A local quasimorphism property for link spectral invariants

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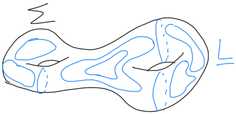


Figure: A Lagrangian link

- Link spectral invariants

$$c_{\underline{L}} : \widetilde{\mathrm{Ham}}(\Sigma) \to \mathbb{R}$$

- Introduced by Cristofaro-Gardiner, Humilière, Mak, Seyfaddini and Smith (built on earlier works of Mak-Smith, Polterovich-Shelukhin).
- They come from computing Lagrangian spectral invariants of a product torus in a symmetric power of a surface.

- CGHMSS use them to prove the simplicity conjecture on any surface (Σ, ω) (possibly $\partial\Sigma \neq \emptyset$):
- $\overline{\text{Ham}}(\Sigma)$:= the group of Hamiltonian homeomorphisms (C^0-limits of Hamiltonian diffeos), compactly supported in the interior of Σ

Theorem (CGHMSS '21)

$\overline{\text{Ham}}(\Sigma)$ *is not simple.*

- Rk: $\text{Hameo}(\Sigma)$ is a proper normal subgroup.

- $c_{\underline{L}}$ is a quasimorphism on $\widetilde{\mathrm{Ham}}(S^2)$:

$$\sup_{\varphi,\psi} \left| c_{\underline{L}}(\varphi\psi) - c_{\underline{L}}(\varphi) - c_{\underline{L}}(\psi) \right| < \infty$$

Plays a key role in proving:

Theorem (CGHMSS '22)

Cal *extends (non-canonically) to a group homomorphism* $\overline{\mathrm{Ham}}(\mathbb{D}) \to \mathbb{R}$

(Recall that $\mathrm{Cal}(\varphi) = \int\int H_t \omega dt$, $\varphi = \Phi_H^1$, $H|_{\partial\Sigma} = 0$)

Theorem (CGHMSS '22)

$\mathrm{Hameo}(S^2)$ *is not simple.*

- Link spectral invariants are not expected to be quasimorphisms on $\mathrm{Ham}(\Sigma)$ for higher genus surfaces Σ.

Main results

- Let D be a topological disc inside a surface (Σ, ω).
- $\mathrm{Ham}_D(\Sigma) :=$ Hamiltonian diffeos of Σ supported in D.

We prove:

Theorem (Mak - T. '23, Local quasimorphism property)

Under some hypotheses on $\underline{L}$, the restriction of $c_{\underline{L}}$ to $\mathrm{Ham}_D(\Sigma) \subset \mathrm{Ham}(\Sigma)$ is a quasimorphism.

And as a consequence:

Theorem (Mak-T. '23)

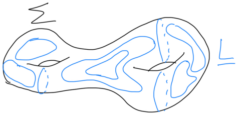
When Σ has boundary, Cal *extends (non-canonically) to a group homomorphism $\overline{\mathrm{Ham}}(\Sigma) \to \mathbb{R}$.*

Theorem (Mak-T. '23)

For Σ closed, $\mathrm{Hameo}(\Sigma)$ *is not simple.*

Link spectral invariants (CGHMSS)

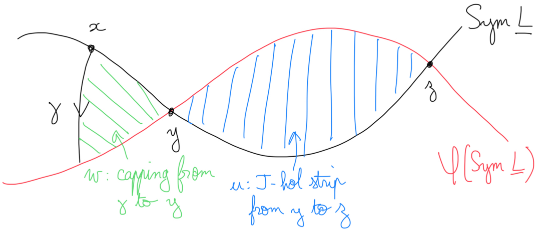
A monotone Lagrangian link on a (closed) surface (Σ, ω) is a disjoint union of circles $\underline{L} = L_1 \cup \ldots \cup L_k$, such that the connected components of $\Sigma \setminus \underline{L}$ are planar and have the same area.



Given a Hamiltonian $H : [0,1] \times \Sigma \to \mathbb{R}$, we want to associate a Floer theoretical invariant $c_{\underline{L}}(H)$.

- $L_1 \times \ldots \times L_k \subset \Sigma^k$ is a Lagrangian submanifold. However, $HF(L_1 \times \ldots \times L_k) = 0$ (displaceable).
- Take the quotient by the permutation group $\mathfrak{S}_k$:
  - get a monotone Lagrangian $\mathrm{Sym}\,\underline{L} \subset \mathrm{Sym}^k(\Sigma)$
  - need to smooth out the symplectic form near the singular locus, following work of Perutz

- $\mathrm{Sym}^k H_t(\{x_1, \dots, x_k\}) := \sum H_t(x_i)$, φ its time-1 map.
- Choose a point x on $\mathrm{Sym}\,\underline{L}$, and denote by γ its trajectory under the flow of $\mathrm{Sym}^k H$. (reference path)
- The Floer complex is generated by equivalence classes of capped intersection points, and the differential counts J-holomorphic strips with Lagrangian boundary conditions.



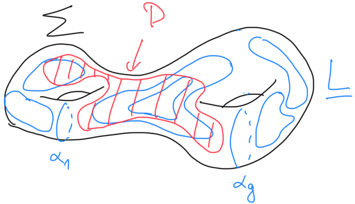
- $HF_*(\mathrm{Sym}\,\underline{L}, \mathrm{Sym}^k(H); \Lambda) \cong H_*(\mathrm{Sym}\,\underline{L}) \otimes \Lambda \cong H_*(T^k) \otimes \Lambda$.
- This is shown by analysing the disc potential.

- For $y \in \mathrm{Sym}\,\underline{L} \cap \varphi(\mathrm{Sym}\,\underline{L})$ and a capping w from γ to y, we define its action:

$$\mathcal{A}_H(y,[w]) := \int_0^1 \mathrm{Sym}\,H_t(x)dt - \int w^*\omega'$$

- $c_{\underline{L}}(H) := \frac{1}{k} c_{\mathrm{Sym}\,\underline{L}}(\mathrm{Sym}^k(H)) = \frac{1}{k} \times \{$ minimal action of a representative of the fundamental class$\}$.
- Homotopy invariance $\implies c_{\underline{L}} : \widetilde{\mathrm{Ham}}(\Sigma) \to \mathbb{R}$.
- When $\Sigma = S^2$, $c_{\underline{L}}$ is a quasimorphism
  - $\mathrm{Sym}^k S^2 \cong \mathbb{C}P^k$ and $QH^*(\mathbb{C}P^k)$ is semi-simple;
  - QH^* semi-simple $\implies$ spectral invariants are quasimorphisms (Entov-Polterovich).
- When $\Sigma \neq S^2$, $QH^*(\mathrm{Sym}^k \Sigma)$ is not semi-simple (Bertram-Thaddeus).

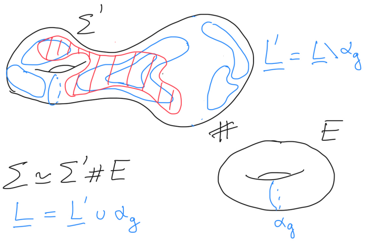
Proof of the local quasimorphism property



Assume $\underline{L}$ is a monotone link with exactly g non-contractible components, that are all away from D. Then:

Theorem (Mak - T. '23, Local quasimorphism property)

The restriction of $c_{\underline{L}}$ *to* $\mathrm{Ham}_D(\Sigma) \subset \mathrm{Ham}(\Sigma)$ *is a quasimorphism.*



Theorem (Mak-T. '23, a Künneth formula for connected sums)

For any Hamiltonian H supported in D, for a small perturbation H_ϵ, we have a natural filtered chain isomorphism:

$$CF_*(\mathrm{Sym}(\underline{L}), \mathrm{Sym}^k H_\epsilon) \cong CF_*(\mathrm{Sym}(\underline{L}'), \mathrm{Sym}^{k-1} H_\epsilon|_{\Sigma'}) \otimes CF_*(\alpha_g, H_\epsilon|_E)$$

In particular, $(k+1)c_{\underline{L}'}(H) = kc_{\underline{L}}(H|_\Sigma)$.

Idea of proof of the Künneth formula

- Step 1: Show there is a filtered isomorphism of Λ-modules

$$f: CF_*(\mathrm{Sym}(\underline{L}'), \mathrm{Sym}^{k-1} H_\epsilon|_{\Sigma'}) \otimes CF_*(\alpha_g, H_\epsilon|_E) \\ \to CF_*(\mathrm{Sym}(\underline{L}), \mathrm{Sym}^k H_\epsilon) \\ x \otimes c \mapsto x \cup \{c\}$$

(also need to identify classes of cappings)

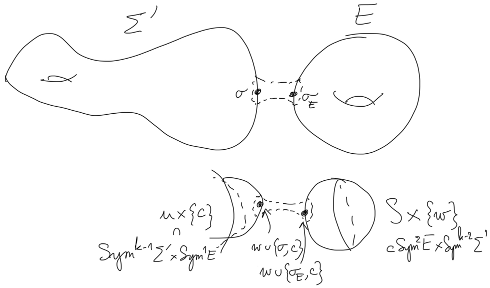
- Step 2: Show it commutes with ∂:

$$\partial f(x \otimes c) = f((\partial x) \otimes c + x \otimes (\partial c))$$

  - $\mathcal{M}_{j_E}(c, c') \cong \mathcal{M}_{J'_T}(x \cup \{c\}, x \cup \{c'\})$ (easy, take product with constant strip at x)
  - $\mathcal{M}_J(x, y) \cong \mathcal{M}_{J'_T}(x \cup \{c\}, y \cup \{c\})$ (difficult, strips can pass through the connected sum region)

$\mathcal{M}_J(x, y) \cong \mathcal{M}_{J'_T}(x \cup \{c\}, y \cup \{c\})$ (following ideas of Ozsváth and Szabó)

- neck-stretching argument, construct a map via gluing



- surjectivity with Gromov compactness

Thank you for your attention!