

Morse theory of Loop Spaces &  
Hecke algebras

jt. w/ Ko Honda, Yin Tian, and Tianyu Guan

Symplectic Zoominar  
May 16, 2025

## Plan for today:

I. Morse theory on path/loop spaces & multiloop homology  
w/ switches

II. Morse multiloop  $A_\infty$ -algebra of  $S^2$

III. Connection w/ higher-dimensional Heegaard Floer homology

## Ia/ Morse theory on path/loop spaces

Consider  $L: [0, \beta] \times TM \rightarrow \mathbb{R}$  smooth Lagrangian on orientable Riemannian mfd  $(M, g)$  which is fiberwise convex & quadratic:

## Ia/ Morse theory on path/loop spaces

Consider  $L: [0, \beta] \times TM \rightarrow \mathbb{R}$  smooth Lagrangian on orientable Riemannian mfd  $(M, g)$  which is fiberwise convex & quadratic:

(L1) There is a continuous function  $\ell_1$  on  $M$  s.t. for any  $(t, x, v) \in [0, \beta] \times TM$   
 $\|\nabla_{vv} L(t, x, v)\| \leq \ell_1(x)$ ,  $\|\nabla_{vx} L(t, x, v)\| \leq \ell_1(x)(1+|v|)$ ,  $\|\nabla_{xx} L(t, x, v)\| \leq \ell_1(1+|v|^2)$

(L2) There is a cont., positive function  $\ell_2$  on  $M$  s.t.

$$\nabla_{vv} L(t, x, v) \geq \ell_2(x) \text{Id} \quad \text{as bilin. forms}$$

where  $\nabla_{vv}, \nabla_{vx}, \nabla_{vx}, \nabla_{xx}$  are the components of the Hessian of  $L$   
w.r.t.  $T^*M = T^h M \oplus T^v M$  given by Levi-Cevita connection  $\nabla$

## Ia/ Morse theory on path/loop spaces

Consider  $L: [0, \beta] \times TM \rightarrow \mathbb{R}$  smooth Lagrangian on orientable Riemannian mfd  $(M, g)$  which is fiberwise convex & quadratic:

(L1) There is a continuous function  $\ell_1$  on  $M$  s.t. for any  $(t, x, v) \in [0, \beta] \times TM$   
 $\|\nabla_{vv} L(t, x, v)\| \leq \ell_1(x)$ ,  $\|\nabla_{vx} L(t, x, v)\| \leq \ell_1(x)(1+|v|)$ ,  $\|\nabla_{xx} L(t, x, v)\| \leq \ell_1(1+|v|^2)$

(L2) There is a cont., positive function  $\ell_2$  on  $M$  s.t.

$$\nabla_{vv} L(t, x, v) \geq \ell_2(x) \text{Id} \quad \text{as bilin. forms}$$

where  $\nabla_{vv}, \nabla_{xv}, \nabla_{vx}, \nabla_{xx}$  are the components of the Hessian of  $L$   
w.r.t.  $T^*M = T^h M \oplus T^v M$  given by Levi-Civita connection  $\nabla$

Examples  $L = \frac{1}{2}|v|^2$ , or  $L = \frac{1}{2}|v|^2 + F(t, x)$

## Ia/ Morse theory on path/loop spaces

Consider  $L: [0, \beta] \times TM \rightarrow \mathbb{R}$  smooth Lagrangian on orientable Riemannian mfd  $(M, g)$  which is fiberwise convex & quadratic:

(L1) There is a continuous function  $\ell_1$  on  $M$  s.t. for any  $(t, x, v) \in [0, \beta] \times TM$   
 $\|\nabla_{vv} L(t, x, v)\| \leq \ell_1(x)$ ,  $\|\nabla_{vx} L(t, x, v)\| \leq \ell_1(x)(1+|v|)$ ,  $\|\nabla_{xx} L(t, x, v)\| \leq \ell_1(x)(1+|v|^2)$

(L2) There is a cont., positive function  $\ell_2$  on  $M$  s.t.

$$\nabla_{vv} L(t, x, v) \geq \ell_2(x) \text{Id} \quad \text{as bilin. forms}$$

where  $\nabla_{vv}, \nabla_{xv}, \nabla_{vx}, \nabla_{xx}$  are the components of the Hessian of  $L$  w.r.t.  $TTM = T^h M \oplus T^v M$  given by Levi-Civita connection  $\nabla$

Examples  $L = \frac{1}{2}|v|^2$ , or  $L = \frac{1}{2}|v|^2 + F(t, x)$

Consider the space  $\Omega^{1,2}(M, q, q')$  (or more generally  $\Omega^{m,2}(M, q, q')$ ) of paths from  $q$  to  $q'$  in  $M$  of finite  $W^{1,2}$ -norm, i.e.  $L^2$ -integr. absolutely cont. paths

- There is an action functional associated with  $L$

$$A_L: \Omega^{1,2}(M, g, g') \rightarrow \mathbb{R}$$

$$A_L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t)) dt$$

- There is an action functional associated with  $L$

$$A_L: \Omega^{1,2}(M, q, q') \rightarrow \mathbb{R}$$

$$A_L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t)) dt$$

- It's crit. pts are solutions of Euler-Lagrange equation

$$\nabla_t (\nabla_v L(t, \gamma(t), \dot{\gamma}(t))) = \nabla_x L(t, \gamma(t), \dot{\gamma}(t))$$

- There is an action functional associated with  $L$

$$A_L: \Omega^{1,2}(M, q, q') \rightarrow \mathbb{R}$$

$$A_L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t)) dt$$

- Its crit. pts are solutions of Euler-Lagrange equation

$$\nabla_t (\nabla_v L(t, \gamma(t), \dot{\gamma}(t))) = \nabla_x L(t, \gamma(t), \dot{\gamma}(t))$$

- For  $L = \frac{1}{2}|v|^2$  these are geodesic paths from  $q$  to  $q'$ .

- There is an action functional associated with  $L$

$$A_L: \Omega^{1,2}(M, q, q') \rightarrow \mathbb{R}$$

$$A_L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t)) dt$$

- It's crit. pts are solutions of Euler-Lagrange equation

$$\nabla_t (\nabla_v L(t, \gamma(t), \dot{\gamma}(t))) = \nabla_x L(t, \gamma(t), \dot{\gamma}(t))$$

- For  $L = \frac{1}{2}|v|^2$  these are geodesic paths from  $q$  to  $q'$ .

Remark  $A_L$  is not always  $C^2$ .

- There is an action functional associated with  $L$

$$A_L: \Omega^{1,2}(M, q, q') \rightarrow \mathbb{R}$$

$$A_L(\gamma) = \int_0^1 L(t, \gamma(t), \dot{\gamma}(t)) dt$$

- It's crit. pts are solutions of Euler-Lagrange equation

$$\nabla_t (\nabla_v L(t, \gamma(t), \dot{\gamma}(t))) = \nabla_x L(t, \gamma(t), \dot{\gamma}(t))$$

- For  $L = \frac{1}{2}|v|^2$  these are geodesic paths from  $q$  to  $q'$ .

Remark  $A_L$  is not always  $C^2$ .

- Nevertheless, works of Abbondandolo-Schwarz ('06, '09, '10) and Abbondandolo-Majer ('10) explain how to make Morse theory for  $A_L$  work.

## Theorem (Abbondandolo-Schwarz 109)

Assume all cr. pts of  $A_L$  are non-degenerate.  
Then there exists a smooth pseudogradient vector field  $X$  on  $\Omega^{1,2}(M, g, g')$ :

## Theorem (Abbondandolo-Schwarz 109)

Assume all cr. pts of  $A_L$  are non-degenerate.

Then there exists a smooth pseudogradient vector field  $X$  on  $\Omega^{1,2}(M, g, g')$ :

(PG1)  $A_L$  is Lyapunov for  $X$  ( $DA_L(p)(X) < 0$  for  $p$  non-crit)

(PG2)  $X$  is Morse for  $A_L$

## Theorem (Abbondandolo-Schwarz 103)

Assume all cr. pts of  $A_L$  are non-degenerate.

Then there exists a smooth pseudogradient vector field  $X$  on  $\Omega^{1,2}(M, g, g')$ :

(PG1)  $A_L$  is Lyapunov for  $X$  ( $DA_L(p)(X) < 0$  for  $p$  non-crit)

(PG2)  $X$  is Morse for  $A_L$

(PG3)  $(A_L, X)$  satisfies Palais-Smale condition

(PG4)  $X$  is forward complete

(PG5)  $X$  is Morse-Smale.

## Theorem (Abbondandolo-Schwarz 109)

Assume all cr. pts of  $A_L$  are non-degenerate.  
Then there exists a smooth pseudogradient vector field  $X$  on  $\Omega^{1,2}(M, g, g')$ :

(PG1)  $A_L$  is Lyapunov for  $X$  ( $DA_L(p)(X) < 0$  for  $p$  non-crit)

(PG2)  $X$  is Morse for  $A_L$

(PG3)  $(A_L, X)$  satisfies Palais-Smale condition

(PG4)  $X$  is forward complete

(PG5)  $X$  is Morse-Smale.

---

Idea of construction: 1) In a chart centered at non-crit. (smooth)

$\delta_0$  set  $X(\delta) = -\nabla A_L(\delta_0)$ .

## Theorem (Abbondandolo-Schwarz '09)

Assume all cr. pts of  $A_L$  are non-degenerate.

Then there exists a smooth pseudogradient vector field  $X$  on  $\Omega^{1,2}(M, g, g')$ :

(PG1)  $A_L$  is Lyapunov for  $X$  ( $DA_L(p)(X) < 0$  for  $p$  non-crit)

(PG2)  $X$  is Morse for  $A_L$

(PG3)  $(A_L, X)$  satisfies Palais-Smale condition

(PG4)  $X$  is forward complete

(PG5)  $X$  is Morse-Smale.

Idea of construction: 1) In a chart centered at non-crit. (smooth)

$\delta_0$  set  $X(\delta) = -\nabla A_L(\delta_0)$

2) In a chart centered at a cr. pt  $\delta_0=0$  set  $X(\delta) = [D^2 A_L(0)](\delta)$

## Theorem (Abbondandolo-Schwarz '09)

Assume all cr. pts of  $A_L$  are non-degenerate.  
Then there exists a smooth pseudogradient vector field  $X$  on  $\Omega^{1,2}(M, g, g')$ :  
(for  $p$  non-crit)  $DA_L(p)(X) < 0$

(PG1)  $A_L$  is Lyapunov for  $X$

(PG2)  $X$  is Morse for  $A_L$

(PG3)  $(A_L, X)$  satisfies Palais-Smale condition

(PG4)  $X$  is forward complete

(PG5)  $X$  is Morse-Smale.

Idea of construction: 1) In a chart centered at non-crit. (smooth)

$\delta_0$  set  $X(\delta) = -\nabla A_L(\delta_0)$

2) In a chart centered at a cr. pt  $\delta_0=0$  set  $X(\delta) = [D^2 A_L(0)](\delta)$

3) Achieve (PG5) by perturbing away from cr. pts.

There is a separable Banach space  $\mathcal{X}$  of such perturbations, which we fix.

Lemma For  $L$  as above one can pick pseudogradient  $X$  such that for all cr. pts  $x_{cr}$  we have

$$W^u(x_{cr}) \rightarrow C^\infty(M, g, g')$$

Sketch: let's show this claim instead for free loops  $\Lambda^{1,2}(M)$  & time-indep.  $L$ .

Lemma For  $L$  as above one can pick pseudogradient  $X$  such that for all cr. pts  $\gamma_{cr}$  we have

$$W^u(\gamma_{cr}) \xrightarrow{\cong} C^\infty(M, q, q')$$

Sketch: let's show this claim instead for free loops  $\Lambda^{1,2}(M)$  & time-indep.  $L$ .

Take  $\gamma \in W^u(\gamma_{cr})$ . We have  $\gamma \in \Lambda^{m,2}(M) \Leftrightarrow \tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow \Lambda^{m-1,2}(M)$   
 $h \mapsto \gamma(\cdot + h)$   
 is a differentiable path

Lemma For  $L$  as above one can pick pseudogradient  $X$  such that for all cr. pts  $\gamma_{cr}$  we have

$$W^u(\gamma_{cr}) \xrightarrow{\cong} C^\infty(M, g, g')$$

Sketch: let's show this claim instead for free loops  $\Lambda^{1,2}(M) \neq$

time-indep.  $L$ .

Take  $\gamma \in W^u(\gamma_{cr})$ . We have  $\gamma \in \Lambda^{m,2}(M) \Leftrightarrow \tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow \Lambda^{m-1,2}(M)$   
 $h \mapsto \gamma(\cdot + h)$   
 is a differentiable path

• Since  $\gamma \in \Lambda^{1,2}(M)$ ,  $\tilde{\gamma}$  is a diff. path in  $L^2$ -topology inside  $W^u(\gamma_{crit})$   
 $W^u(\gamma_{cr})$  is finite dim., so  $W^{1,2} \rightarrow L^2$  restricts to an embedding

Lemma For  $L$  as above one can pick pseudogradient  $X$  such that for all cr. pts  $\gamma_{cr}$  we have

$$W^u(\gamma_{cr}) \xrightarrow{\cong} C^\infty(M, q, q')$$

Sketch: let's show this claim instead for free loops  $\Lambda^{1,2}(M)$  & time-indep.  $L$ .

Take  $\gamma \in W^u(\gamma_{cr})$ . We have  $\gamma \in \Lambda^{m,2}(M) \Leftrightarrow \tilde{\gamma} : (-\varepsilon, \varepsilon) \rightarrow \Lambda^{m-1,2}(M)$   
 $h \mapsto \gamma(\cdot + h)$   
 is a differentiable path

• Since  $\gamma \in \Lambda^{1,2}(M)$ ,  $\tilde{\gamma}$  is a diff. path in  $L^2$ -topology inside  $W^u(\text{crit})$ .  
 $W^u(\text{crit})$  is finite dim., so  $W^{1,2} \rightarrow L^2$  restricts to an embedding  
 $\Rightarrow \tilde{\gamma}$  is diff path in  $\Lambda^{1,2}(M) \Rightarrow \gamma \in \Lambda^{2,2}(M)$ .

## Operations from string topology

Consider cr-pts.  $\gamma_1 \in \Omega^{1,2}(M, q_0, q_1)$ ,  $\gamma_2 \in \Omega^{1,2}(M, q_1, q_2)$

Pontrjagin product  $\cdot : CM_{-*}(\Omega^{1,2}(M, q_1, q_2)) \otimes CM_{-*}(\Omega^{1,2}(M, q_0, q_1)) \rightarrow CM_{-*}(\Omega^{1,2}(M, q_0, q_2))$

can be defined via

## Operations from string topology

Consider cr. pts.  $\delta_1 \in \Omega^{1,2}(M, q_0, q_1)$ ,  $\delta_2 \in \Omega^{1,2}(M, q_1, q_2)$

Pontrjagin product  $\cdot : CM_{-*}(\Omega^{1,2}(M, q_1, q_2)) \otimes CM_{-*}(\Omega^{1,2}(M, q_0, q_1)) \rightarrow CM_{-*}(\Omega^{1,2}(M, q_0, q_2))$

can be defined via

$$\delta_1 \cdot \delta_2 = \sum_{\substack{\delta_0\text{-cr.pt.} \\ \text{in } \Omega^{1,2}(M, q_0, q_2)}} \# \mathcal{M}(\delta_2, \delta_1, \delta_0) \cdot \delta_0$$

where  $\mathcal{M}(\delta_2, \delta_1, \delta_0) = \# C(w^u(\delta_2) \times w^u(\delta_1)) \cap w^s(\delta_0)$   
w/  $C : \Omega^{1,2}(M, q_1, q_2) \times \Omega^{1,2}(M, q_0, q_1) \rightarrow \Omega^{1,2}(M, q_0, q_2)$  - standard concatenation

# Operations from string topology

Consider cr. pts.  $\delta_1 \in \Omega^{1,2}(M, q_0, q_1)$ ,  $\delta_2 \in \Omega^{1,2}(M, q_1, q_2)$

Pontrjagin product  $\cdot : CM_{-*}(\Omega^{1,2}(M, q_1, q_2)) \otimes CM_{-*}(\Omega^{1,2}(M, q_0, q_1)) \rightarrow CM_{-*}(\Omega^{1,2}(M, q_0, q_2))$

can be defined via

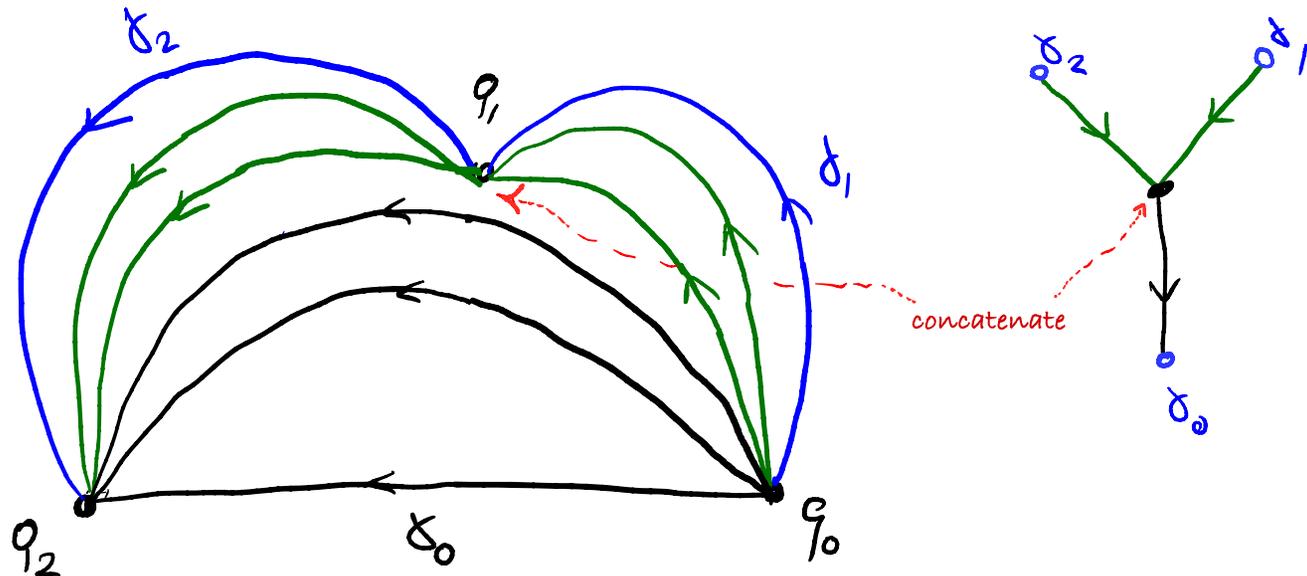
$$\delta_1 \cdot \delta_2 = \sum_{\substack{\delta_0 \text{-cr.pt.} \\ \text{in } \Omega^{1,2}(M, q_0, q_2)}} \# M(\delta_2, \delta_1, \delta_0) \cdot \delta_0$$

where  $M(\delta_2, \delta_1, \delta_0) = C(W^u(\delta_2) \times W^u(\delta_1)) \cap W^s(\delta_0)$

w/  $C: \Omega^{1,2}(M, q_1, q_2) \times \Omega^{1,2}(M, q_0, q_1) \rightarrow \Omega^{1,2}(M, q_0, q_2)$  - standard concatenation

Cartoon:

$M = \mathbb{R}^2$



## $A_\infty$ -operations

$$\mathcal{M}_{\text{Morse}}^d : CM_{-*}(\Omega^{1,2}(M, q_{d-1}, q_d)) \otimes \dots \otimes CM_{-*}(\Omega^{1,2}(M, q_0, q_1)) \longrightarrow CM_{-*}(\Omega^{1,2}(M, q_0, q_d))$$
$$\gamma_d \otimes \dots \otimes \gamma_1 \longmapsto \sum \# \mathcal{M}(\gamma_d, \dots, \gamma_1; \gamma_0) \cdot \gamma_0$$

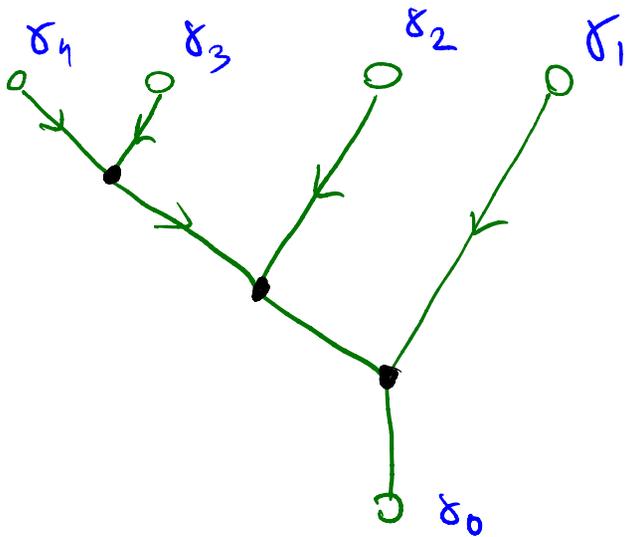
# $A_\infty$ -operations

$$M_{\text{Morse}}^d : CM_{-*}(\Omega^{1,2}(M, q_{d-1}, q_d)) \otimes \dots \otimes CM_{-*}(\Omega^{1,2}(M, q_0, q_1)) \longrightarrow CM_{-*}(\Omega^{1,2}(M, q_0, q_d))$$

$$\gamma_d \otimes \dots \otimes \gamma_1 \longmapsto \sum \# M(\gamma_d, \dots, \gamma_1; \gamma_0) \cdot \gamma_0$$

Elements of  $M(\gamma_d, \dots, \gamma_1; \gamma_0)$  are Morse flow trees  
w/ concatenations at vertices

Cartoon



$d=4$

# $A_\infty$ -operations

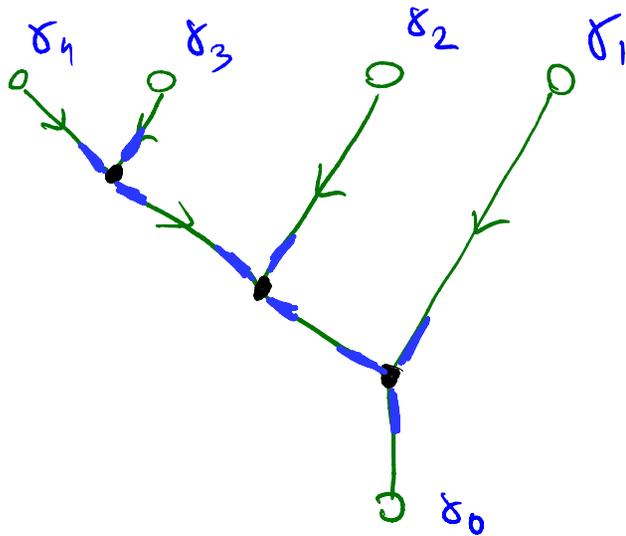
$$M_{\text{Morse}}^d : CM_{-*}(\Omega^{1,2}(M, g_{d-1}, g_d)) \otimes \dots \otimes CM_{-*}(\Omega^{1,2}(M, g_0, g_1)) \longrightarrow CM_{-*}(\Omega^{1,2}(M, g_0, g_d))$$

$$\delta_d \otimes \dots \otimes \delta_1 \longmapsto \sum \# M(\delta_d, \dots, \delta_1; \delta_0) \cdot \delta_0$$

Elements of  $M(\delta_d, \dots, \delta_1; \delta_0)$  are Morse flow trees  
w/ concatenations at vertices

Cartoon

$d=4$



- Achieve transversality by perturbing in blue regions ( $\exists \epsilon$ )
- In finite dimensional case Morse flow trees were studied by Fukaya-Oh ('97), Abouzaid ('09, '11), Mescher ('18)
- Necessary gluing results were shown by Wehrheim ('12)

# $A_\infty$ -operations

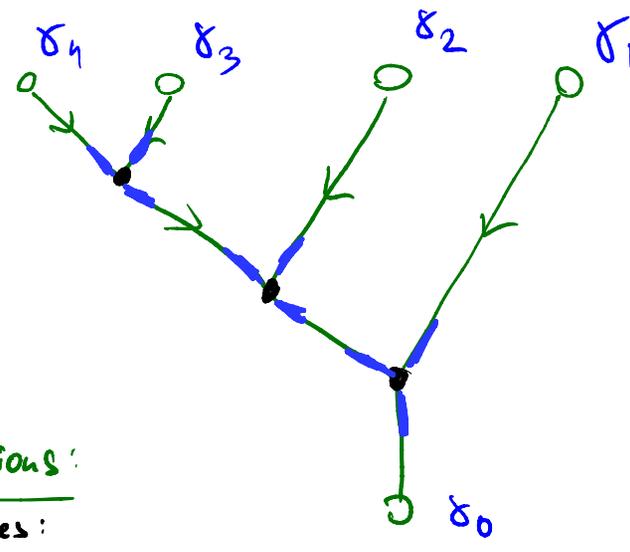
$$M_{Morse}^d : CM_{-*}(\Omega^{1,2}(M, q_{d-1}, q_d)) \otimes \dots \otimes CM_{-*}(\Omega^{1,2}(M, q_0, q_1)) \rightarrow CM_{-*}(\Omega^{1,2}(M, q_0, q_d))$$

$$\delta_d \otimes \dots \otimes \delta_1 \mapsto \sum \# M(\delta_d, \dots, \delta_1; \delta_0) \cdot \delta_0$$

Elements of  $M(\delta_d, \dots, \delta_1; \delta_0)$  are Morse flow trees  
w/ concatenations at vertices

Cartoon

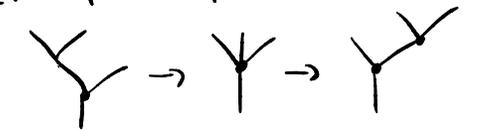
$d=4$



- Achieve transversality by perturbing in blue regions
- In finite dimensional case Morse flow trees were studied by Fukaya-Oh ('97), Abouzaid ('09, '11), Mescher ('18)
- Necessary gluing results were shown by Wehrheim ('12)

Proving  $A_\infty$ -relations:

In 1-param. families:



## Ib | Multiloops and switching maps

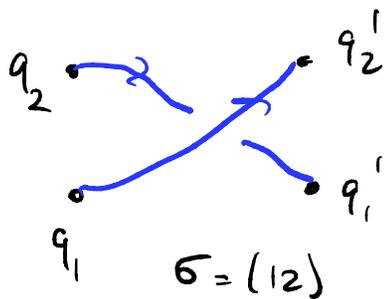
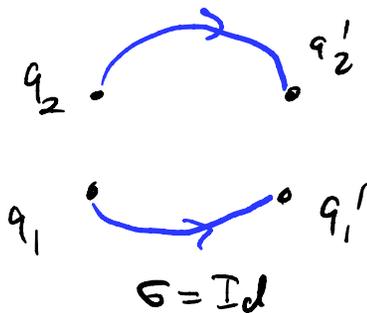
Fix  $k \in \mathbb{Z}_{\geq 1}$ . Consider tuples  $q = (q_1, \dots, q_k)$ ,  $q' = (q'_1, \dots, q'_k) \in \text{Conf}_k(M)$ .

# Ib Multiloops and switching maps

Fix  $k \in \mathbb{Z}_{\geq 1}$ . Consider tuples  $q = (q_1, \dots, q_k)$ ,  $q' = (q'_1, \dots, q'_k) \in \text{Conf}_k(M)$ .

Set  $\Omega^{m,2}(M, q, q') = \bigsqcup_{\sigma \in \mathcal{S}_k} \prod_{i=1}^k \Omega^{m,2}(M, q_i, q'_{\sigma(i)})$ ,  $\mathcal{S}$ -critical multiloops

Cartoon:  $k=2$

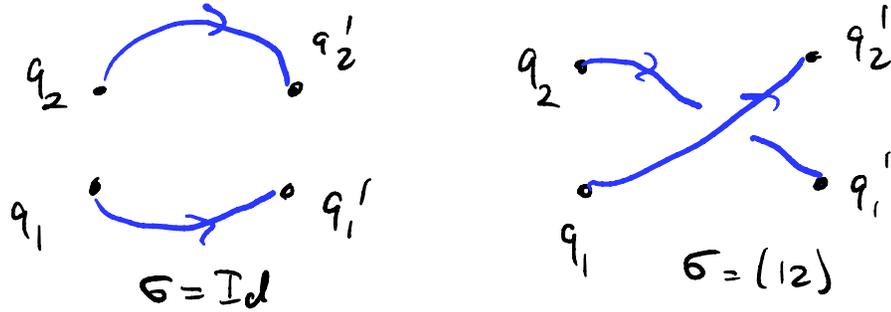


# Ib) Multiloops and switching maps

Fix  $k \in \mathbb{Z}_{\geq 1}$ . Consider tuples  $q = (q_1, \dots, q_k)$ ,  $q' = (q'_1, \dots, q'_k) \in \text{Conf}_k(M)$ .

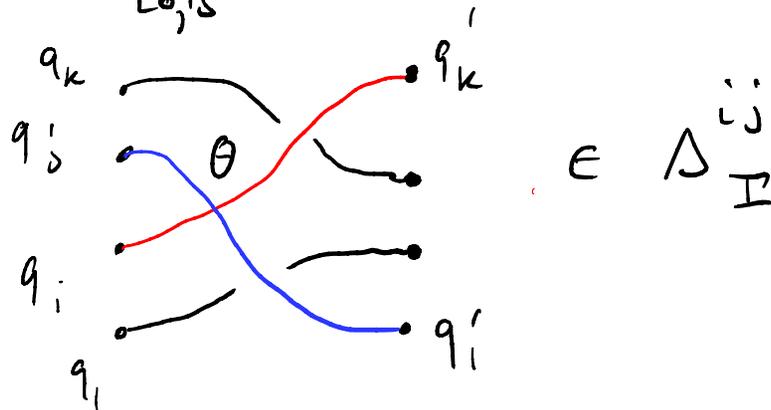
Set  $\Omega^{m,2}(M, q, q') = \bigsqcup_{\sigma \in S_k} \prod_{i=1}^k \Omega^{m,2}(M, q_i, q'_{\sigma(i)})$ ,  $\mathcal{P}$ -critical multiloops

Cartoon:  $k=2$



Diagonals given  $\{i, j\} \in \binom{[k]}{2}$ , where  $[k] = \{1, \dots, k\}$ , there is a  $C^{m-1}$ -submanifold

$$\Delta_{\mathbb{I}}^{ij} \subseteq \mathbb{I} \times \Omega^{m,2}(M, q, q') = \left\{ (\theta, \underline{x}) \mid x_i(\theta) = x_j(\theta) \right\}$$

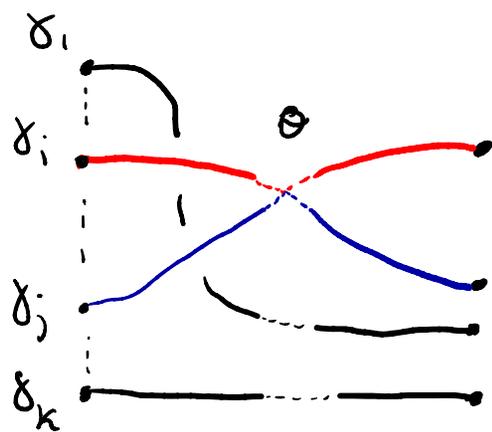
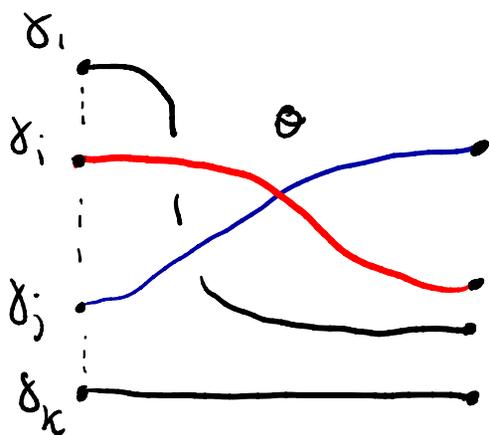


More generally there are diagonals  $\Delta_{I_k}^{i\bar{j}} \subseteq I_1 \times \dots \times I_e \times \Omega^{m,2}(M, \mathfrak{g}, \mathfrak{g}')$ .

Switching maps  $sw_{I_k}^{i\bar{j}}: \Delta_{I_k}^{i\bar{j}} \rightarrow \Delta_{I_k}^{i\bar{j}}$

$(\underline{\theta}, \underline{\delta}) \mapsto (\underline{\theta}, \delta_1(\mathcal{V}(\cdot)), \dots, \delta_i(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \# \delta_j(\mathcal{V}(\cdot))|_{[\theta_k, 1]}, \dots, \delta_j(\mathcal{V}(\cdot))|_{[\theta, \theta_k]}, \# \delta_i(\mathcal{V}(\cdot))|_{[\theta_k, 1]}, \dots, \delta_k)$

Cartoon:

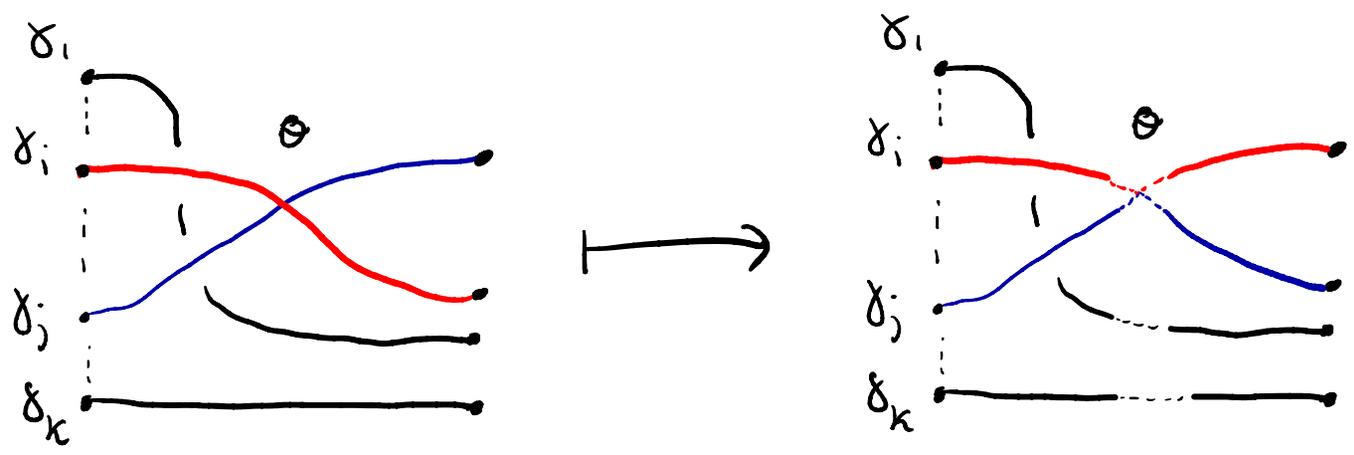


More generally there are diagonals  $\Delta_{I_k}^{ij} \subseteq I_1 \times \dots \times I_e \times \Omega^{m,2}(M, \mathfrak{g}, \mathfrak{g}')$ .

Switching maps  $sw_{I_k}^{ij}: \Delta_{I_k}^{ij} \rightarrow \Delta_{I_k}^{ij}$

$$(\underline{\theta}, \underline{\mathfrak{g}}) \mapsto (\underline{\theta}, \delta_1(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \dots, \delta_i(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \# \delta_j(\mathcal{V}(\cdot))|_{[\theta_k, 1]}, \dots, \delta_j(\mathcal{V}(\cdot))|_{[\theta_k, 1]}, \# \delta_i(\mathcal{V}(\cdot))|_{[\theta_k, 1]}, \dots, \delta_k)$$

Cartoon:



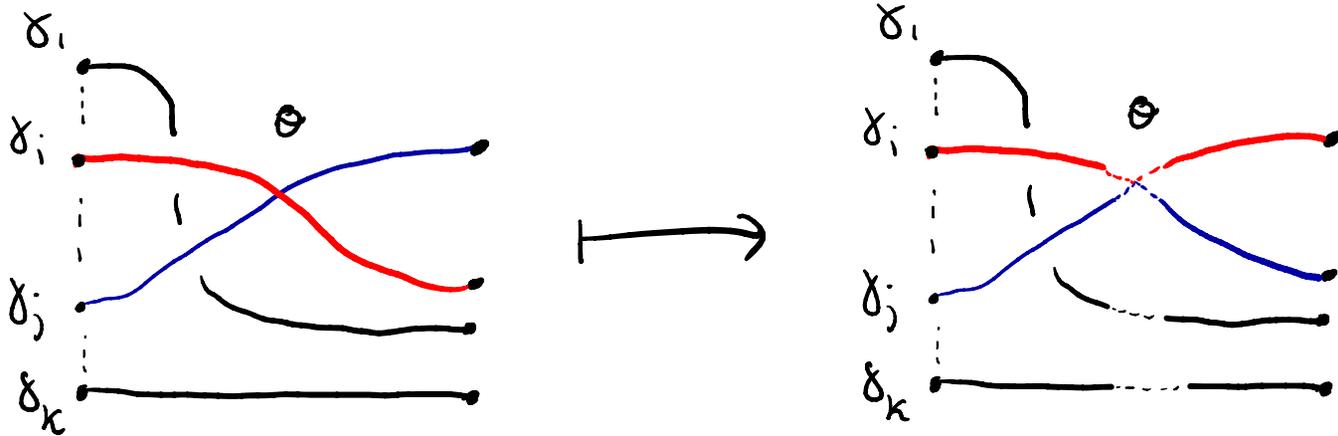
Remark 1)  $\mathcal{V}(\cdot)$  is a reparametrization w/ vanishing derivatives at  $\theta_k$ .  
 2)  $\mathcal{V}$  depends on  $\underline{\theta}$  and  $sw_{I_k}^{ij}$  is smooth on  $Conf_e([0,1]) \times \Omega^{m,2}$

More generally there are diagonals  $\Delta_{I_k}^{i_0} \subseteq I_1 \times \dots \times I_e \times \Omega^{m,2}(M, \mathfrak{g}, \mathfrak{g}')$ .

Switching maps  $sw_{I_k}^{i_0}: \Delta_{I_k}^{i_0} \rightarrow \Delta_{I_k}^{i_1}$

$(\underline{\theta}, \underline{\delta}) \mapsto (\underline{\theta}, \delta_1(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \dots, \delta_i(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \dots, \delta_j(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \dots, \delta_i(\mathcal{V}(\cdot))|_{[0, \theta_k]}, \dots, \delta_k)$

Cartoon:



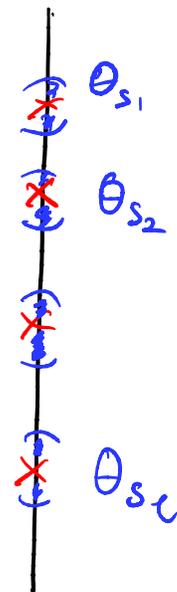
Remark 1)  $\mathcal{V}(\cdot)$  is a reparametrization w/ vanishing derivatives at  $\theta_k$ .  
 2)  $\mathcal{V}$  depends on  $\underline{\theta}$  and  $sw_{I_k}^{i_0}$  is smooth on  $Conf_e([0,1]) \times \Omega^{m,2}$   
 3) restriction of  $\mathcal{V}$  to  $\Delta \times \Omega^{m,2}(M, \mathfrak{g}, \mathfrak{g}')$  is also smooth, where  $\Delta$  is any element of stratification of  $[0,1]^e$ , corresponding to partitions of  $[1]$

# Morse flow lines w/ switchings:

Defn Given two critical multipaths  $\underline{\delta}, \underline{\delta}' \in \mathcal{P}$ , set  $\mathcal{M}_e(\underline{\delta}, \underline{\delta}')$  to be:  
 Pick  $\underline{\tau} = (\tau_1, \dots, \tau_e)$  w/  $\tau_i \in \binom{[k]}{2}$  &  $\underline{s} = (s_1, \dots, s_e)$  a permutation of  $(1, \dots, e)$ .

An MFLS is a tuple  $\underline{\Gamma} = ((\Gamma_0, \dots, \Gamma_e), \underline{\theta}, \underline{\tau}, \underline{s})$  where

- $\Gamma_0: (-\infty, 0] \rightarrow \Omega^{1,2}(M, g, g')$
  - $\Gamma_i: [0, \ell_i] \rightarrow \Omega^{1,2}(M, g, g')$
  - $\Gamma_e: [0, +\infty) \rightarrow \Omega^{1,2}(M, g, g')$
  - $\Gamma_0(-\infty) = \underline{\delta}, \Gamma_0(+\infty) = \underline{\delta}'$
- } gradient trajectories for perturbed  $X$



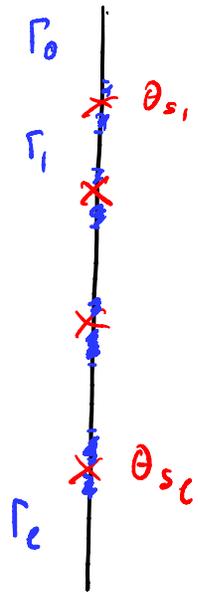
$$\mathcal{S} w_{\tau_j}^{\tau_j}(\underline{\theta}, \Gamma_j(\ell_j)) = (\underline{\theta}, \Gamma_{j+1}(0))$$

# Morse flow lines w/ switchings

Defn Given two critical multipaths  $\underline{\delta}, \underline{\delta}' \in \mathcal{P}$ , set  $\mathcal{M}_e(\underline{\delta}, \underline{\delta}')$  to be:  
 Pick  $\underline{\tau} = (\tau_1, \dots, \tau_e)$  w/  $\tau_i \in \binom{[k]}{2}$  &  $\underline{s} = (s_1, \dots, s_e)$  a permutation of  $(1, \dots, e)$ .

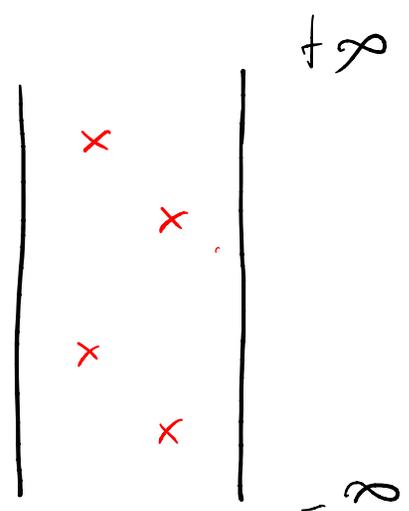
An MFLS is a tuple  $\underline{\Gamma} = ((\Gamma_0, \dots, \Gamma_e), \underline{\theta}, \underline{\tau}, \underline{s})$  where

- $\Gamma_0: (-\infty, 0] \rightarrow \Omega^{1,2}(M, g, g')$
  - $\Gamma_i: [0, l_i] \rightarrow \Omega^{1,2}(M, g, g')$
  - $\Gamma_e: [0, +\infty) \rightarrow \Omega^{1,2}(M, g, g')$
  - $\Gamma_0(-\infty) = \underline{\delta}, \Gamma_0(+\infty) = \underline{\delta}'$
- } gradient trajectories for perturbed  $X$



$$\mathcal{S}w_{\Gamma_{s_j}}^{\tau_j}(\underline{\theta}, \Gamma_j(l_j)) = (\underline{\theta}, \Gamma_{j+1}(0))$$

Remark Secretly switching markers live in the moduli of marked strips.



## Morse flow trees w/ switches:

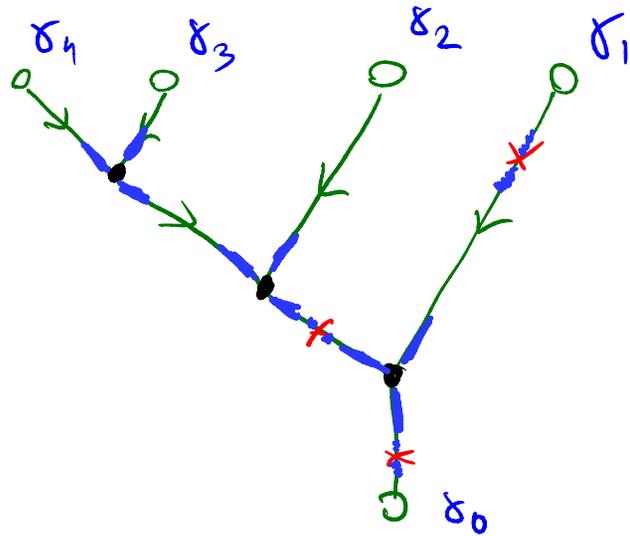
Given multipaths  $\underline{\sigma}_d, \dots, \underline{\sigma}_1, \underline{\sigma}_0$  and a combinatorial type of a tree  $\mathcal{T}$   
there are spaces  $M_{\mathcal{T}, e}(\underline{\sigma}_d, \dots, \underline{\sigma}_1, \underline{\sigma}_0)$  of MFT's w/  $e$  switches

## Morse flow trees w/ switches:

Given multipaths  $\underline{\delta}_d, \dots, \underline{\delta}_1, \underline{\delta}_0$  and a combinatorial type of a tree  $\mathcal{T}$  there are spaces  $M_{\mathcal{T}, \ell}(\underline{\delta}_d, \dots, \underline{\delta}_1, \underline{\delta}_0)$  of MFT's w/  $\ell$  switches

$$d=4$$

$$\ell=3$$



- Multipath concatenation at vertices.

- Perturb near switches & vertices

## Theorem (Honda-K.-Tian-Yuan)

a) There exists a choice of perturbation data such that spaces

$\mathcal{M}_e(\underline{\delta}, \underline{\delta}')$  and  $\mathcal{M}_{T,e}(\vec{\delta}; \underline{\delta}_0)$  are smooth manifolds.

of dimensions  $|\underline{\delta}'| - |\underline{\delta}| - (n-2)l$  &  $|\underline{\delta}_0| - |\underline{\delta}_1| - \dots - |\underline{\delta}_d| - (n-2)l + (d-2)$ .

# Theorem (Honda-K-Tian-Yuan)

a) There exists a choice of perturbation data such that spaces  $M_e(\underline{x}, \underline{x}')$  and  $M_{T,e}(\underline{\sigma}'; \underline{\sigma}_0)$  are smooth manifolds.

of dimensions  $|\underline{x}'| - |\underline{x}| - (n-2)e$  &  $|\underline{\sigma}_0| - |\underline{\sigma}_1| - \dots - |\underline{\sigma}_d| - (n-2)e + (d-2)$ .

b) Set  $CM_{-*}^0(\Omega^{1,2}(M, g, g'))$  to be a graded  $\mathbb{Z}$ -module generated by  $\mathcal{P}$ , and set  $CM_{-*}(\Omega^{1,2}(M, g, g')) = CM_{-*}^0 \otimes \mathbb{Z}[[\hbar]]$ ,  $|\hbar| = 2-n$

Then  $M'_{Morse}: CM_{-*}(\Omega^{1,2}(M, g, g')) \rightarrow CM_{-*}(\Omega^{1,2}(M, g, g'))$   

$$x \mapsto \sum \frac{\# M_e(\underline{x}, \underline{x}')}{e!} \hbar^e \cdot \underline{x}'$$
it lands in  $\mathbb{Z}$  due to symmetry

is a differential.

# Theorem (Honda-K.-Tian-Yuan)

a) There exists a choice of perturbation data such that spaces  $\mathcal{M}_e(\underline{x}, \underline{x}')$  and  $\mathcal{M}_{T,e}(\underline{\sigma}; \underline{\sigma}_0)$  are smooth manifolds.

of dimensions  $|\underline{x}'| - |\underline{x}| - (n-2)e$  &  $|\underline{\sigma}_0| - |\underline{\sigma}_1| - \dots - |\underline{\sigma}_d| - (n-2)e + (d-2)$ .

b) Set  $CM_{-*}^0(\Omega^{1,2}(M, g, g'))$  to be a graded  $\mathbb{Z}$ -module generated by  $\mathcal{P}$ , and set  $CM_{-*}(\Omega^{1,2}(M, g, g')) = CM_{-*}^0 \otimes \mathbb{Z}[[\hbar]]$ ,  $|\hbar| = 2-n$

Then  $M'_{Morse}: CM_{-*}(\Omega^{1,2}(M, g, g')) \rightarrow CM_{-*}(\Omega^{1,2}(M, g, g'))$   

$$\gamma \mapsto \sum \frac{\# \mathcal{M}_e(\underline{x}, \underline{x}')}{e!} \hbar^e \cdot \underline{x}'$$

is a differential.

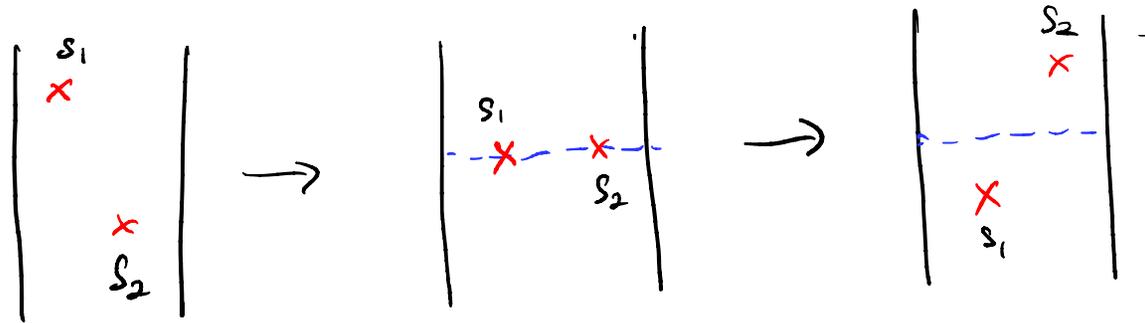
c) There are  $A_\infty$ -operations

$$M^d_{Morse}(\underline{\sigma}_d, \dots, \underline{\sigma}_1) = \sum_{\substack{T\text{-tree} \\ \text{type}}} \# \frac{\mathcal{M}_{T,e}(\underline{\sigma}_d, \dots, \underline{\sigma}_1; \underline{\sigma}_0)}{e!} \hbar^e \cdot \underline{\sigma}_0$$

which make  $CM_{-*}(\Omega^{1,2}(M, g))$  into an  $A_\infty$ -algebra.

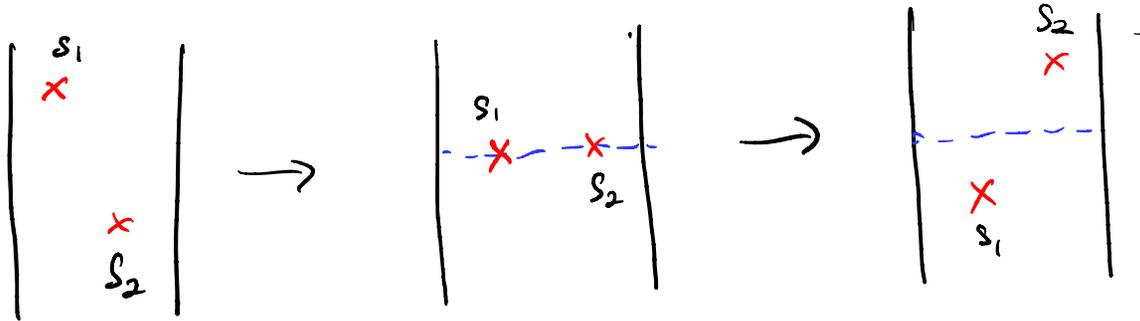
Sketch:

b) The only nontrivial new bdry phenomenon in 1-param. families are "swaps" of switches:

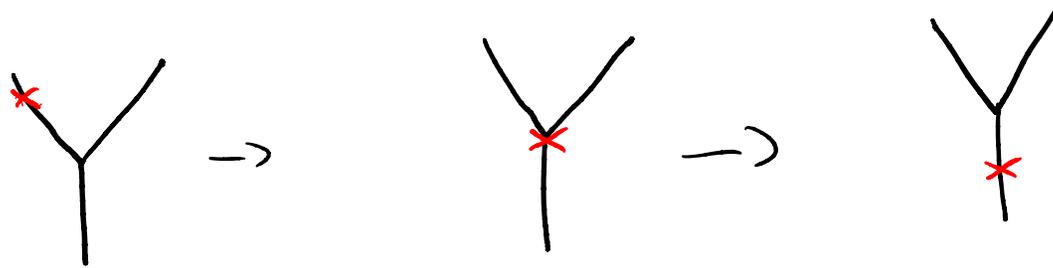


Sketch:

b) The only nontrivial new bdry phenomenon in 1-param. families are "swaps" of switches:



c) For trees new phenomenon is:



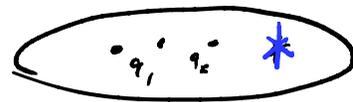
## II | Morse multiloop $A_\infty$ -algebra for $S^2$ :

$$q = (q_1, \dots, q_k) \in \Sigma \quad \downarrow \text{surface}$$

Definition (Morton-Samuelson)

Braid surface Hecke algebra

$$BSk_k(\Sigma, q)$$



is a quotient of  $\mathbb{Z}[\hbar, c^{\pm 1}]$   $B_{\mathbb{N}_k}(\Sigma, q, *) \leftarrow$  braid group  
subject to:

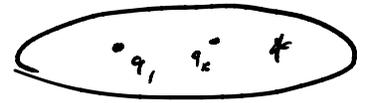
# II | Morse multiloop $A_\infty$ -algebra for $S^2$ :

$$g = (g_1, \dots, g_k) \in \Sigma \quad \downarrow \text{surface}$$

Definition (Morton-Samuelson)

Braid surface Hecke algebra

$$BSk_k(\Sigma, g)$$



is a quotient of  $\mathbb{Z}\langle h, c^{\pm 1} \rangle$  subject to:

$$B_{M_k}(\Sigma, g, *) \leftarrow \text{braided group}$$

1) HOMFLY-PT skein relation

$$\begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \nearrow \\ \swarrow \end{array} = h \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

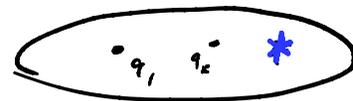
# II | Morse multiloop $A_\infty$ -algebra for $S^2$ :

$q = (q_1, \dots, q_k) \in \Sigma$  ↓ surface

Definition (Morton-Samuelson)

Braid surface Hecke algebra

$$BSk_k(\Sigma, q)$$



is a quotient of  $\mathbb{Z}[\hbar, c^{\pm 1}]$   $B_{M_k}(\Sigma, q, *) \leftarrow$  braid group

1) HOMFLY-PT skein relation

$$\begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \searrow \\ \nearrow \end{array} = \hbar \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

2) marked point relation

$$\begin{array}{c} \curvearrowright \\ \uparrow \\ \downarrow \end{array} = c^2 \begin{array}{c} \uparrow \\ \uparrow^* \end{array}$$

# II | Morse multiloop $A_\infty$ -algebra for $S^2$ :

$q = (q_1, \dots, q_k) \in \Sigma$  ↓ surface

Definition (Morton-Samuelson)

Braid surface Hecke algebra

$$BSk_k(\Sigma, q)$$



is a quotient of  $\mathbb{Z}[\hbar, c^{\pm 1}]$  subject to:

$$B_{M_k}(\Sigma, q, *) \leftarrow \text{braided group}$$

1) HOMFLY-PT skein relation

$$\begin{array}{c} \nearrow \\ \searrow \end{array} - \begin{array}{c} \searrow \\ \nearrow \end{array} = \hbar \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

2) marked point relation

$$\begin{array}{c} \curvearrowright \\ \uparrow \\ \downarrow \end{array} = c^2 \begin{array}{c} \uparrow \\ \uparrow^* \end{array}$$

Morton-Samuelson (19) show that

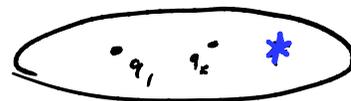
$$BSk_k(\mathbb{T}^2, *) \cong \text{to } gl_k \text{ double affine Hecke algebra}$$

# II | Morse multiloop $A_\infty$ -algebra for $S^2$ :

$g = (g_1, \dots, g_k) \in \Sigma$  <sup>surface</sup>

Definition (Morton-Samuelson)

$B\mathcal{S}k_k(\Sigma, g)$



Braid surface Hecke algebra

is a quotient of  $\mathbb{Z}[\hbar, c^{\pm 1}]$   $B\mathcal{M}_k(\Sigma, g, *) \leftarrow$  braid group

subject to:

1) HOMFLY-PT skein relation

$$\begin{array}{c} \nearrow \searrow \\ \nwarrow \nearrow \end{array} - \begin{array}{c} \nwarrow \nearrow \\ \nearrow \nwarrow \end{array} = \hbar \begin{array}{c} \uparrow \\ \downarrow \end{array}$$

2) marked point relation

$$\begin{array}{c} \curvearrowright \\ \uparrow \\ \downarrow \end{array} = c^2 \begin{array}{c} \uparrow \\ \uparrow \end{array}$$

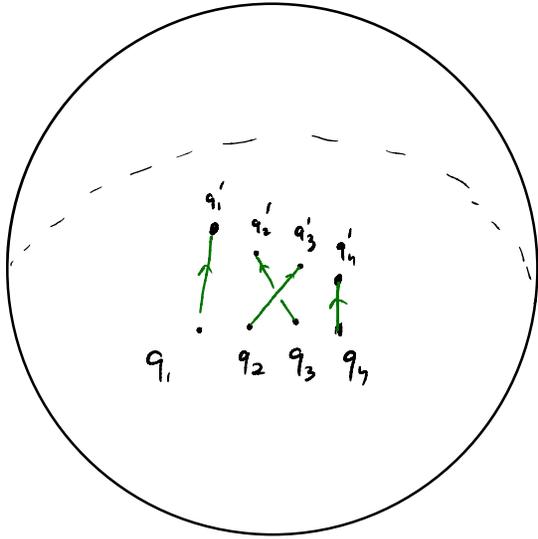
Morton-Samuelson ('19) show that  $B\mathcal{S}k_k(\mathbb{T}^2, *) \cong$  to  $gl_k$  double affine Hecke algebra

Theorem (Honda-Tian-Juan, '22) For closed, orient. surface  $\Sigma$  of genus  $g > 0$

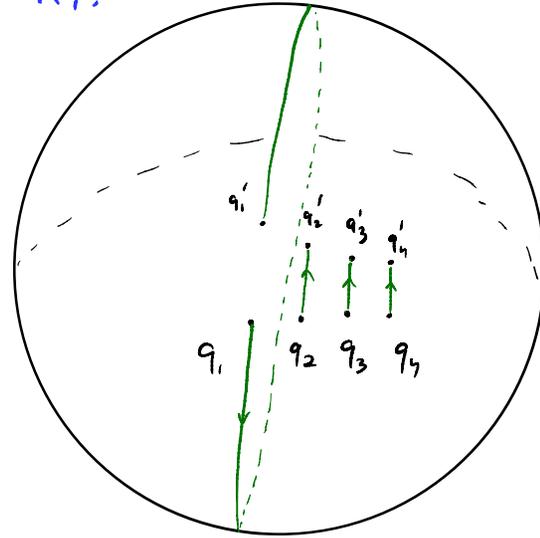
$$CM_{-*}(\Omega^{1,2}(\Sigma, g)) \underset{g.i.}{\cong} B\mathcal{S}k_k(\Sigma, g) \underset{c=1}{\otimes} \mathbb{Z}[\hbar]$$

Theorem (Honda-K.-Tian-Yuan) The  $A_\infty$ -algebra  $CM_*(S^2, \mathfrak{g})$  is quasi-equivalent to the unital dga  $\mathfrak{H}_k$  over  $\mathbb{Z}\langle T, X \rangle$  w/ generators  $T_1, \dots, T_{k-1}$  and  $X_i$  w/  $|T_i| = 0, |X_i| = -1$  subject to:

$T_i$ :



$X_i$ :



Theorem (Honda-K.-Tian-Yuan) The  $A_\infty$ -algebra  $CM_*(S^2, \mathfrak{g})$  is quasi-equivalent to the unital dga  $\mathcal{H}_k$  over  $\mathbb{Z}[\hbar]$  w/ generators

$T_1, \dots, T_{k-1}$  and  $x_i$  w/  $|T_i| = 0, |x_i| = -1$  subject to:

(1)  $T_i^2 = 1 + \hbar T_i$ , (2)  $T_i T_{i+\hbar} T_i = T_{i+\hbar} T_i T_{i+\hbar}$   $1 \leq i \leq k-2$ , (3)  $T_i T_j = T_j T_i$   $|i-j| > 1$

(4)  $x_i T_j = T_j x_i$ ,  $2 \leq j$  (5)  $x_i T_i^{-1} x_i T_i^{-1} + T_i^{-1} x_i T_i^{-1} x_i T_i^{-1} = 0$

Theorem (Honda-K.-Tian-Yuan) The  $A_\infty$ -algebra  $CM_{\hbar}(S^2, \mathfrak{g})$  is quasi-equivalent to the unital dga  $\mathcal{H}_k$  over  $\mathbb{Z}\langle\hbar\rangle$  w/ generators  $T_1, \dots, T_{k-1}$  and  $x_i$  w/  $|T_i| = 0, |x_i| = -1$  subject to:

(1)  $T_i^2 = 1 + \hbar T_i$ , (2)  $T_i T_{i+\hbar} T_i = T_{i+\hbar} T_i T_{i+\hbar}$   $1 \leq i \leq k-2$ , (3)  $T_i T_j = T_j T_i$   $|i-j| > 1$

(4)  $x_i T_j = T_j x_i$ ,  $2 \leq j$ , (5)  $x_1 T_1^{-1} x_1 T_1^{-1} + T_1^{-1} x_1 T_1^{-1} x_1 T_1^{-1} = 0$

$dT_i = 0$ ,  $dx_i = T_1 T_2 \dots T_{k-2} T_{k-1}^2 T_{k-2} \dots T_1 - 1$ .

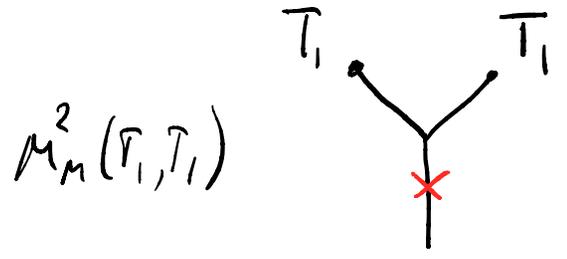
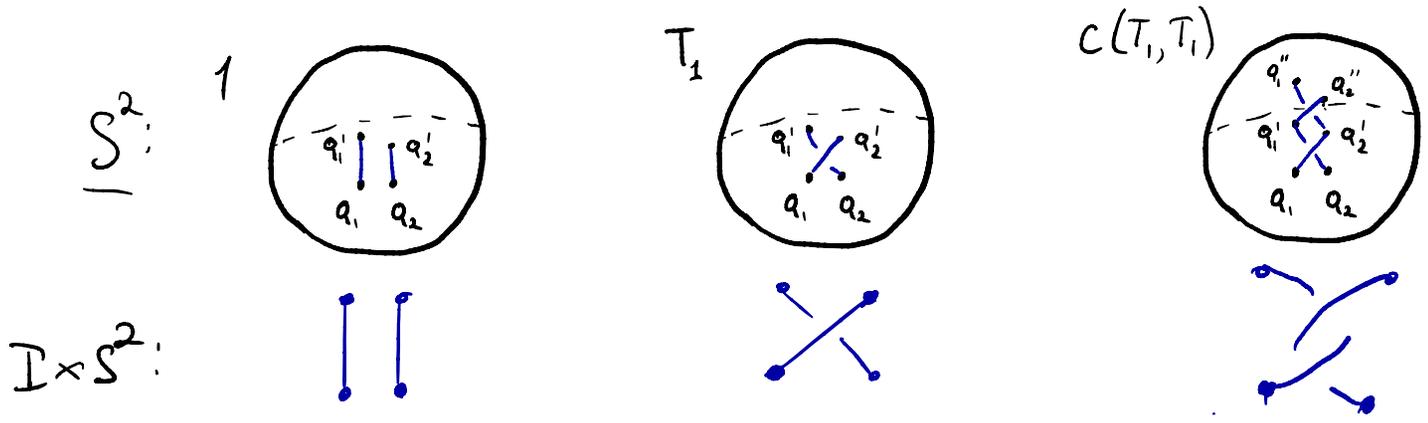
Question: Does one recover this dga as a derived shein algebra of  $S^2$ , e.g. by computing appropriate factorization homology as in works of Gunningham-Safronov-Jordan & Ben-Zvi - Brochier-Jordan?

## Verifying relations

1)  $T_1^2 = 1 + \hbar T_1$ ,  $k=2$ ; we check that  $\mu_{\text{Morse}}^2(T_1, T_1) = 1 + \hbar T_1$

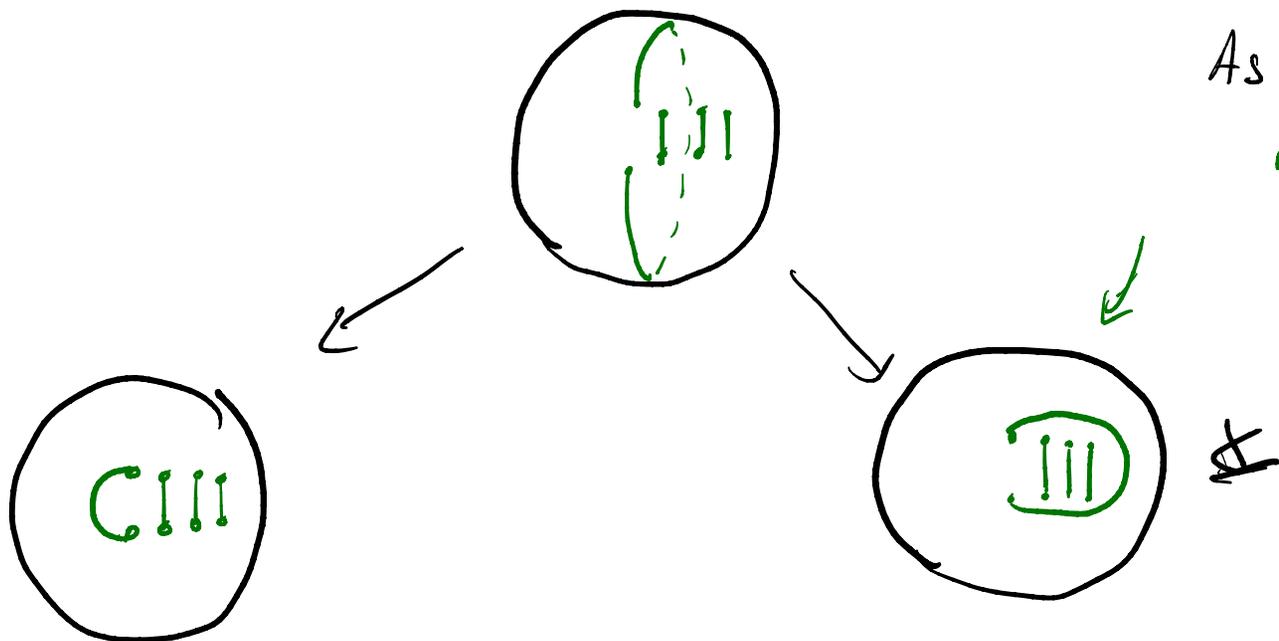
# Verifying relations

1)  $T_1^2 = 1 + \hbar T_1$ ,  $k=2$ ; we check that  $M_{\text{Morse}}^2(T_1, T_1) = 1 + \hbar T_1$



2)

$$dx_i = T_1 \dots T_{k-2} T_{k-1}^2 T_{k-2} \dots T_1 - 1$$



As a braid :



One can show that "full resolution" of  $\underline{\delta}$  is equal to the class of  $\underline{\delta}$  in the braid skein algebra.

## Higher $A_\infty$ -operations

- It is natural to expect that  $\mu_d = 0$  for  $d \geq 3$ , but even for  $k=1$  it is not immediate for round metric.

## Higher $A_\infty$ -operations

- It is natural to expect that  $M_d = 0$  for  $d \geq 3$ , but even for  $k=1$  it is not immediate for round metric.

- Cure: consider Finsler metrics of Katok examples:

$\lambda \in [0, 1)$

$$F_\lambda = \frac{\sqrt{(1 - \lambda^2 \sin^2 r) dr^2 + \sin^2 r d\varphi^2} - \lambda \sin^2 r d\varphi}{1 - \lambda^2 \sin^2 r}$$

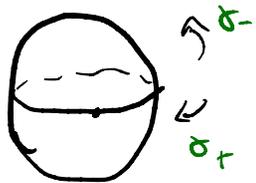
in polar  
geodes. coord  
 $(r, \varphi) \in (0, \pi) \times [0, 2\pi]$

## Higher $A_\infty$ -operations

- It is natural to expect that  $\mu_d = 0$  for  $d \geq 3$ , but even for  $k=1$  it is not immediate for round metric.

- Cure: consider Finsler metrics of Katok examples:

$t \in [0, 1)$   $F_\lambda = \frac{\sqrt{(1 - \lambda^2 \sin^2 r) dr^2 + \sin^2 r d\varphi^2} - \lambda \sin^2 r d\varphi}{1 - \lambda^2 \sin^2 r}$  ← in polar geodes. coord  $(r, \varphi) \in (0, \pi) \times [0, 2\pi]$



$\sigma_\pm$  - shortest geodesics in each direction

$$\text{Length}_{F_\lambda}(\sigma_+^m) = \frac{2\pi m}{1-\lambda}, \quad \text{Length}_{F_\lambda}(\sigma_-^m) = \frac{2\pi m}{1+\lambda}$$

So below some index  $N(\lambda)$  we have  $\text{length}_{F_\lambda}(\gamma) = C \cdot \text{ind}(\gamma)$

Denote by  $A_{n,k}$  the  $A_\infty$ -alg associated w/  $\lambda = \frac{1}{n} + \epsilon$ .

Using the above "linearity" we can show that for  $(A_{i,k})_{i \in \mathbb{Z}_{\geq 1}}$ :

$$(A) \quad M_i^d : \bigoplus^d A_{i,k} [S] \rightarrow A_{i,k} [S-d+2] \text{ vanishes for } 0 \geq S \geq (d-2)-i$$

Using the above "linearity" we can show that for  $(A_{i,k})_{i \in \mathbb{Z}_{\geq 1}}$ :

$$(A) \quad M_i^d : A_{i,k}^{\oplus d}[S] \rightarrow A_{i,k}[S-d+2] \text{ vanishes for } 0 \geq S \geq (d-2)-i$$

$$(B) \quad \text{There are q.-e. } \mathcal{F}_i : A_{i-1,k} \rightarrow A_{i,k} \text{ s.t.}$$

$$\mathcal{F}_i^1 : A_{i-1,k}[S] \rightarrow A_{i,k}[S] \text{ is an isom for } S \geq -i$$

Using the above "linearity" we can show that for  $(A_{i,k})_{i \in \mathbb{Z}_{\geq 1}}$ :

(A)  $M_i^d : A_{i,k}^{\oplus d}[S] \rightarrow A_{i,k}[S-d+2]$  vanishes for  $0 \geq S \geq (d-2)-i$

(B) There are q.-e.  $F_i : A_{i-1,k} \rightarrow A_{i,k}$  s.t.

$F_i^1 : A_{i-1,k}[S] \rightarrow A_{i,k}[S]$  is an isom for  $S \geq -i$

(C)  $F_i^k : A_{i-1,k}^{\oplus k}[S] \rightarrow A_{i,k}[S-k+1]$  vanishes for  $S \geq (k-1)-i$

$F_i^k$ 's are given by counting two-colored trees as introduced by Mazur (121)

Using the above "linearity" we can show that for  $(A_{i,k})_{i \in \mathbb{Z}_{\geq 1}}$ :

(A)  $M_i^d : A_{i,k}^{\oplus d}[S] \rightarrow A_{i,k}[S-d+2]$  vanishes for  $0 \geq S \geq (d-2)-i$

(B) There are q.-e.  $F_i : A_{i-1,k} \rightarrow A_{i,k}$  s.t.

$F_i^1 : A_{i-1,k}[S] \rightarrow A_{i,k}[S]$  is an isom for  $S \geq -i$

(C)  $F_i^k : A_{i-1,k}^{\oplus k}[S] \rightarrow A_{i,k}[S-k+1]$  vanishes for  $S \geq (k-1)-i$

$F_i^1$ 's are given by counting two-colored trees as introduced by Mazur (121)

(A)+(B)+(C)  $\Rightarrow$

$$\lim_{i \rightarrow \infty} A_{i,k} \cong H_k$$

III | Connection w/ higher-dimensional Heegaard Floer An-algebra of cotangent fibers.

$H: T^*M \rightarrow \mathbb{R}$  - Hamiltonian given by  $H(q, p) = \frac{1}{2}|p|^2$

Denote  $T_q^*M = \bigsqcup_{i=1}^k T_{q_i}^*M$ ,  $CW_0^*(T_q^*M) = \bigoplus_{\sigma \in \mathcal{S}_k} \bigotimes_{i=1}^k CW^*(T_{q_i}^*M, T_{q_{\sigma(i)}}^*M)$

$CW^*(T_q^*M) = CW_0^*(T_q^*M) \hat{\otimes} \mathbb{Q}[[\hbar]]$ ,  $|\hbar| = 2 - 4$

III | Connection w/ higher-dimensional Heegaard Floer An algebra of cotangent fibers.

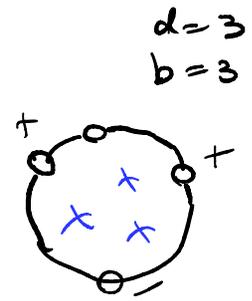
$H : T^*M \rightarrow \mathbb{R}$  - Hamiltonian given by  $H(q,p) = \frac{1}{2}|p|^2$

Denote  $T_q^*M = \bigsqcup_{i=1}^k T_{q_i}^*M$ ,  $CW_0^*(T_q^*M) = \bigoplus_{\sigma \in S_k} \bigotimes_{i=1}^k CW^*(T_{q_i}^*M, T_{q_{\sigma(i)}}^*M)$

$CW^*(T_q^*M) = CW_0^*(T_q^*M) \hat{\otimes} \mathbb{Q}[[\hbar]]$ ,  $|\hbar| = 2-h$

Hurwitz-type moduli &  $M_{Floer}^d$ :

$R^{d,b}$  - moduli space of  $(d+1)$ -punctured disks w/  $b$  interior marked points



$R_{k,\chi}^d$  - moduli space of  $k$ -fold simply branched covers  $F \rightarrow \dot{D}$  over  $(d+1)$ -punctured disk  $\dot{D}$ , w/  $\chi(F) = \chi$ , branch pts =  $(k-\chi)$  marked pts in  $\dot{D}$

III | Connection w/ higher-dimensional Floer An-algebra of cotangent fibers.

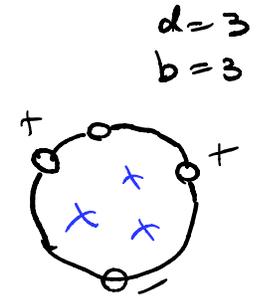
$H : T^*M \rightarrow \mathbb{R}$  - Hamiltonian given by  $H(q, p) = \frac{1}{2} |p|^2$

Denote  $T_g^* M = \bigsqcup_{i=1}^k T_{q_i}^* M$ ,  $CW_0^*(T_g^* M) = \bigoplus_{\sigma \in S_k} \bigotimes_{i=1}^k CW^*(T_{q_i}^* M, T_{q_{\sigma(i)}}^* M)$

$CW^*(T_g^* M) = CW_0^*(T_g^* M) \hat{\otimes} \mathbb{Q}[[\hbar]]$ ,  $|\hbar| = 2 - n$

Hurwitz-type moduli &  $M_{Floer}^d$ :

$R^{d,b}$  - moduli space of  $(d+1)$ -punctured disks w/  $b$  interior marked points



$R_{k,\chi}^d$  - moduli space of  $k$ -fold simply branched covers  $F \rightarrow \mathring{D}$  over  $(d+1)$ -punctured disk  $\mathring{D}$ , w/  $\chi(F) = \chi$ , branch pts =  $(k-\chi)$  marked pts in  $\mathring{D}$

Example

$(\text{disk with 2 branch points} \rightarrow \text{disk with 1 marked point}) \in R_{2,1}^1$

## $A_\infty$ -operations

$$\mu_{\text{Floer}}^d : (CW^*(T_{\mathbb{Z}}^*M))^{\otimes d} \rightarrow CW^*(T_{\mathbb{Z}}^*M)$$

$$\mu_{\text{Floer}}^d(\underline{y}_d, \dots, \underline{y}_1) = \sum_{\#} \frac{\# M_{k,x}(\underline{y}_d, \dots, \underline{y}_1, \underline{y}_0)}{(k-x)!} \cdot \hbar^{(k-x)} \cdot \underline{y}_0$$

where  $\underline{y}_i$  are tuples of Hamiltonian chords.

## $A_\infty$ -operations

$$\mu_{\text{Floer}}^d : (CW^*(T_g^*M))^{\otimes d} \rightarrow CW^*(T_g^*M)$$

$$\mu_{\text{Floer}}^d(\underline{y}_d, \dots, \underline{y}_1) = \sum \# \frac{\mathcal{M}_{k,x}(\underline{y}_d, \dots, \underline{y}_1; \underline{y}_0)}{(k-x)!} \cdot h^{(k-x)} \cdot \underline{y}_0$$

where  $\underline{y}_i$  are tuples of Hamiltonian chords.

$\mathcal{M}_{k,x}(\underline{y}_d, \dots, \underline{y}_1; \underline{y}_0)$  is a moduli space of curves

$$u = (\pi, v) : \dot{F} \rightarrow \dot{D} \times T^*M \quad \text{s.t.}$$

1)  $(\pi : \dot{F} \rightarrow \dot{D}) \in \mathcal{R}_{k,x}^d$

2)  $v$  solves perturbed  $\bar{J}$ -holic equation defined in terms of Floer data dependent on  $(\pi : \dot{F} \rightarrow \dot{D})$

## Theorem (Honda-K. Tian-Yuan)

There is a quasi-equivalence

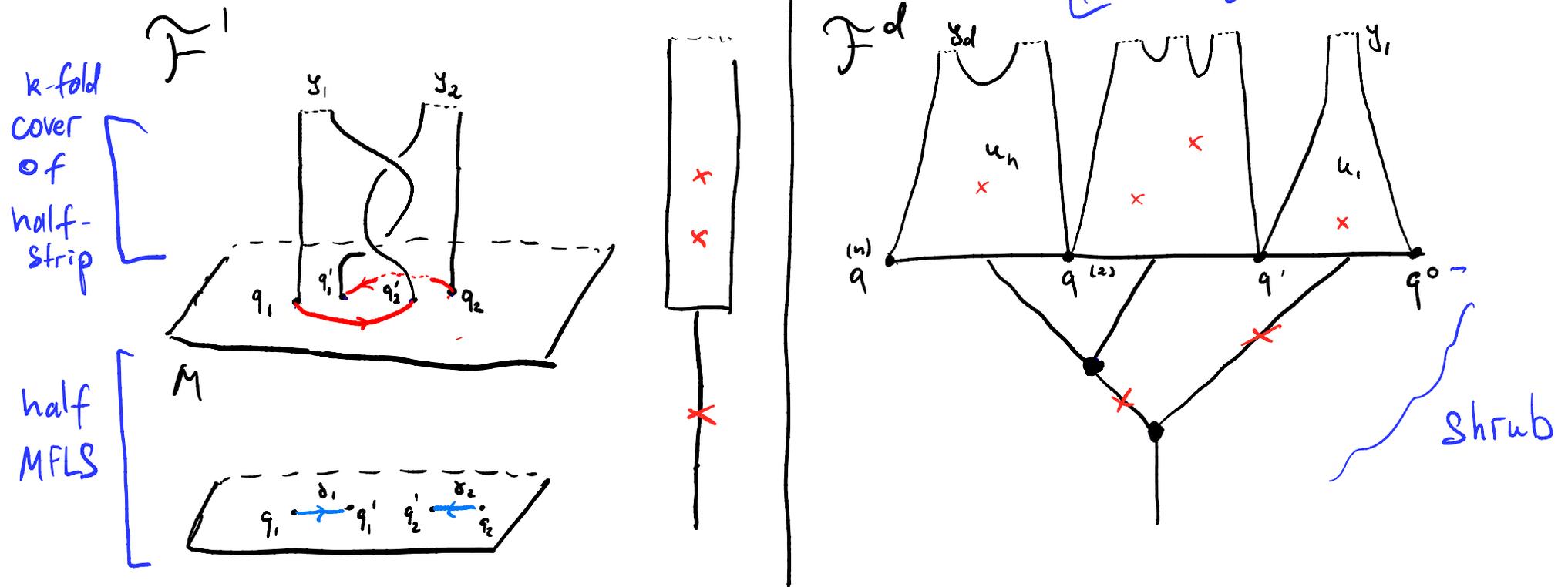
$$\mathcal{F}: CW^*(T_g^*M) \rightarrow CM_*(\Omega^{1,2}(M,g))$$

# Theorem (Honda-K-Tian-Yuan)

There is a quasi-equivalence

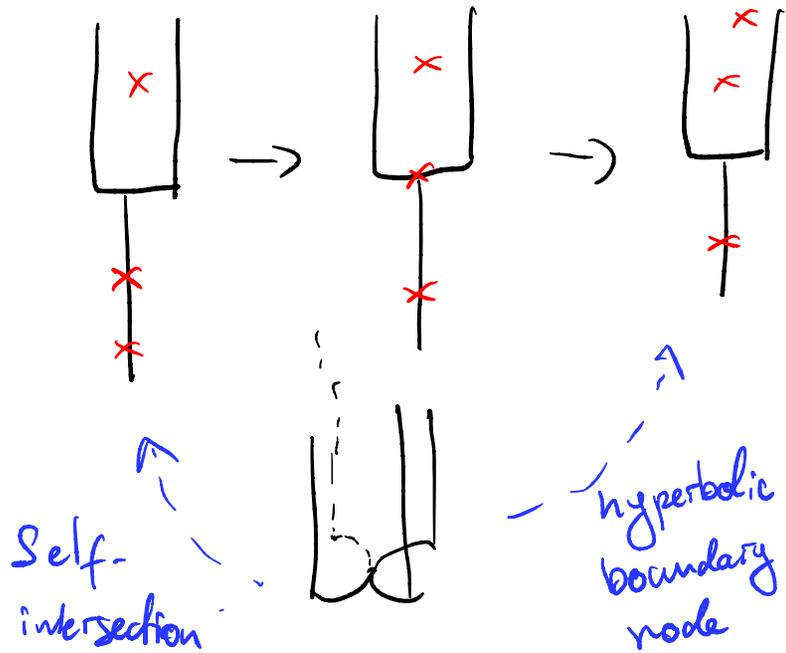
$$\mathcal{F}: CW^*(T_q^*M) \rightarrow CM_*(\Omega^{1,2}(M, g))$$

Pictorially:



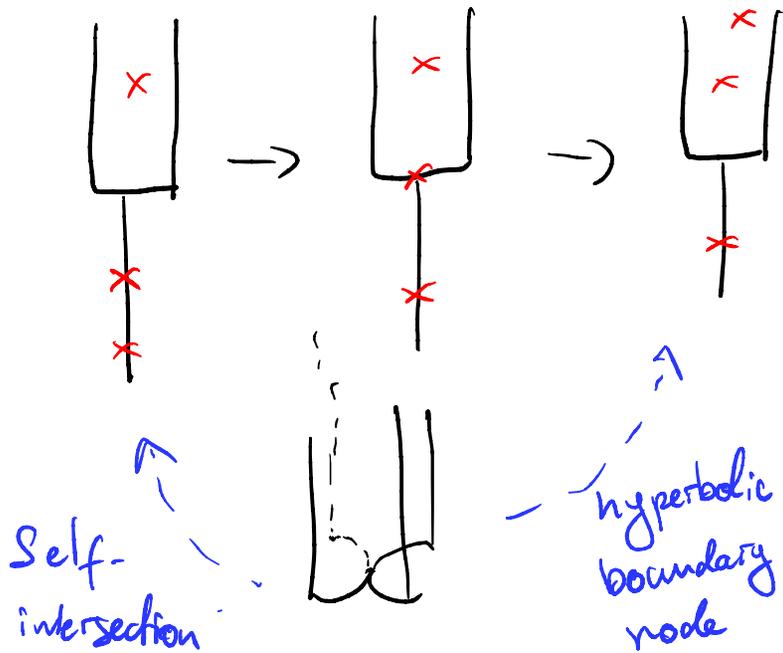
Sketch: 1) For  $k=1$  it is a result of Abouzaid ('12) and is a variation of Viterbo isomorphism.

2) Interesting boundary degeneration:



Sketch: 1) For  $k=1$  it is a result of Abouzaid ('12) and is a variation of Viterbo isomorphism.

2) Interesting boundary degeneration:



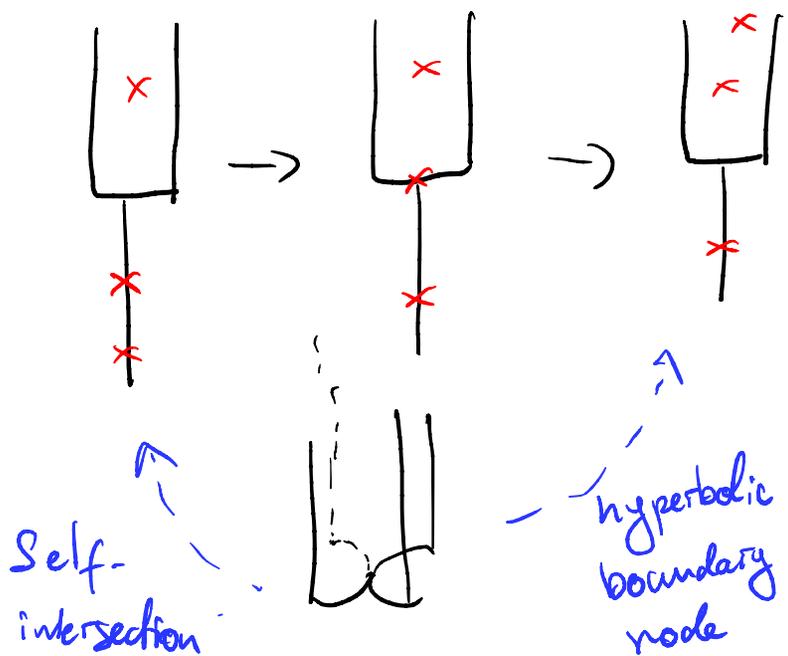
• local model:  $(z, w) \in \mathbb{C}^2$   
 $R \in \mathbb{R}_{\geq 0}$

$$\begin{cases} zw = e^{-R} \\ \text{Im } z \geq 0 \end{cases}$$

• Accurate proof of this "classical" result is given by Hirsch-Hugtenburg ('25)

Sketch: 1) For  $k=1$  it is a result of Abouzaid ('12) and is a variation of Viterbo isomorphism.

2) Interesting boundary degeneration:



• local model:

$$\begin{cases} zw = e^{-R} \\ \text{Im } z \geq 0 \end{cases}$$

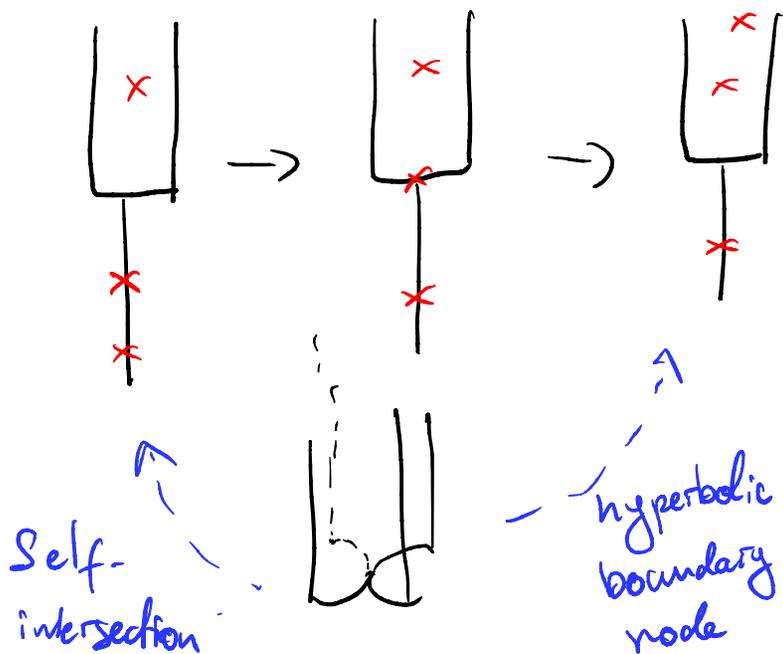
$$\begin{aligned} (z, w) &\in \mathbb{C}^2 \\ R &\in \mathbb{R}_{\geq 0} \end{aligned}$$

• Accurate proof of this "classical" result is given by Hirsch-Hugtenburg ('25)

3) This shows that  $\mathcal{F}'$  is a chain map. Also it is a deformation of  $\mathcal{F}'|_{\hbar=0}$  which is quasi-isom by above. Hence  $\mathcal{F}'$  is q.-i.

Sketch: 1) For  $k=1$  it is a result of Abouzaid ('12) and is a variation of Viterbo isomorphism.

2) Interesting boundary degeneration:



• local model:  $(z, w) \in \mathbb{C}^2$   
 $R \in \mathbb{R}_{\geq 0}$   
 $\begin{cases} zw = e^{-R} \\ \text{Im } z \geq 0 \end{cases}$

• Accurate proof of this "classical" result is given by Hirsch-Hugtenburg ('25)

3) This shows that  $\mathcal{F}'$  is a chain map. Also it is a deformation of  $\mathcal{F}'|_{\hbar=0}$  which is quasi-isom by above. Hence  $\mathcal{F}'$  is q.-i.

4)  $\mathcal{F}$  is an  $A_\infty$ -functor: few more gluing results are needed and these are similar to gluing results in works of Abouzaid ('10, '11)

Thank  
you!

# Excision

