Parametrised Whitehead torsion of families of nearby Lagrangians

Joint work-in-progress with Sylvain Courte

1. context
2. results
3. parametrised Whitehead torsion

(4. methods)

1. context

- Let Q be a closed n-mfld.

<u>Conj</u> (Nearby Lagrangian conjecture / NLC)
If $L \subseteq T^*Q$ is a closed exact Lagrangian
nearby
submfld, then L is Hamiltonian isotopic to 0-section.

NLC $\implies$ Any such L is diffeo to Q.

<u>Rmk</u> "Local" case of studying Lagrangians in arbitrary X.

What's known:

Whole NLC known for:
- S^1 (Folklore)
- S^2 (Hind)
- $\mathbb{RP}^2$ (Hind-Ainsonnault-Wu)
- T^2 (Rizell-Goodman-Ivrii)

What's known for any Q:

Any nearby Lag is ~~htpy equiv~~ simple htpy equiv to Q. (Abouzaid-Kragh, Guillermou)

For some specific Q:

Some restrictions on smooth structures (framed cobordism).

(Abouzaid
Ekholm-Kragh-Smith
Torricelli
Abouzaid-Alvarez-Gavela-Courte-Kragh
A.-Smith)

Conj (**Strong** NLC):

The space of nearby Lagrangians in T^*Q $=: \mathrm{Lag}(T^*Q)$

is contractible.

- Ordinary NLC $\iff$ $\mathrm{Lag}(T^*Q)$ path-connected.
- Tautological fibre bundle $\mathbb{L} \longrightarrow \mathrm{Lag}(T^*Q)$

  Fibre over L is L.

Strong NLC $\Rightarrow$ fibre bundle is trivial:

$$\begin{tikzcd}
\mathbb{L} \arrow[rr, "\text{htpy equiv}", "\text{Diffeo}"'] \arrow[dr] & & \mathrm{Lag}(T^*Q) \times Q \arrow[dl] \\
& \mathrm{Lag}(T^*Q) &
\end{tikzcd}$$

What we know

Parametrised Whitehead torsion:

$$w: \mathrm{Lag}(T^*Q) \longrightarrow \mathcal{H}(Q)$$

stable h-cobordism space.

Thm (Côté–P., in progress):

1. w is 0 on π_0 [Abouzaid–Kragh, simple $\simeq$].
2. w is 0 on π_1
3. w is 0 on all π_* if $\chi(Q) = 0$.

what does ω detect?

1. $L \longrightarrow Q$ is a simple $\simeq$

$\Rightarrow$ If Q is a 3-Dim lens space, L must be diffeo to Q.

2. Consider $Q = T^n$, $n \geq 6$.

$\rightsquigarrow$ Hamiltonian diffeo $\Phi : T^*T^n \longrightarrow T^*T^n$ sending 0-section T^n to itself.

Consider $\phi := \Phi|_{0\text{-section}} \in \mathrm{Diff}(T^n)$.

Question: must ϕ be trivial in $MCG(T^n)$?

Is ϕ isotopic to Id?

<u>Thm</u> (Hatcher, Hsiang–Sharpe): $MCG(T^n) \cong \underbrace{GL_n(\mathbb{Z})}_{\text{Linear isometries}} \ltimes \underbrace{\left(\mathbb{Z}/2\right)^{\infty}}_{\text{Stuff we can detect.}} \times \left(\text{Finite group}\right).$

<u>Cor</u> (π_1 part of thm):

Projection of ϕ to 2^{nd} component vanishes.

$L \subseteq T^*O^n$

- Proj to 1^{st} component is 0

Parametrised Whitehead torsion $\omega : \mathrm{Lag}(T^*Q) \to \mathcal{H}(Q)$.

- Fix a closed n-mfld Q.

Def an <u>h-cobordism</u> on Q is a compact $(n+1)$-mfld W, with $\partial W \cong Q \sqcup Q'$, such that inclusions $Q, Q' \hookrightarrow W$ are $\simeq$.

Def The <u>h-cobordism space</u> $H(Q)$ is the space of such.

Ex An element of $\pi_i H(Q)$ is a fibre bundle $\{W_t\}_{t \in S^i}$, each W_t an h-cob on Q.

Def The <u>stable</u> h-cobordism space of Q

$$\boxed{\mathcal{H}(Q)} := \varinjlim \left(H(Q) \xrightarrow[\cdot\times I]{} H(Q\times I) \xrightarrow[\cdot\times I]{} H(Q\times I^2) \xrightarrow[\cdot\times I]{} \cdots \right)$$

Target of ω

Thm (Waldhausen–Jahren–Rognes):

$$\pi_* \mathcal{H}(pt) \cong \underbrace{K_{*-1}(\mathbb{S}) / \pi_{*-1}\mathbb{S}}_{\text{Algebraic.}}$$

Ex

$$\pi_0(\mathcal{H}(pt)) \cong 0$$
$$\pi_1 \qquad = 0$$
$$\pi_2 \qquad = \mathbb{Z}/2$$
$$\vdots$$

$\mathcal{H}(\cdot)$

$$\begin{tikzcd} pt \arrow[r] \arrow[rr, bend right, "="'] & Q \arrow[r] & pt \end{tikzcd}$$

Def $\omega : \mathrm{Lag}(T^*Q) \longrightarrow \mathcal{H}(Q)$.

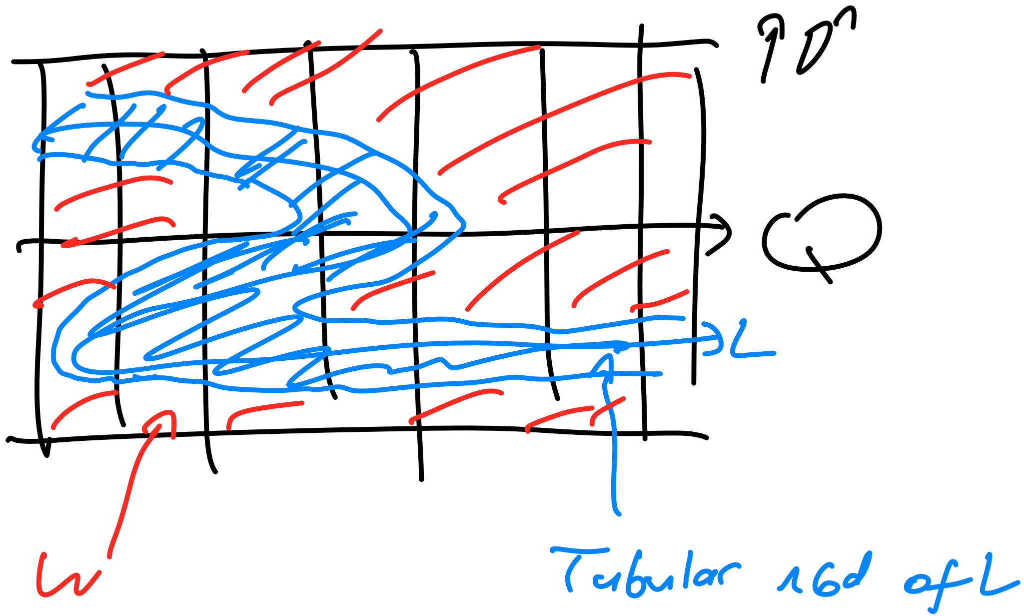
- Assume Q is framed $D^*Q \cong Q \times D^n$
- Let $L \subseteq D^*Q$ be a nearby Lagrangian.
- Let $W = Q \times D^n \setminus$ tubular nbd of L

  $\partial W = Q \times S^{n-1} \sqcup \ldots$

Let $W' \approx W$ with $Q \times D^n$ glued back in

$\rightsquigarrow W'$ is an h-cobordism on $Q \times D^n$.
$\in H(Q \times D^n) \subseteq \mathcal{H}(Q)$.

Def $\omega(L) =$ this W'.



Idea: If $\omega \cong Q \times S^{n-1} \times I$ (i.e. $\omega(L)$ trivial),

$Q \times D^n \underset{\text{diffeo}}{\cong}$ Tub nbd of L

"ω = obstruction for L, Q to be stably diffeo"

Thm (CP, in progress):

1.
$$\begin{tikzcd} Lag(T^*Q) \arrow[r, "\omega"] \arrow[d, "S"'] & \mathcal{H}(Q) \\ \mathcal{H}(pt) \arrow[ur, "\cdot \times Q"'] & \end{tikzcd}$$

what does this mean?

Parametrised Whitehead torsion factors up to weak homotopy

Restricts $\omega(\pi_i)$ if $\pi_i \mathcal{H}(pt) = 0$

2.
$$\begin{tikzcd} \mathcal{H}(pt) \arrow[r, "\cdot \times Q"] \arrow[d, "\cdot \chi(Q)"'] & \mathcal{H}(Q) \\ \mathcal{H}(pt) \arrow[ur, "\text{inclusion}"'] & \end{tikzcd}$$

$\cdot \times Q$ factors too:

Restrictions if $\chi(Q) = 0$.

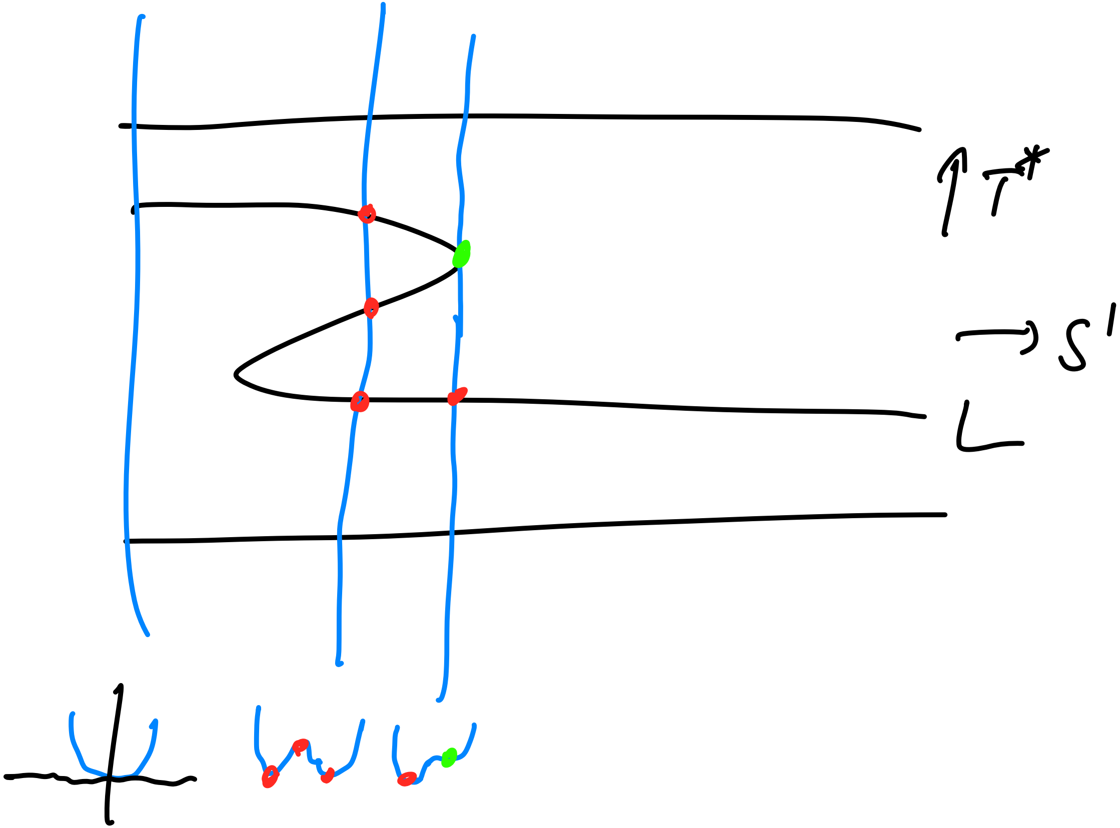
Methods ($\rightsquigarrow$ constructing δ)

Thm (Abouzaid–Courte–Guillermou–Kragh '20):

Any nearby Lag $L \subseteq T^*Q$ admits a twisted generating function, "behaves nice at ∞".

Recall An (untwisted) GF for L is a map $f: Q \times \mathbb{R}^l \longrightarrow \mathbb{R}$ $\quad \{f_x : \mathbb{R}^l \longrightarrow \mathbb{R}\}_{x \in Q}$

s.t. $L \cong \bigcup_x \mathrm{Crit}(f_x)$.



T^*
S^1
L

Def The <u>difference function</u> of f:

$$F : Q \times \mathbb{R}^{2c} \longrightarrow \mathbb{R}$$

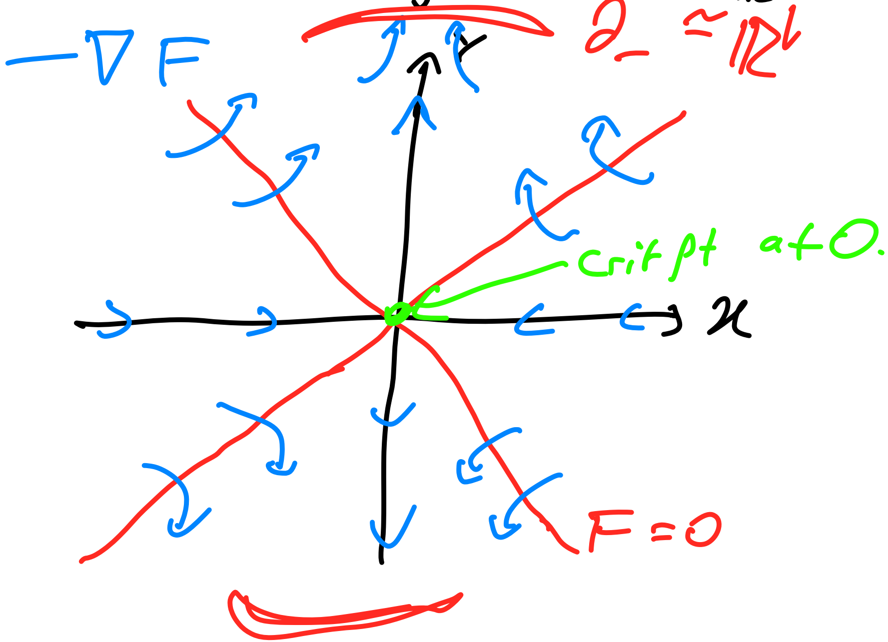
$$(x, v_1, v_2) \mapsto f(x, v_1) - f(x, v_2).$$

- Even if f is twisted, F is not.

- "niceness at ∞": Each $F_x : \mathbb{R}^{2c} \longrightarrow \mathbb{R}$ is $F_x =$ (homogeneous deg 2 function) + (thing w/ bounded derivative).

<u>Ex</u>: · $x_1^2 + \dots + x_k^2 - x_{k+1}^2 - \dots$

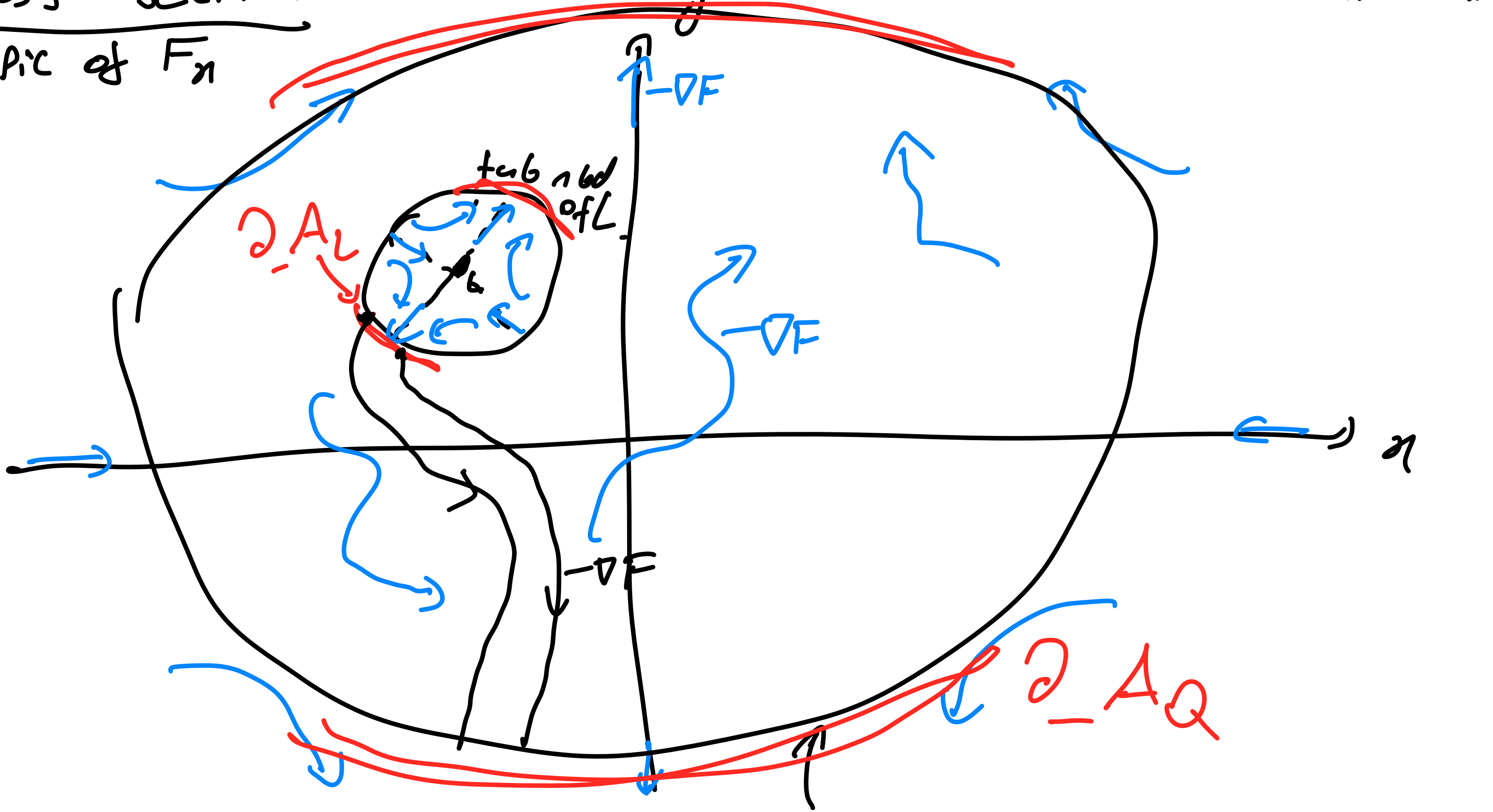
Local model for $F : \mathbb{R}^2 \to \mathbb{R}$ is $x^2 - y^2$.



- Crit $F \cong L$, Morse-Bott critical set.
  In $Q \times \mathbb{R}^{2l}$

Cross-section For more general L: $F: Q \times \mathbb{R}^{2l} \to \mathbb{R}$

Pic of F_η



Def

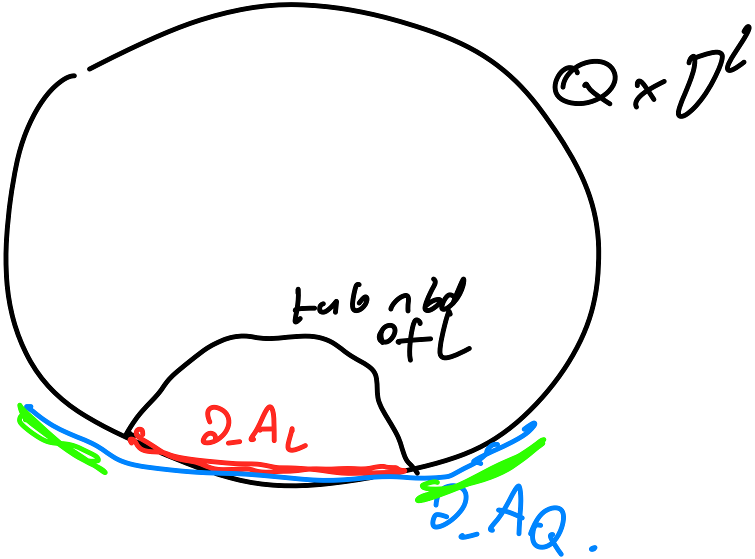
$\partial_- A_Q = \{$ subset of $\partial(Q \times D^l)$ where $-\nabla F$ points out $\}$

$\partial_- A_L = \{$ ——— ∂(tub nbd of L) ——— $\}$

Both are bundles over Q and L, respectively. Fibres $\cong S^{l-1} \times D^l$

- If I apply flow of $-\nabla F$ to $\partial_- A_L$, it lands in $\partial_- A_Q$

$\rightsquigarrow$ can assume $\partial_- A_L \subseteq \partial_- A_2$.



$W := \partial_- A_M \setminus \partial_- A_L$ [This represents $w(L)$].

Find:
$$\begin{tikzcd} \partial_- A_L \arrow[r, hook, "\cong"] \arrow[d, "\cong"] & \partial_- A_Q \\ L \times S^{l-1} \arrow[r, "\cong"] & Q \times S^{l-1} \arrow[u, dashed, "G."'] \end{tikzcd}$$

Take $\partial_- A_Q \setminus$ tub nbd of G

$\rightsquigarrow$ a fibre bundle over Q.

Fibres are h-cobordisms on a disc

__Def__ $\delta(L) :=$ these fibres.