# Systolic $S^1$-index and characterization of non-smooth Zoll convex bodies

Symplectic zoominar

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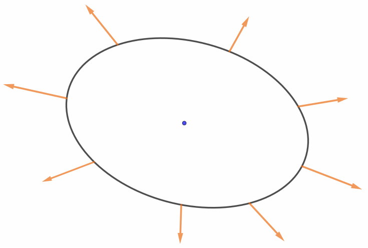
# Motivation

$\lambda_0 := \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$-standard Liouville form on $\mathbb{R}^{2n}$.

$\omega_0 := d\lambda_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$ - standard symplectic form.

$C \subset \mathbb{R}^{2n}$- A bounded convex domain (convex body) with a smooth boundary whose interior contains the origin.



- Vector field $r = \frac{1}{2}\sum_{j=1}^{n}(x_j\partial_{x_j} + y_j\partial_{y_j})$transverse to the $\partial C$.
- $r$ is the Liouville vector field for $\lambda_0$.

$\Bigg\} \implies$

$\implies (C, \lambda_0)$ - Liouville domain.

# Motivation

$R$ - Reeb vector field of $(\partial C, \lambda_0)$ given by the system of equations

$$\bullet \; i_R \lambda_0 = 1, \quad i_R \omega_0 = 0, \text{ on } \partial C.$$

- $C$- *Zoll* if all of its Reeb orbits on $(\partial C, \lambda_0)$ are closed and share a common minimal period.
- $C$ smooth and strongly convex if sectional curvature positive everywhere on $\partial C$.

## Corollary (Ginzburg–Gurel–Mazzucchelli '21)

*Smooth and strongly convex body* $C \subset \mathbb{R}^{2n}$ *is Zoll if and only if*

$$s_1(C) = s_n(C)$$

* $(s_k(C))_{k \in \mathbb{N}}$-Ekeland-Hofer spectral invariants ('87).

## Theorem (M. '24)

*Let* $C \subset \mathbb{R}^{2n}$ *be a convex body. Then, for all* $k \in \mathbb{N}$,

$$c_k^{GH}(C) = s_k(C).$$

* $(c_k^{GH}(C))_{k \in \mathbb{N}}$ - Gutt-Hutchings capacities ('18).

# Motivation

## Corollary (M. '24)

*Smooth and strongly convex body* $C \subset \mathbb{R}^{2n}$ *is Zoll if and only if*

$$c_1^{GH}(C) = c_n^{GH}(C).$$

## Corollary (M. '24)

*Let* $C \subset \mathbb{R}^{2n}$ *be a smooth and strongly convex body whose interior is symplectomorphic to the interior of a Ball. Then* $C$ *is Zoll.*

# Spaces of generalized and centralized generalized systoles

- $i_R\lambda_0 = 1, \quad i_R\omega_0 = 0,$ on $\partial C$. - Reeb vector field

$\mathrm{Char}(\partial C)$ - set of closed Reeb orbits.

$\sigma(\partial C)$ - spectrum of $\partial C$ (set of periods of closed Reeb orbits).

$H_C : \mathbb{R}^{2n} \to \mathbb{R}$ - positivly two homogenious function such that $H_C|_{\partial C} = 1$.

- $R(x) = J_0 \nabla H_C(x), \quad x \in \partial C,$

where $J_0 : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ is the standard almost complex structure.

- $\gamma : \mathbb{R}/T\mathbb{Z} \to \partial C, \quad \dot\gamma(t) = J_0 \nabla H_C(\gamma(t))$ -closed Reeb orbit of period $T > 0$.

$$\mathcal{A}(x) = \int_x \lambda_0 = \frac{1}{2}\int_0^T \langle x(t), J_0 \dot x(t)\rangle dt \; - \; \text{Action of the loop } x : \mathbb{R}/T\mathbb{Z} \to \mathbb{R}^{2n}.$$

By reparametrizing

$$\mathrm{Char}(\partial C) = \{\gamma : \mathbb{T} \to \partial C \mid \dot\gamma(t) = T J_0 \nabla H_C(\gamma(t)), \; T > 0\}$$

where $\mathbb{T} := \mathbb{R}/\mathbb{Z}$.

- $\sigma(\partial C) = \{\mathcal{A}(\gamma) \mid \gamma \in \mathrm{Char}(\partial C)\}.$

# Spaces of generalized and centralized generalized systoles

$C$ any convex body $\implies$ $H_C : \mathbb{R}^{2n} \to \mathbb{R}$ convex function.

The subdifferential of $H_C$ at $x \in \mathbb{R}^{2n}$ is given by

$$\partial H_C(x) = \{\eta \in \mathbb{R}^{2n} \mid H_C(y) - H_C(x) \geqslant \langle \eta, y - x \rangle, \quad \forall y \in \mathbb{R}^{2n}\}.$$

**Generalized closed characteristics:**

$$\mathrm{Char}(\partial C) := \{\gamma : \mathbb{T} \to \partial C \text{ a. c.} \mid \dot{\gamma}(t) \in T J_0 \partial H_C(\gamma(t)) \text{ a.e., for some } T > 0\}.$$

**Generalized systoles:**

$\mathrm{Sys}(\partial C)$ - subset of $\mathrm{Char}(\partial C)$ with minimal action.

- $\pi_0(\gamma) = \gamma - \int_{\mathbb{T}} \gamma(t) dt$ $(\gamma \mapsto \dot{\gamma})$

**Centralized generalized systoles:**

$$\mathrm{Sys}_0(\partial C) := \pi_0(\mathrm{Sys}(\partial C))$$

$\mathrm{Sys}(\partial C)$, $\mathrm{Sys}_0(\partial C)$ $-$ $S^1$ spaces $(\theta \cdot \gamma = \gamma(\cdot - \theta))$.

# $S^1$-equivariant cohomology

$X$ - $S^1$ space.

$ES^1 = S^\infty$, with action given by:

$$\theta \cdot (z_1, z_2, \cdot) = (e^{2\pi i\theta} z_1, e^{2\pi i\theta} z_2, \dots), \qquad \theta \in \mathbb{T}$$

.

- $X \times ES^1 \simeq X$ and $S^1$ acts freely on $X \times ES^1$.

$$H^*_{S^1}(X) := H^*(X \times_{S^1} ES^1)$$

$X \times_{S^1} ES^1 := X \times ES^1 / S^1$

- If $S^1$ acts freely on $X$ then $H^*_{S^1}(X) \cong H^*(X/S^1)$.

# Fadell-Rabinowitz index

$BS^1 = ES^1/S^1 = \mathbb{C}P^\infty.$

$$\pi_2 : X \times_{S^1} ES^1 \to BS^1, \quad \pi_2([x,y]) = [y]$$

$$\pi_2^* : H^*(BS^1) \to H^*_{S^1}(X)$$

$H^*(BS^1) \cong \mathbb{F}[e], \quad |e| = 2$

**The Fadell-Rabinowitz index:** $\mathrm{ind}_{FR}(X;\mathbb{F}) = \sup\limits_{k\geqslant 0}\{k+1 \mid \pi_2^* e^k \neq 0\}.$

$\mathrm{ind}_{FR}(\varnothing;\mathbb{F}) = 0$

$F : X \to Y$- $S^1$-equivariant $\implies \mathrm{ind}_{\mathrm{FR}}(X) \leqslant \mathrm{ind}_{\mathrm{FR}}(Y).$

$\pi_0 - S^1$-equivariant $\implies \mathrm{ind}_{\mathrm{FR}}(\mathrm{Sys}(\partial C)) \leqslant \mathrm{ind}_{\mathrm{FR}}(\mathrm{Sys}_0(\partial C)).$

# Main results

## Definition

*Let $C \subset \mathbb{R}^{2n}$ be a convex body, and let $C'$ be any translation of $C$ whose interior contains the origin. We define the systolic $S^1$-index of $C$ as*

$$\mathrm{ind}^{S^1}_{\mathrm{sys}}(C) = \mathrm{ind}_{\mathrm{FR}}(\mathrm{Sys}_0(\partial C')),$$

*where the topology on $\mathrm{Sys}_0(\partial C')$ is induced by the uniform norm.*

## Theorem

*Let $C \subset \mathbb{R}^{2n}$ be a convex body whose interior contains the origin. Then, it holds that*

$$\mathrm{ind}_{\mathrm{FR}}(\mathrm{Sys}_0(\partial C)) = \max\{k \in \mathbb{N} \mid c^{GH}_k(C) = c^{GH}_1(C)\}.$$

*Moreover, if $C$ is strictly convex or $\partial C$ is smooth, we have*

$$\mathrm{ind}_{\mathrm{FR}}(\mathrm{Sys}_0(\partial C)) = \mathrm{ind}_{\mathrm{FR}}(\mathrm{Sys}(\partial C)).$$

*In particular, $\mathrm{ind}^{S^1}_{\mathrm{sys}}(C)$ is well-defined for a convex body $C \subset \mathbb{R}^{2n}$, and it holds*

$$\mathrm{ind}^{S^1}_{\mathrm{sys}}(C) = \max\{k \in \mathbb{N} \mid c^{GH}_k(C) = c^{GH}_1(C)\}.$$

*Gutt–Ramos ('24) $\implies$ $\mathrm{ind}^{S^1}_{\mathrm{sys}}(C) = \max\{k \in \mathbb{N} \mid c^{EH}_k(C) = c^{EH}_1(C)\}$

# Main results

## Corollary

*The systolic $S^1$-index of convex bodies is a symplectic invariant. More precisely, if the interior of a convex body $C_1$ is symplectomorphic to the interior of a convex body $C_2$, then*

$$\mathrm{ind}_{\mathrm{sys}}^{S^1}(C_1) = \mathrm{ind}_{\mathrm{sys}}^{S^1}(C_2).$$

- $\mathrm{Conv}(\mathbb{R}^{2n})$ - Set of all convex bodies in $\mathbb{R}^{2n}$ endowed with Hausdorf-distance topology.

Function

$$\mathrm{ind}_{\mathrm{sys}}^{S^1} : \mathrm{Conv}(\mathbb{R}^{2n}) \to \mathbb{N}$$

well defined ($c_k^{GH}(C) \to \infty$ as $k \to \infty$).

- $\mathrm{ind}_{\mathrm{sys}}^{S^1}$ upper semicontinuous.
- For every convex body $C$, we have $\mathrm{ind}_{\mathrm{sys}}^{S^1}(C) \leqslant 4n^3$.
- If $C = -C$, then $\mathrm{ind}_{\mathrm{sys}}^{S^1}(C) \leqslant 2n^2$.
- For every $S^1$-invariant convex body $C$ it holds $\mathrm{ind}_{\mathrm{sys}}^{S^1}(C) \leqslant n$
- For every smooth convex body $C$, it holds that $\mathrm{ind}_{\mathrm{sys}}^{S^1}(C) \leqslant n$.

# Main results

## Definition

*A convex body* $C \subset \mathbb{R}^{2n}$ *is a generalized Zoll convex body if it satisfies*

$$\mathrm{ind}_{\mathrm{sys}}^{S^1}(C) \geqslant n.$$

## Theorem

*The following statements hold:*

1. *A convex body* $C$ *is generalized Zoll if and only if* $c_1^{GH}(C) = c_n^{GH}(C)$.
2. *A convex body* $C$ *with a smooth boundary is generalized Zoll if and only if it is Zoll.*
3. *The space of generalized Zoll convex bodies is closed in the space of all convex bodies with respect to the Hausdorff distance topology.*

# Main results

## Corollary

*If the interior of a convex body $C \subset \mathbb{R}^{2n}$ is symplectomorphic to the interior of a ball, then $C$ is a generalized Zoll convex body. In particular, if the boundary of $C$ is smooth, then $C$ is Zoll.*

- The only $S^1$-invariant generalized Zoll convex body is a ball.

## Conjecture (Viterbo '00)

*For any convex body $C \subset \mathbb{R}^{2n}$*

$$c_1^{GH}(C) \leqslant n!\,\mathrm{Vol}(C).$$

*Moreover, the equality holds if and only if interior of $C$ is symplectomorphic to the interior of a ball.*

- Viterbo conjecture fails for $n \geqslant 2$ (P. Haim-Kislev, Y. Ostrover '24)

*Thank you for your attention!*