On certain C^0-aspects of contactomorphism groups

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Based on a joint work with B. Serraille

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Assume that there exists a symplectic embedding $\Phi : B^{2n}(r) \to Z^{2n}(R)$. Then $r \leq R$.

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Still, there exist non-squeezing phenomena.

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Assume $\exists k \in \mathbb{N}$ such that $r \leq k \leq R$. Then $B^{2n}(R) \times S^1$ cannot be squeezed into $B^{2n}(r) \times S^1$.

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For $r, R \geq 1$, the condition on k was removed by Chiu '17.

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Goal 1: Relate contact (non)-squeezing to properties of $\mathrm{Cont}_{0,c}(\cdot)$.

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We will now sketch the proof of this result!

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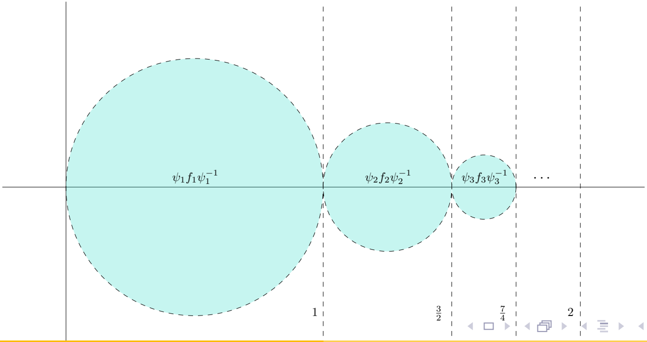

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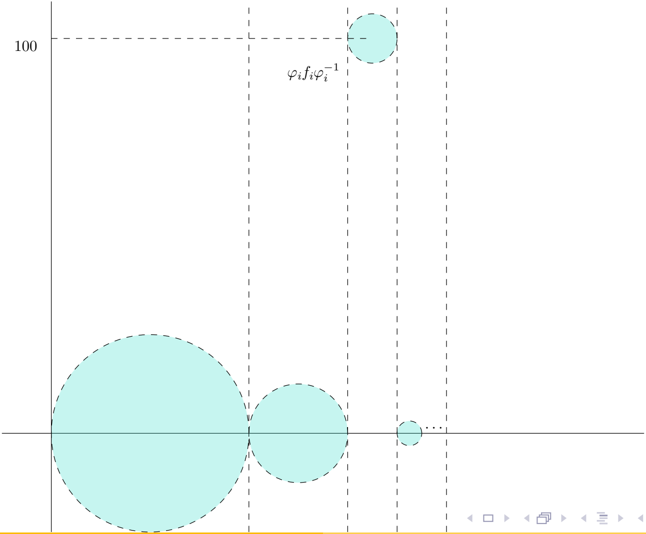

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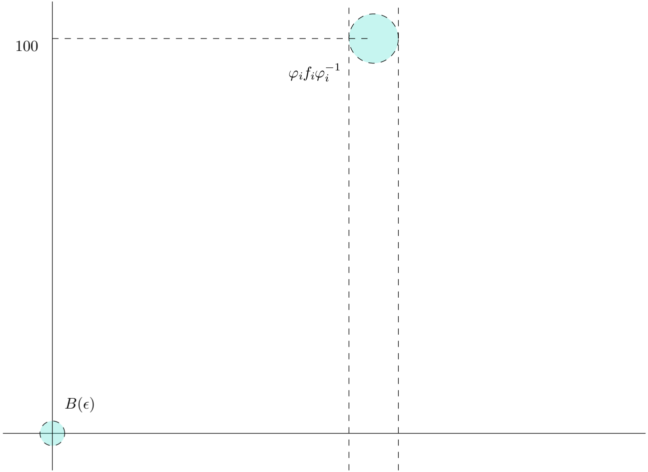

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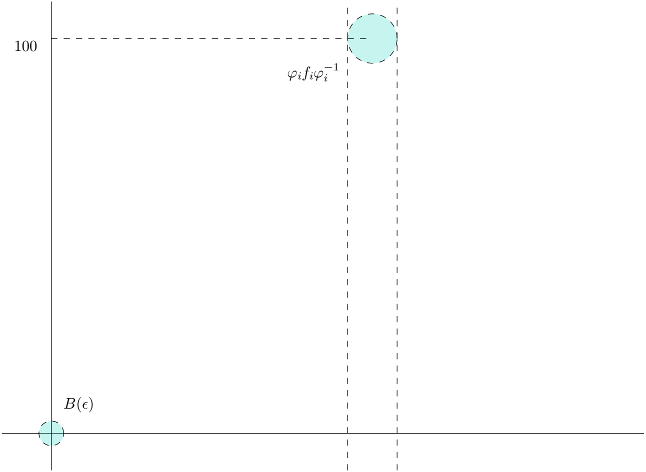

Letting $\epsilon \to 0$ proves that $f_i \in \overline{\mathrm{Conj}}(g)$. □

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This violates contact non-squeezing. □

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Utopia: Construct a C^0-continuous, conjugation-invariant norm on $\overline{\mathrm{Cont}}_{0,c}(\cdot)$.

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Theorem (Burago, Ivanov, Polterovich '08)

A conjugation-invariant norm on $\mathrm{Cont}_{0,c}(Y)$ can never be continuous with respect to any reasonable topology on $\mathrm{Cont}_{0,c}(Y)$.

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Instead of continuity, we will show that γ is C^0-**locally bounded**.

Locally bounded: There exists a C^0-neighbourhood of id on which γ is bounded.

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Let $\phi \in \text{Cont}_{0,c}(\mathbb{R}^{2n} \times S^1)$ and $d_{C^0}(\phi, id) < \frac{1}{2}$. Then

$$\gamma(\phi) \leq 2.$$

Moreover, γ is C^0-lower semicontinuous.

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Let us see some applications!

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This is impossible since $\tilde{\gamma}$ is unbounded. □

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For every $\phi \in \mathrm{Cont}_{0,c}(Y)$ there exist $\phi_1, \ldots, \phi_N$ such that $\phi_i \in \mathrm{Cont}_{0,c}(U_{k_i})$ and

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We get that

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Corollary (Serraille, S. '24)

For $\phi \in \mathrm{Cont}_{0,c}(\mathbb{R}^{2n} \times S^1)$ it holds

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C^0-continuity in various cases has been proven by Seyfaddini, Buhovsky-Humilière-Seyfaddini, Shelukhin and Kawamoto.

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Theorem (Viterbo '92)

There exists a conjugation-invariant norm $\gamma : \mathrm{Ham}_c(\mathbb{R}^{2n}) \to \mathbb{R}$.
This norm is unbounded and C^0-continuous.

Viterbo's construction uses generating functions.

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C^0-continuity in various cases has been proven by Seyfaddini, Buhovsky-Humilière-Seyfaddini, Shelukhin and Kawamoto.

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Our result is analogous to C^0-continuity. We also prove C^0-continuity of spectral invariants of Albers and Merry.

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Thaking integer parts gives us $\gamma(\phi) \leq 2$.

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Proof. By previous $|\tilde{\gamma}(\phi) - \tilde{\gamma}(\psi)| \leq 2d_{cc}(\phi, \psi)$. □

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Question 5: Is d_{cc} bounded for $(\overline{\text{Cont}}_{0,c}(\widehat{W} \times S^1), d_{C^0})$? What about $(\overline{\text{Cont}}_{0,c}(\widehat{W} \times \mathbb{R}), d_{C^0})$?

Thank you for your attention!