$ is H so $\ker g \to H$ must be epic. Therefore H and H' are exactly the images (i.e., homologies), the naturality of the isomorphism follows from the fact that it is enough to have one little square to commute to get the map of homologies. $\square$

Theorem 2.4.20. ***Zig-Zag Lemma.*** *A short exact sequence* $0 \to A' \to A \to A'' \to 0$ *of complexes gives rise to a long exact sequence*

$$\cdots \to H_n(A') \to H_n(A) \to H_n(A'') \xrightarrow{\partial} H_{n-1}(A') \to \cdots$$

of homologies.

Proof. The only thing unclear is the existence of ∂. Using the previous lemmas, we obtain the following commutative diagram with exact rows.

$$\begin{array}{ccccccccc}
 & & H_n(A') & \longrightarrow & H_n(A) & \longrightarrow & H_n(A'') & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \operatorname{coker} d_{A',n+1} & \longrightarrow & \operatorname{coker} d_{A,n+1} & \longrightarrow & \operatorname{coker} d_{A'',n+1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker d_{A',n-1} & \longrightarrow & \ker d_{A,n-1} & \longrightarrow & \ker d_{A'',n-1} & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & H_{n-1}(A') & \longrightarrow & H_{n-1}(A) & \longrightarrow & H_{n-1}(A'') & &
\end{array}$$

The exactness of the two middle rows can be confirmed with a straightforward diagram chasing argument. Now the existence of ∂ follows from the snake lemma. □

Remark 2.4.21. By going over the necessary lemmas again, one can show that the map ∂ is natural in the following sense: if we have the following commutative diagram of complexes with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & A'' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & B'' & \longrightarrow & 0
\end{array}$$

then the maps of homologies make

$$\begin{array}{ccccccccccc}
\cdots & \longrightarrow & H_n(A') & \longrightarrow & H_n(A) & \longrightarrow & H_n(A'') & \to & H_{n-1}(A') & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(B') & \longrightarrow & H_n(B) & \longrightarrow & H_n(B'') & \to & H_{n-1}(B') & \longrightarrow & \cdots
\end{array}$$

commute.