Persistence of unknottedness of Lagrangian intersections

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Double bubble plumbings

Let Q_0 and Q_1 be two 3-spheres such that $Z = Q_0 \cap Q_1$ is a circle, with embeddings $\kappa_i : Z \hookrightarrow Q_i$. Given any identification

$$\eta : \nu_{Z/Q_0} \xrightarrow{\simeq} \nu_{Z/Q_1},$$

we can define a Stein manifold $W_\eta(\kappa_0, \kappa_1)$ as the plumbing $T^*Q_1 \#_{Z,\eta} T^*Q_2$.

More precisely, there are open neighborhoods $U(Z)_i$ of $D^*Z \subset D^*Q_i$, which are symplectomorphic to disc subbundles of $\nu_{Z/Q_i} \otimes \mathbb{C}$ over Z. We can glue the Weinstein neighborhoods using the map

$$\eta \otimes \sqrt{-1} : U(Z)_1 \xrightarrow{\simeq} U(Z)_2,$$

and the result is our plumbing $W_\eta(\kappa_0, \kappa_1)$.

Double bubble plumbings

Conversely, if (M, ω) is any 6-dimensional symplectic manifold containing a pair of Lagrangian 3-spheres Q_0, Q_1 meeting cleanly along a circle Z, a neighborhood of $Q_0 \cup Q_1 \subset M$ is symplectically equivalent to a Stein subdomain of some $W_\eta(\kappa_0, \kappa_1)$.

We will be interested in the case where $\kappa_i : Z \hookrightarrow Q_i$, $i = 0, 1$ are unknots, and simply write W_η for $W_\eta(\kappa_0, \kappa_1)$ in this case. W_η is called a *double bubble plumbing* by Smith-Wemyss.

Since η is a $\mathbb{Z}$-torsor, we obtain a sequence of Stein manifolds W_k, $k \in \mathbb{Z}$. Our convention is that when $k = 0$, the Morse-Bott surgery of Q_0 and Q_1 along Z gives $S^1 \times S^2$, and when $k \neq 0$, the Morse-Bott surgery gives the Lens space $L(|k|, 1)$. Since W_k and W_{-k} are symplectomorphic, we can assume $k \in \mathbb{Z}_{\geq 0}$.

Persistence of unknottedness

Theorem (Asplund-L)

Let (M, ω) be a 6-dimensional symplectic manifold, and $Q_0, Q_1 \subset M$ two Lagrangian spheres intersecting cleanly along a circle Z that is unknotted in both Q_0 and Q_1. Then there is no nearby Hamiltonian diffeomorphism ϕ supported in the Stein neighborhood of $Q_0 \cup Q_1$ so that $\phi(Q_0) \cap \phi(Q_1)$ is clean and is knotted in either Q_0 or Q_1.

Remark

- *There is no obstruction to changing the knot type of Z if one allows smooth isotopies.*
- *Smith-Wemyss (2023) and Ganatra-Pomerleano (2021) proved the theorem by imposing additional assumptions on the identification $\eta : \nu_{Z/Q_0} \xrightarrow{\simeq} \nu_{Z/Q_1}$ of normal bundles, so that the Stein neighborhood of $Q_0 \cup Q_1$ is a subdomain of W_0 or W_1.*

Morse-Bott fibration

There is a Morse-Bott fibration

$$\pi : W_k \to \mathbb{C}$$

with general fiber $(\mathbb{C}^*)^2$, and three singular fibers isomorphic to $(\mathbb{C} \vee \mathbb{C}) \times \mathbb{C}^*$ at $-1, 0, 1$. The monodromy maps are fibered Dehn twists which act trivially on the homology of the fiber, so we can label the homology classes of the meridian and longitude by $a, b \in H_1(T^2; \mathbb{Z})$ consistently in the fibration. With respect to this labeling, the vanishing cycles for $\pi^{-1}(-1)$, $\pi^{-1}(0)$ and $\pi^{-1}(1)$ are given respectively by

$$a, b, a + kb.$$

There is an A_∞-category $\mathcal{W}(W_k, \pi)$ associated to π, called the *fiberwise wrapped Fukaya category* (Abouzaid-Auroux 2024). Localizing $\mathcal{W}(W_k, \pi)$ gives the (fully) wrapped Fukaya category.

Morse-Bott fibration

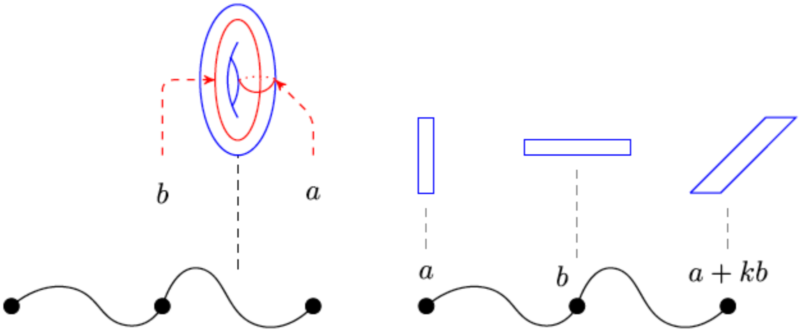


Figure: Vanishing cycles for the Morse-Bott fibration $\pi : W_k \to \mathbb{C}$ (figure borrowed from Smith-Wemyss' paper)

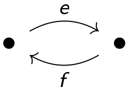
Wrapped Fukaya category

The wrapped Fukaya category $\mathcal{W}(W_k;\mathbb{K})$ is generated by two cotangent fibers L_0 and L_1 over Q_0 and Q_1, respectively. We form the endomorphism A_∞-algebra

$$\mathcal{W}_k := \bigoplus_{i,j=0,1} CW^*(L_i, L_j).$$

Proposition (Asplund-L)

Let $\mathbb{K}$ be any field and $k \geq 1$. The Fukaya A_∞-algebra $\mathcal{W}_k$ is quasi-isomorphic to the Ginzburg dg algebra $\mathcal{G}_k$ associated to the 2-cycle quiver



with potential

$$w_k = efe\left(1 + (fe+1) + \cdots + (fe+1)^{k-1}\right) f.$$

Admissible arcs

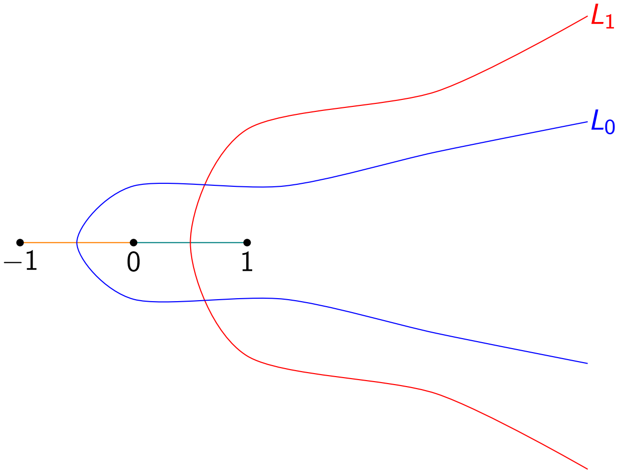


Figure: Projections of the zeros Q_0, Q_1 and the cotangent fibers L_0, L_1 under $\pi : W_k \to \mathbb{C}$.

Mirror symmetry

Consider the cA_2 singularity

$$R_k = \frac{\mathbb{K}[u,v,x,y]}{(uv - xy\left((x+1)^k + y - 1\right))}, \ k \geq 1.$$

Let $Y_k \to Spec(R_k)$ be its crepant (partial) resolution.

Theorem (Asplund-L)

Let $\mathbb{K}$ be any field and $k \geq 1$. We have an equivalence

$$D^{perf}\mathcal{W}(W_k;\mathbb{K}) \simeq D^b Coh(Y_k)/\langle \mathcal{O}_{Y_k} \rangle.$$

When $\mathrm{char}(\mathbb{K}) = 0$, combined with the work of Hua-Keller (2024), we get the following version of closed-string mirror symmetry:

$$SH^0(W_k;\mathbb{K}) \cong \mathcal{T}_k,$$

where the right-hand side is the Tjurina algebra of R_k.

Exact Calabi-Yau structures

The wrapped Fukaya category $\mathcal{W}(M;\mathbb{K})$ of any Weinstein manifold M with $c_1(M) = 0$ has a smooth CY structure (Ganatra 2023). In particular, there is a class $\zeta \in HC^-_{-n}(\mathcal{W}(M;\mathbb{K}))$ whose image under

$$\iota : HC^-_{-n}(\mathcal{W}(M;\mathbb{K})) \to HH_{-n}(\mathcal{W}(M;\mathbb{K}))$$

induces a quasi-isomorphism

$$\mathcal{W}(M;\mathbb{K}) \simeq \mathcal{W}^\vee(M;\mathbb{K})[n]$$

between the diagonal bimodule and its (degree shifted) derived dual.

The smooth CY structure is called *exact* if it can be lifted to a class $\tilde{\beta} \in HC_{-n+1}(\mathcal{W}(M;\mathbb{K}))$, so that under Connes' map

$$B : HC_{-n+1}(\mathcal{W}(M;\mathbb{K})) \to HH_{-n}(\mathcal{W}(M;\mathbb{K}))$$

it satisfies $B(\tilde{\beta}) = \iota(\zeta)$.

Cyclic quasi-dilations

A *cyclic quasi-dilation* is a pair $(\tilde{b}, h) \in SH^1_{S^1}(M;\mathbb{K}) \times SH^0(M;\mathbb{K})^\times$ consisting of a class $\tilde{b} \in SH^1_{S^1}(M;\mathbb{K})$, whose image under the marking map

$$\mathbf{B} : SH^*_{S^1}(M;\mathbb{K}) \to SH^{*-1}(M;\mathbb{K})$$

hits the invertible element $h \in SH^0(M;\mathbb{K})^\times$.

This notion generalizes that of a *quasi-dilation* introduced by Seidel-Solomon (2012), in which case the class $\tilde{b} \in SH^1_{S^1}(M;\mathbb{K})$ lifts to $b \in SH^1(M;\mathbb{K})$ along the erasing map

$$\mathbf{I} : SH^*(M;\mathbb{K}) \to SH^*_{S^1}(M;\mathbb{K}).$$

Proposition (L)

Let M be a Weinstein manifold with $c_1(M) = 0$. Then $\mathcal{W}(M;\mathbb{K})$ is exact CY if and only if M admits a cyclic quasi-dilation over $\mathbb{K}$.

Existence of cyclic quasi-dilations

Lemma

W_k admits a cyclic quasi-dilation over any field $\mathbb{K}$.

Proof.

$\mathcal{W}(W_k;\mathbb{K})$ can be identified with the Ginzburg dg algebra $\mathcal{G}_k$ over any field $\mathbb{K}$, and the latter is exact CY. □

When $h = 1 \in SH^0(M;\mathbb{K})$, a cyclic quasi-dilation is called a *cyclic dilation*. The notion is introduced independently by Z. Zhou under the name of *k-dilations*. A Liouville manifold M admits a cyclic dilation if and only if the first Gutt-Hutchings capacity of the corresponding Liouville domain $\overline{M}$ is finite.

However, the existence of a cyclic dilation is in general sensitive to $\mathrm{char}(\mathbb{K})$. For example, T^*S^2 does not admit a (cyclic) dilation if $\mathrm{char}(\mathbb{K}) = 2$, but it admits a (cyclic) quasi-dilation over any field $\mathbb{K}$.

A conjecture

Conjecture

The Stein manifold W_k *admits a dilation* $b \in SH^1(W_k; \mathbb{K})$ *over a field* $\mathbb{K}$ *with* $\mathrm{char}(\mathbb{K}) = p$ *if* $p = 0$ *or* p *does not divide* k*.*

Ganatra-Pomerleano proved the existence of a dilation for W_1 over $\mathbb{F}_3$ using their log PSS map.
On the other hand, it is easy to see if $p|k$, then W_k does not admit a dilation over $\mathbb{F}_p$.

Theorem (Cieliebak-Mohnke, Fukaya, Irie, L)

Let M *be any Liouville manifold with* $c_1(M) = 0$*, such that the first Gutt-Hutchings capacity of* $\overline{M}$ *is finite. Then any closed oriented aspherical Lagrangian submanifold* $L \subset M$ *bounds a holomorphic disc of Maslov index* 2*.*
It follows that when $\dim(M) = 6$*, any oriented prime Lagrangian* $L \subset M$ *is either spherical or* $S^1 \times \Sigma_g$ *for* $g \geq 0$*.*

Excluding aspherical Lagrangians

Lemma

There is no oriented closed exact Lagrangian $K(\pi, 1)$ in W_k.

Proof.

This uses only the existence of a cyclic quasi-dilation $(\tilde{b}, h)$ over a field $\mathbb{K}$ of characteristic 0.
If $h = 1$, this is a direct consequence of (S^1-equivariant) Viterbo functoriality. Otherwise, suppose there is such a Lagrangian $L \subset W_k$ and consider the map

$$SH^0(W_k; \mathbb{K}) \to SH^0(T^*L; \mathbb{K}) \cong \mathbb{K}[\pi_1(L)].$$

Using the fact that there is no non-trivial central units in $\mathbb{K}[\pi_1(L)]$ (Öinert, 2024), we can show that $\langle h^n \rangle_{n \in \mathbb{Z}}$ span an infinite-dimensional subspace in $SH^0(W_k; \mathbb{K})$, which contradicts with the fact that the dimension of $SH^0(W_k; \mathbb{K})$ is the Tjurina number of R_k, which is in particular finite. □

Spherical 3-manifolds

Let Q be a spherical 3-manifold, then $\pi_1(Q)$ is either cyclic or a central extension of a dihedral, tetrahedral, octahedral, or icosahedral group by a cyclic group of even order.

- $\pi_1(Q)$ is cyclic, then Q is a Lens space.
- $\pi_1(Q)$ is a product of $\mathbb{Z}_m$ with a group of order $4n$, where $m \geq 1$, $n \geq 2$ and $\gcd(m, 2n) = 1$. In this case, Q is a prism manifold.
- $\pi_1(Q)$ is a product of $\mathbb{Z}_m$ with a group of order 24, where $m \geq 1$ and $\gcd(m, 6) = 1$. In this case, we call Q a spherical manifold of type **T**.
- $\pi_1(Q)$ is a product of $\mathbb{Z}_m$, where $\gcd(m, 6) = 1$, with the binary octahedral group $2O$. In this case, we call Q a spherical manifold of type **O**.
- $\pi_1(Q)$ is a product of $\mathbb{Z}_m$, where $\gcd(m, 30) = 1$, with the binary icosahedral group $2I$. In this case, we call Q a spherical manifold of type **I**.

Nonexistence of types **T**, **O**, **I**

Proposition

Let $k \geq 1$. If $L \subset W_k$ is a spherical Lagrangian submanifold, then L cannot be of types **T**, **O** *or* **I**.

The proof starts with the map on free loop space homologies

$$H_*(\mathcal{L}L; \mathbb{K}) \to H_*(\mathcal{L}B\pi_1(L); \mathbb{K})$$

induced by the classifying map $L \to B\pi_1(L)$. Composing with Viterbo restriction gives

$$SH^*(W_k; \mathbb{K}) \to H_{3-*}(\mathcal{L}B\pi_1(L); \mathbb{K}),$$

which is compatible with product structures.

We claim that there is always a homomorphism $\pi_1(L) \to \Gamma$, where Γ is a *non-trivial finite group with trivial center*.

Constructing the maps

- When L is type **T**, consider the quotient $\pi_1(L) \to \pi_1(L)/Z(\pi_1(L)) \cong A_4$ and take $\Gamma = A_4$.
- When L is type **O**, consider the composition $\pi_1(L) \to 2O \to S_4$ and take $\Gamma = S_4$.
- When L is type **I**, consider the composition $\pi_1(L) \to 2I \to A_5$ and take $\Gamma = A_5$.

In any case, we get a map

$$H_*(\mathcal{L}B\pi_1(L);\mathbb{K}) \to H_*(\mathcal{L}B\Gamma;\mathbb{K}).$$

Composing with the previous map we obtain

$$\vartheta : SH^*(W_k;\mathbb{K}) \to H_{3-*}(\mathcal{L}B\Gamma;\mathbb{K}).$$

In the same vein, we have a map

$$\tilde{\vartheta} : SH^*_{S^1}(W_k;\mathbb{K}) \to H^{S^1}_{3-*}(\mathcal{L}B\Gamma;\mathbb{K}).$$

Group homologies

The maps ϑ and $\tilde{\vartheta}$ fit into the commutative diagram:

$$\begin{array}{ccc} SH^*_{S^1}(W_k;\mathbb{K}) & \xrightarrow{\tilde{\vartheta}} & H^{S^1}_{3-*}(\mathcal{L}B\Gamma;\mathbb{K}) \\ \mathbf{B}\downarrow & & \downarrow\mathbf{B}_{\mathcal{L}} \\ SH^{*-1}(W_k;\mathbb{K}) & \xrightarrow{\vartheta} & H_{4-*}(\mathcal{L}B\Gamma;\mathbb{K}) \end{array}$$

We claim that we can choose $\mathbb{K}$ so that there are isomorphisms

$$H_3(L;\mathbb{K}) \cong H_3(B\pi_1(L);\mathbb{K}) \cong H_3(B\Gamma;\mathbb{K}).$$

This follows from the computation of group homologies:

$$H_3(BA_4;\mathbb{Z}) \cong H_3(A_4;\mathbb{Z}) \cong \mathbb{Z}_6,$$

$$H_3(BS_4;\mathbb{Z}) \cong H_3(S_4;\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4,$$
$$H_3(BA_5;\mathbb{Z}) \cong H_3(A_5;\mathbb{Z}) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5.$$

Deriving the contradiction

Take $\mathbb{K} = \mathbb{F}_3$ when $\Gamma = A_4$ or S_4, and $\mathbb{K} = \mathbb{F}_2$ when $\Gamma = A_5$. In any case, there is a cyclic quasi-dilation $(\tilde{b}, h) \in SH^1_{S^1}(W_k; \mathbb{K}) \times SH^0(W_k; \mathbb{K})^\times$. Our choices of $\mathbb{K}$ ensure that we get a central unit

$$\vartheta(h) \in H_3(\mathcal{L}B\Gamma; \mathbb{K}) \cong Z(\mathbb{K}[\Gamma]).$$

Lemma

$\vartheta(h) = 1$.

Since $\mathbf{B}_{\mathcal{L}}$ preserves the homotopy classes of loops, $\vartheta(h) = 1$ implies that $\tilde{\vartheta}(\tilde{b}) \in H_2^{S^1}(\mathcal{L}B\Gamma; \mathbb{K})$ lies in the space of contractible loops in $B\Gamma$. Since $B\Gamma$ is a $K(\Gamma, 1)$-space, the space of contractible loops retracts onto constant loops, and we have

$$\vartheta(h) = \mathbf{B}_{\mathcal{L}}\left(\tilde{\vartheta}(\tilde{b})\right) = 0,$$

a contradiction.

Integral group rings

The proof of the claim $\vartheta(h) = 1$ requires passing to integral lifts. More precisely, our computations of the wrapped Fukaya categories of W_k holds over $\mathbb{Z}$, and one can use this to argue that $\vartheta(h)$ lifts to a central unit

$$\vartheta(h)_{\mathbb{Z}} \in Z(\mathbb{Z}[\Gamma]).$$

Thus it suffices to show that the image of $\vartheta(h)_{\mathbb{Z}}$ is trivial under the map $\mathbb{Z}[\Gamma] \to \mathbb{K}[\Gamma]$. When $\Gamma = S_4$ or A_4, all integral central units are trivial. On the other hand, Li-Parmenter (1997) showed that the group of central units in $\mathbb{Z}[A_5]$ is generated by a single element

$$49 + 26C_1 - 10C_2 - 16C_4,$$

where the C_i's are sums of elements in certain conjugacy classes in A_5. This projects to the identity in $\mathbb{F}_2[A_5]$.

Lens space and prism manifolds

The exclusion of certain Lens spaces and prism manifolds follows from similar arguments used in the cases of aspherical manifolds and spherical manifolds of types **T**, **O** and **I**. Here we need the existence of a cyclic quasi-dilation over $\mathbb{F}_p$.

Proposition

Suppose $k \geq 1$, and $L \subset W_k$ is a Lagrangian submanifold that is diffeomorphic to a Lens space or a prism manifold. If p is a prime factor of $|\pi_1(L)|$, then p also divides k.

Conjecture

Every oriented prime exact Lagrangian submanifold $L \subset W_k$ is diffeomorphic to S^3 or $L(k,1)$.

By classifying the fat-spherical objects in $\mathcal{F}(W_p; \mathbb{F}_p)$, Smith-Wemyss proved the non-existence of exact Lagrangian $S^1 \times S^2$'s with vanishing Maslov class in W_p.

Dehn surgeries

We will use the following deep result proved using the monopole Floer homology.

Theorem (Kronheimer-Mrowka-Ozsváth-Szabó, 2007)

Let $S^3_r(\kappa)$ be the Dehn surgery of slope $r \in \mathbb{Q}$ along a knot $\kappa \subset S^3$. If there is an orientation-preserving diffeomorphism $S^3_r(\kappa) \cong S^3_r(\mu)$, where μ is the unknot, then κ must be the unknot.

Suppose there is a Hamiltonian isotopy which changes the knot types of the unknots to $\kappa_0 : Z' \to Q'_0$ and $\kappa_1 : Z' \to Q'_1$, then there is a Weinstein embedding $W_n(\kappa_0, \kappa_1) \subset W_k$. Let $K \subset W_k$ be the Morse-Bott surgery of Q'_0 and Q'_1 along Z', it is unique up to diffeomorphism. It is an observation due to Smith-Wemyss that $n = k$.

First consider the case where only one of κ_0 and κ_1, say κ_0 is non-trivial, then K is diffeomorphic to the Dehn surgery on κ_0 with slope k.

Finishing the proof

Howie (2002) proved that K has a prime decomposition with at most 3 summands, and if there were actually 3 summands, one of them would be a Poincaré homology sphere. Since the latter possibility has been ruled out by our previous arguments, we conclude that K is either

(i) a Lens space or a prism manifold such that every prime factor of $|\pi_1(K)|$ also divides k, or

(ii) a connected sum of two spherical manifolds in (i).

In case (i), we can use known results (e.g. Ni-Zhang 2018) to show that K must be a Lens space, so κ_0 is an unknot by Kronheimer-Mrowka-Ozsváth-Szabó's theorem.
In case (ii), Gordon-Luecke (1987) proved that K must be a connected sum of two non-trivial Lens spaces. One can use the existence of a cyclic quasi-dilation over $\mathbb{F}_p$, where p divides $|\pi_1(K)|$, and similar argument as before to get a contradiction.

Finishing the proof

Finally, assume that both κ_0 and κ_1 are non-trivial knots, then the Morse-Bott surgery K is irreducible and contains an incompressible torus (Hedden-Kim-Mark-Park 2019). Since $D^*K \subset W_k$ admits a cyclic quasi-dilation (over a field $\mathbb{K}$ of characteristic 0) by Viterbo functoriality, one can show (using Kotschick-Neofytidis 2013) that K is finitely covered by

- $S^1 \times \Sigma_g$, where $g \geq 1$, or
- a connected sum of $S^1 \times S^2$ (including S^3).

Since we have ruled out the possibility that K is aspherical, only the second case is possible. On the other hand, since K is irreducible and $H_1(K;\mathbb{Z}) = \mathbb{Z}_k$, it must be spherical, which contradicts with the fact that it contains an incompressible torus.