Symplectic Zoominar 11/04/2025:

The Lagrangian Ceresa cycle

Goal: • Introduce an equivalence relation on Lags in a sympl mfld "algebraic Lagrangian cobordism"

• Show it is strictly weaker than (cyl.) Lag cobordism ← easy
stronger than homological equivalence ← "Lagrangian Ceresa cycle"

* Thm: $\exists$ homol. trivial Lagrangian $L_C - L_{\varphi(C)}$ in a symplectic 6-torus which is not oriented algebraic Lag. nullcobordant.

Why? In AG, Z, Z' k-cycles in Y.

$Z \overset{\text{rat}}{\sim} Z'$ "families of cycles over $\mathbb{P}^1$"

↑ strictly stronger

↕ Sheridan-Smith M.S

cylindrical Lag. cobordism

$Z \overset{\text{alg.}}{\sim} Z'$ "families of cycles over curves"

↑ strictly stronger than homological equiv.

"Ceresa cycle"

↕

? ← alg Lag. cobordism

B polarised tropical torus

↳ real torus + full rank lattice $T^*_{\mathbb{Z}}B \subseteq T^*B$

↳ "has a lot of tropical curves"

$X(B) := T^*B / T^*_{\mathbb{Z}}B$ polarised sympl. torus

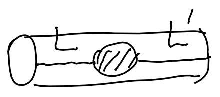
* def: L, L' Lags in (X, ω) are alg. Lag. cob if $\exists$ pst $X(B)$ w. $p, q \in B$ & a Lag $\Gamma \subseteq \overline{X(B)} \times X$ st

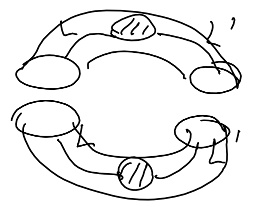
(i) $\pi_B(\Gamma) = B$

(ii) $\Gamma(F_p) = L$ & $\Gamma(F_q) = L' = \pi_X(\Gamma \cap (F_q \times X))$

↑ fibre in $X(B)$ over $p \in B$

* ex: (a) cyl. Lag cob





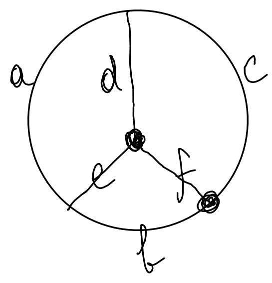
alg Lag cob

(b) $X = X(B)$, $\Delta \subseteq \overline{X(B)} \times X(B)$ is an alg Lag cob btw F_p & F_q $\forall p, q \in B$.

~~not~~ but if $p \neq q$, then F_p & F_q are not oriented (cyl.) Lag cob.

"Lagrangian Ceresa cycle"

C =



$C \hookrightarrow \mathrm{Jac}(C)$ ← polarised tropical 3-torus involut° φ

$C - \varphi(C)$ "trop. Ceresa cycle of C" nullhomologous

Zharkov (Ceresa): For generic such curve, $C - \varphi(C)$ is not alg. trivial.

$X(\mathrm{Jac}(C)) \supseteq L_C - L_{\varphi(C)}$ "Lagrangian Ceresa cycle" nullhomologous

$\downarrow$ ⇣ Lag lift

$\mathrm{Jac}(C) \supseteq C - \varphi(C)$

* Thm: For generic such C, $L_C - L_{\varphi(C)}$ is not oriented alg. Lag. nullcobordant.

↳ Sketch of proof:

* Prop: If L_C & $L_{\varphi(C)}$ are oriented ~~Lag.~~ alg. Lag. cobordant, $\partial\gamma = L_C - L_{\varphi(C)}$, then $\int_\gamma \omega^2 \in \text{periods}(\omega^2)$

↑ expressed in terms of edge lengths a, b, c, d, e & f

From Zharkov's construction, build such a 4-chain γ_0, and show generically $\int_{\gamma_0} \omega^2 \notin \text{periods}(\omega^2)$.

"$\square$"