# Contact non-squeezing in various closed prequantizations

Pierre-Alexandre Arlove

Université de Strasbourg

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# Symplectic Manifolds

## Theorem (Symplectic Darboux Theorem : No local invariants!)

*Let $(W, \omega)$ be a symplectic manifold. Then for any $p \in W$ there exists a chart $\phi : U \hookrightarrow \mathbb{R}^{2n}$ in a neighborhood of $p$ so that $\phi_* \omega = \omega_0 = \sum_{i=1}^{n} \mathrm{d}x_i \wedge \mathrm{d}y_i = \mathrm{d}\lambda_0$ where $\lambda_0 = -\sum_{i=1}^{n} y_i \mathrm{d}x_i$.*

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## Theorem (Gromov's non-squeezing 85')

*There exists a symp. emb.* $(\mathbb{B}^{2n+2}(r), \omega_0) \hookrightarrow (\mathbb{R}^{2n} \times \mathbb{B}^2(R), \omega_0)$ *if and only if* $r \leq R$.

# Contact manifolds

## Definition

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## Examples

1. Contact $\to$ Symplectic : $(M, \ker\alpha) \overset{\text{Symplectization}}{\to} (M \times \mathbb{R}, d(e^\theta \alpha))$

2. Symplectic $\to$ Contact : $(W, \omega = \mathrm{d}\lambda) \overset{\text{Prequantization}}{\to} \begin{cases} (W \times \mathbb{R}, \ker(\mathrm{d}z + \lambda)) \\ (W \times \mathbb{R}/\mathbb{Z}, \ker(\mathrm{d}z + \lambda)) \end{cases}$

# No local invariants and existence of squeezing phenomena

## Theorem (Contact Darboux Theorem)

*Let $(M, \xi)$ be a cooriented contact manifold. Then for any $p \in M$ there exists a chart $\phi : U \hookrightarrow \mathbb{R}^{2n+1}$ in a neighborhood of $p$ so that $\phi_* \xi = \xi_0 :=$* $\ker \left( \mathrm{d}z - \sum_{i=1}^{n} y_i dx_i \right)$.

## Lemma (Squeezing Phenomena in Darboux charts)

$\phi_a : (\mathbb{R}^{2n+1}, \xi_0) \to (\mathbb{R}^{2n+1}, \xi_0)$, $(x, y, z) \mapsto (ax, ay, a^2 z)$ *is a contactomorphism for any* $a > 0$.

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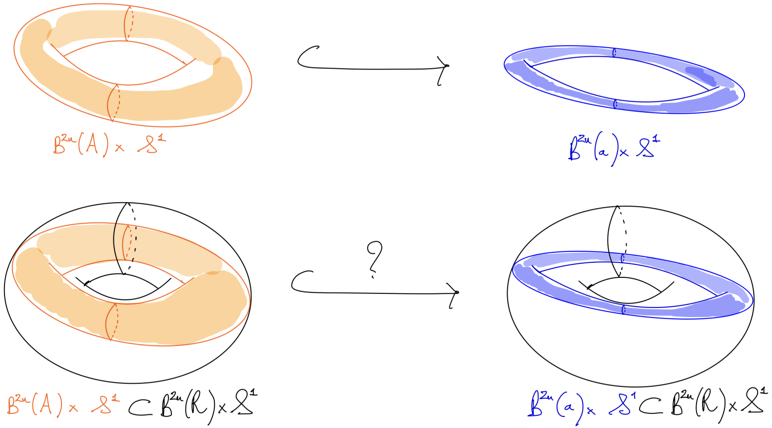
### Corollary

*No hope of Gromov's type non-squeezing : for any $R > 0$ and any contact manifold $(M, \xi)$ there exists a contact embedding of $(\mathbb{B}^{2n+1}(R), \xi_0) \hookrightarrow (M, \xi)$.*

# Existence of contact non-squeezing?

## Definition

A bounded set $U \subset M$ can be contactly squeezed in $V \subset M$ inside $(M, \xi)$ if $\exists$ a cont. emb. $(U, \xi|_U) \hookrightarrow (V, \xi|_V)$ that can be extended to a contactomorphism $\phi \in \mathrm{Cont}_0(M, \xi)$.



# Contact non-squeezing at large scale

$$\mathrm{d}z - \sum_{i=1}^{n} y_i dx_i \text{ on } \mathbb{R}^{2n+1} \text{ descends to a contact form } \alpha_0 \text{ on } \mathbb{R}^{2n} \times \underbrace{\mathbb{R}/(2\pi\mathbb{Z})}_{S^1}.$$

### Theorem (Eliashberg-Kim-Polterovich 06', Sandon 11')

*$\mathbb{B}^{2n}(R) \times S^1$ cannot be contactly squeezed in $\mathbb{R}^{2n-2} \times \mathbb{B}^2(r) \times S^1$ inside $(\mathbb{R}^{2n} \times S^1, \ker \alpha_0)$ if there exists $j \in \mathbb{N}_{>0}$ so that $\pi r^2 < 2\pi j < \pi R^2$.*

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### Remark

1. If $\pi R^2,\ \pi r^2 \in (0, 2\pi)$ then there are contact squeezing (for $n > 1$).
2. Similar results for $(W \times S^1, \ker(\mathrm{d}z - \lambda))$ when $(W, \omega = \mathrm{d}\lambda)$ Liouville manifolds - which are necessarily open - (Albers-Merry 13').

# Prequantization

If $(M, \ker \alpha)$ a cont. manifold, then the equation $\begin{cases} \alpha(X) = 1 \\ \mathrm{d}\alpha(X, \cdot) = 0 \end{cases}$ admits a unique solution $R_\alpha$ with flow $(\phi_\alpha^t)_{t\in\mathbb{R}}$.

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## Definition

A $\mathbb{R}/2\pi\mathbb{Z}$-principal bundle $\pi_1 : (M_1, \xi_1) \to (W, \omega)$ is a prequantization if there is a contact form $\alpha_1$ supporting $\xi_1$, i.e. $\ker \alpha_1 = \xi_1$ s.t. :

1. $\pi_1^* \omega = \mathrm{d}\alpha_1$
2. the Reeb flow of $\alpha_1$, is Zoll of minimal period $2\pi$, induces the $\mathbb{R}/2\pi\mathbb{Z}$-action.

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## Remark

1. $p_k : (M_1, \xi_1) \to (M_k := M_1/\mathbb{Z}_k, \xi_k)$ where $\mathbb{Z}_k \simeq \{[\frac{2\pi j}{k}] \mid j \in \mathbb{Z}\} \subset \mathbb{R}/2\pi\mathbb{Z}$ and $p_k^*\xi_k = \xi_1$ (resp. $p_k^*\alpha_k = \alpha_1$).
2. $\pi_k : (M_k, \xi_k = \ker(k\alpha_k)) \to (W, k\omega)$ prequantization where $\pi_k \circ p_k = \pi_1$.

# First example: the sphere

## Example

$$M_1 := \mathbb{S}^{2n+1} \subset \mathbb{R}^{2n+2} \text{ and } \ker\left(\alpha_1 := \left(\sum_{i=1}^{n+1}(x_i dy_i - y_i dx_i)\right)|_{\mathbb{S}^{2n+1}}\right) = \xi_1.$$

$$\mathbb{R}^{2n+2} \simeq \mathbb{C}^{n+1}, \ ((x_1, y_1), \cdots, (x_{n+1}, y_{n+1})) \mapsto (z_1 = x_1 + iy_1, \cdots, z_n = x_n + iy_{n+1})$$

$$\phi^t_{\alpha_1}(z_1, \cdots, z_{n+1}) = e^{it}(z_1, \cdots, z_n)$$

$$W := \mathbb{S}^{2n+1}/\mathbb{S}^1 = \mathbb{CP}^n \text{ and } M_k := \mathbb{S}^{2n+1}/\mathbb{Z}_k = L_k^{2n+1}.$$

## Lemma

*If $(M_1, \xi_1 = \ker \alpha_1)$ a cont. manifold s.t. the $\alpha_1$-Reeb flow is $2\pi$-Zoll, then*

1. $\mathrm{d}\alpha_1$ *descends to a symp. form $\omega$ on $W := M_1/(\phi^t_{\alpha_1})$ and*
2. $\pi_1 : (M_1, \xi_1) \to (W, \omega)$ *is a prequantization.*

# Main Theorem

Let $\pi_1 : (M_1, \xi_1) \to (W, \omega)$ be one of these prequantizations:

1. The sphere over the complex projective space described above
2. $(M_1, \xi_1)$ closed strongly orderable cont. manifold.

## Theorem A (A. 24')

*Let* $\Psi : (\mathbb{B}^{2n}(R), \omega_0) \hookrightarrow (W, \omega)$ *be a symp. emb. for some* $R > 0$.

*For any* $a_1 \in (0, R]$, $a_2 \in (0, \frac{R}{\sqrt{3}})$ *and* $k \in \mathbb{N}_{\geq 2}$, *if there exists* $j \in \mathbb{N}_{>0}$ *s.t.* $j < k$ *and*

$$\pi a_2^2 \leq \frac{2\pi}{k} \cdot j < \pi a_1^2,$$

*then* $B_k(a_1)$ *cannot be contactly squeezed in* $B_k(a_2)$ *inside* $(M_k, \xi_k)$,

*where* $B_k(a) := {\pi_k}^{-1}(\Psi(\mathbb{B}^{2n}(a))) \subset$ *for all* $a \in (0, R]$.

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1. $(X, g)$ a closed Rieman. man. whose geo. flow is $2\pi$-Zoll
$\Rightarrow (S_g T^*X, \ker \lambda_{\mathrm{can}})$ is strongly orderable and the Reeb flow is $2\pi$-Zoll
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   Any $\mathbb{R}/2\pi\mathbb{Z}$-bundle $\pi_1 : M_1 \to W$ whose Euler class projects to $-[\omega/2\pi]$ by the map $H^2(W, \mathbb{Z}) \to H^2_{dR}(W, \mathbb{R})$ admits a contact form $\alpha_1$ whose Reeb flow induces the $\mathbb{R}/2\pi\mathbb{Z}$-action and s.t. $\pi^*\omega = \mathrm{d}\alpha_1$.

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   If $\langle [\omega], \pi_2(W) \rangle = 0$ then $(M_1, \ker \alpha_1)$ is strongly orderable
(Colin-Chantraine-Dimitroglou Rizell).

# A contact capacity...

For any $T \in \mathbb{R}_{>0}$ and $x \in \mathbb{R}$, the $T$-integer part of $x$ is by definition

$$\lceil x \rceil_T := T \left\lceil \frac{x}{T} \right\rceil, \text{ the smallest multiple of } T \text{ greater or equal to } x.$$

## Theorem B (A. 24')

*For any $k \geq 2$ there exists $C_k : \{\text{open sets of } M_k\} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ s.t.*

1. $C_k(U) \leq C_k(V)$ *if* $U \subset V$
2. $\lceil C_k(U) \rceil_{\frac{2\pi}{k}} = \lceil C_k(\phi(U)) \rceil_{\frac{2\pi}{k}}$ *where and* $\phi \in \mathrm{Cont}_0(M_k, \xi_k)$
3. $C_k(B_k(r)) \leq \pi r^2$ *if* $r \in (0, R/\sqrt{3})$
4. $C_k(B_k(r)) \geq \min\{2\pi, \pi r^2\}$

*where* $B_k(r) := \pi_k^{-1}(\Psi(\mathbb{B}^{2n}(r)))$ *for any* $\Psi : (\mathbb{B}^{2n}(R), \omega_0) \hookrightarrow (W, \omega)$ *and* $r \in (0, R]$.

# ...coming from a map $\widetilde{\mathrm{Cont}_0} \to \mathbb{R}$

The contact size of $U$ will be the supremum of the size of a contact isotopy starting at the identity supported in $U$, i.e. $C(U) := \sup\{c((\phi_t)) \mid \mathrm{Supp}(\phi_t) \subset U\}$ where

$$c : \{(\phi_t) \subset \mathrm{Cont}_0(M, \xi) \mid \phi_0 = \mathrm{id}\} \to \mathbb{R}.$$

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$c(\{\phi_t\})$ depends only on the homotopy class $\{\phi_t\}$
$\Rightarrow$ $c$ descends to a map on

$$\widetilde{\mathrm{Cont}_0}(M, \xi) := \{(\phi_t) \subset \mathrm{Cont}_0(M, \xi) \mid \phi_0 = \mathrm{id}\}/\sim$$

where $(\phi_t) \sim (\psi_t)$ if $(\phi_t \circ \psi_t^{-1})$ is a contractible loop in $\mathrm{Cont}_0(M, \xi)$.

# Associate numbers to a cont. isotopy: the Spectrum

$(M, \xi = \ker \alpha)$ a closed cooriented contact manifold and $[(\varphi_t)] = \widetilde{\varphi} \in \widetilde{\mathrm{Cont}_0}(M, \xi)$.

## Definition

1. $x \in M$ is a discriminant point of $\widetilde{\varphi}$ if $\varphi_1(x) = x$ and $\varphi_1^*(\alpha_x) = \alpha_x$.
2. $x \in M$ is a $\alpha$-translated point of $\widetilde{\varphi}$ with translation $T \in \mathbb{R}$ if $x$ is a disc. point of $(\phi_\alpha^T)^{-1} \circ \varphi_1$.
3. $\mathrm{Spec}^\alpha(\widetilde{\varphi}) := \{ T \in \mathbb{R} \mid (\phi_\alpha^T)^{-1} \circ \varphi_1 \text{ has a disc. point}\}$

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## Lemma

*For any $\widetilde{\varphi}, \widetilde{\psi} \in \widetilde{\mathrm{Cont}_0}(M, \xi)$ :*

1. *$x \in M$ a disc. point of $\widetilde{\varphi} \Leftrightarrow \psi_1(x)$ a disc. point of $\widetilde{\psi}\widetilde{\varphi}\widetilde{\psi}^{-1}$*
2. *$\mathrm{Spec}^\alpha(\widetilde{\varphi}) \subset \mathbb{R}$ is nowhere dense.*

# The Spectrum as length of Reeb chords

$$(M, \ker \alpha) \to (M \times M \times \mathbb{R}, \Xi := \ker(\beta := \alpha_2 - e^{\theta}\alpha_1)) \text{ and } \phi^t_\beta(x, y, \theta) = (x, \phi^t_\alpha(y), \theta).$$

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$$\phi \in \mathrm{Cont}_0(M, \xi) \to gr_\alpha(\phi) := \{(x, \phi(x), g(x)) \mid x \in M\} \subset M \times M \times \mathbb{R} \text{ where } \phi^*\alpha = e^g\alpha.$$

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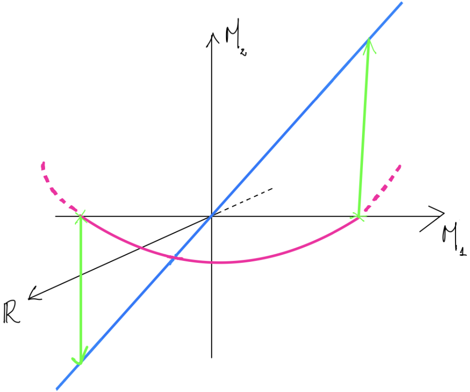
$\beta$-Reeb chord of length $T$ between $gr_\alpha(\varphi_1)$ and $gr_\alpha(\mathrm{id}) \leftrightarrow \alpha$-$T$- translated point of $\widetilde{\varphi}$.



# The Spectral Selectors

## Theorem C (Allais-Sandon-A. 24', A. 24')

*If the prequantization* $(M_1, \xi_1)$ *is* $(\mathbb{S}^{2n+1}, \xi_1)$ *(resp. is str. ord.) then for all* $k \geq 2$ *(resp.* $k \geq 1$*) there exists a* $C^\infty$*-cont. map* $c_k : \widetilde{\mathrm{Cont}_0}(M_k, \xi_k) \to \mathbb{R}$ *which is* $\alpha_k$*-spectral, i.e.* $c_k(\widetilde{\varphi}) \in \mathrm{Spec}^{\alpha_k}(\widetilde{\varphi})$*, and which satisfies moreover :*

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3. $\widetilde{\varphi} \preceq \widetilde{\psi} \Rightarrow c_k(\widetilde{\varphi}) \leq c_k(\widetilde{\psi})$ *(whr.* $\widetilde{\varphi} \preceq \widetilde{\psi}$ *if* $\exists [(\phi_t)] = \widetilde{\varphi}\widetilde{\psi}^{-1}$, $\alpha_k(\frac{d}{dt}\phi_t) \geq 0$*).*

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# The Spectral Selectors

## Theorem C (Allais-Sandon-A. 24', A. 24')

*If the prequantization* $(M_1, \xi_1)$ *is* $(\mathbb{S}^{2n+1}, \xi_1)$ *(resp. is str. ord.) then for all* $k \geq 2$ *(resp.* $k \geq 1$*) there exists a* $C^\infty$*-cont. map* $c_k : \widetilde{\mathrm{Cont}_0}(M_k, \xi_k) \to \mathbb{R}$ *which is* $\alpha_k$*-spectral, i.e.* $c_k(\widetilde{\varphi}) \in \mathrm{Spec}^{\alpha_k}(\widetilde{\varphi})$*, and which satisfies moreover :*

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   - 6.2 *If* $(M_1, \xi_1)$ *str. ord. preq then* $c_k(\widetilde{\varphi}) \geq c_1(\mathcal{L}_k(\widetilde{\varphi}))$.

# Proof of Theorem C 1: Givental's non linear Maslov index

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## Theorem (Givental 90', Granja-Karshon-Pabiniak-Sandon 17')

*For $k \geq 2$, the GNLMI* $\mu : \widetilde{\mathrm{Cont}_0}(L_k^{2n+1}, \xi_k) \to \mathbb{Z}$ *is s.t. for any* $(\widetilde{\varphi}_t)$, $s_1 < s_2 \in [0,1]$

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**Construction of Givental's non linear Maslov index.**

$(\phi_t) \subset \mathrm{Cont}_0(L_k^{2n+1}, \xi_k)$ one associates $(\overline{\phi_t}) := (\mathcal{L}_k(\phi_t))$ and $(f_t : L_k^{2N+1} \to \mathbb{R})$ s.t.

$$\{x \in L_k^{2N+1} \mid \mathrm{d}_x f_t = 0,\ f_t(x) = 0\} \overset{1-1}{\leftrightarrow} \{\pi(x) \in L_k^{2n+1} \mid x \text{ is a disc. point of } \overline{\phi}_t\},$$

The abundance of homology classes in $L_k^{2N+1}$ allows to compare and quantify the difference of topology between $\{f_s \leq 0\}$ and $\{f_t \leq 0\}$. □

# Proof of Theorem C 2: Order Spectral Selectors

$(M, \ker \alpha) \to (M \times M \times \mathbb{R}, \Xi := \ker(\beta := \alpha_2 - e^{\theta}\alpha_1)).$

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$(\Lambda_t) \to (h_t : \Lambda_t \to \mathbb{R})$ s.t. for any param. $(\Phi_t)$ of $(\Lambda_t)$, i.e. $\Phi_t(\Lambda_0) = \Lambda_t$

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### Theorem (Allais-A. 23', A. 24')

*If $(M, \xi)$ is strongly orderable, i.e. no contractible loop $(\Lambda_t)$ based at $gr_\alpha(\mathrm{id})$ s.t. $\mathrm{Length}_+^\beta((\Lambda_t)) < 0$, then $c_k$ satisfies the properties of Theorem C.*

## Theorem C $\Rightarrow$ Theorem B

Let $U \subset M_k$ and $\widetilde{\mathrm{Cont}_0}(U, \xi_k) := \{\widetilde{\varphi} \in \widetilde{\mathrm{Cont}_0}(M_k, \xi_k) \mid \widetilde{\varphi} = [(\phi_t)],\ \mathrm{Supp}(\phi_t) \subset U\}$

where $\mathrm{Supp}(\phi_t) := \overline{\{x \in M_k \mid \phi_t(x) \neq x\}}$.

$$C_k(U) := \sup\{c_k(\widetilde{\varphi}) \mid \widetilde{\varphi} \in \widetilde{\mathrm{Cont}_0}(U, \xi_k)\}.$$

### Theorem B (Again)

*For any $k \geq 2$, $C_k : \{\text{open sets of } M_k\} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ s.t.*

1. $C_k(U) \leq C_k(V)$ *if* $U \subset V$
2. $\lceil C_k(U) \rceil_{\frac{2\pi}{k}} = \lceil C_k(\phi(U)) \rceil_{\frac{2\pi}{k}}$ *where and* $\phi \in \mathrm{Cont}_0(M_k, \xi_k)$
3. $C_k(B_k(r)) \leq \pi r^2$ *if* $r \in (0, R/\sqrt{3})$
4. $C_k(B_k(r)) \geq \min\{2\pi, \pi r^2\}$

*where* $B_k(r) := \pi_k^{-1}(\Psi(\mathbb{B}^{2n}(r)))$ *for any* $\Psi : (\mathbb{B}^{2n}(R), \omega_0) \hookrightarrow (W, \omega)$ *and* $r \in (0, R]$.

# Periodicity of Reeb flow ⇒ conj. invariance

## Lemma

$\lceil c_k(\widetilde{\varphi}) \rceil_{\frac{2\pi}{k}} = \left\lceil c_k(\widetilde{\psi}\widetilde{\varphi}\widetilde{\psi}^{-1}) \right\rceil_{\frac{2\pi}{k}}$. *In particular* $\lceil C_k(U) \rceil_{\frac{2\pi}{k}} = \lceil C_k(\psi(U)) \rceil_{\frac{2\pi}{k}}$.

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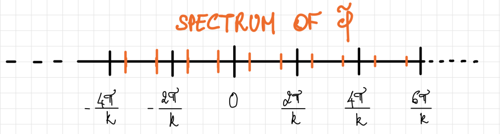
**Proof.**

1. $\widetilde{\varphi}$ has no disc. point $\Leftrightarrow \mathrm{Spec}^{\alpha_k}(\widetilde{\varphi}) \cap \frac{2\pi}{k}\mathbb{Z} = \emptyset$.



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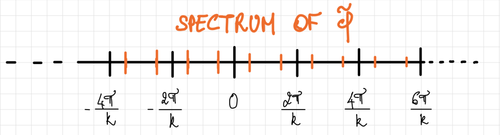


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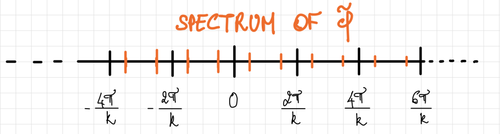


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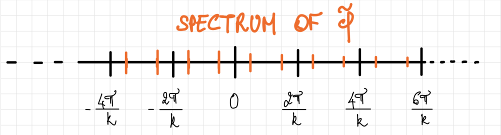


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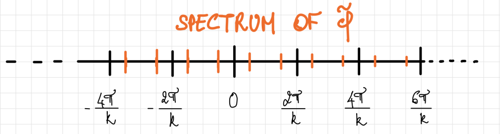


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2. Argument by density:
   for any $\widetilde{\varphi}$ there exists $(\widetilde{\varphi}_n)$ s.t. $\widetilde{\varphi}_n \to \widetilde{\varphi}$, $\widetilde{\varphi}_n \preceq \widetilde{\varphi}$ and $\widetilde{\varphi}_n$ has no disc. point.

# Energy-Capacity inequality: $C_k(U) \leq c_k(\widetilde{\psi}^{-1}) + \left\lceil c_k(\widetilde{\psi}) \right\rceil_{\frac{2\pi}{k}}$

Definition

$[(\varphi_t)] := \widetilde{\varphi} \in \widetilde{\mathrm{Cont}_0}(M, \xi)$ $\alpha$-displaces $U \subset M$ if $\varphi_1(\mathcal{O}_\alpha(U)) \cap \mathcal{O}_\alpha(U) = \emptyset$,
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$\widetilde{\varphi}$ supported in $U$ then $\mathrm{Spec}^\alpha(\widetilde{\psi}\widetilde{\varphi}) = \mathrm{Spec}^\alpha(\widetilde{\psi}) \overset{\text{nwh.dense} + C^0}{\Rightarrow} c_k(\widetilde{\psi}\widetilde{\varphi}) = c_k(\widetilde{\psi})$.

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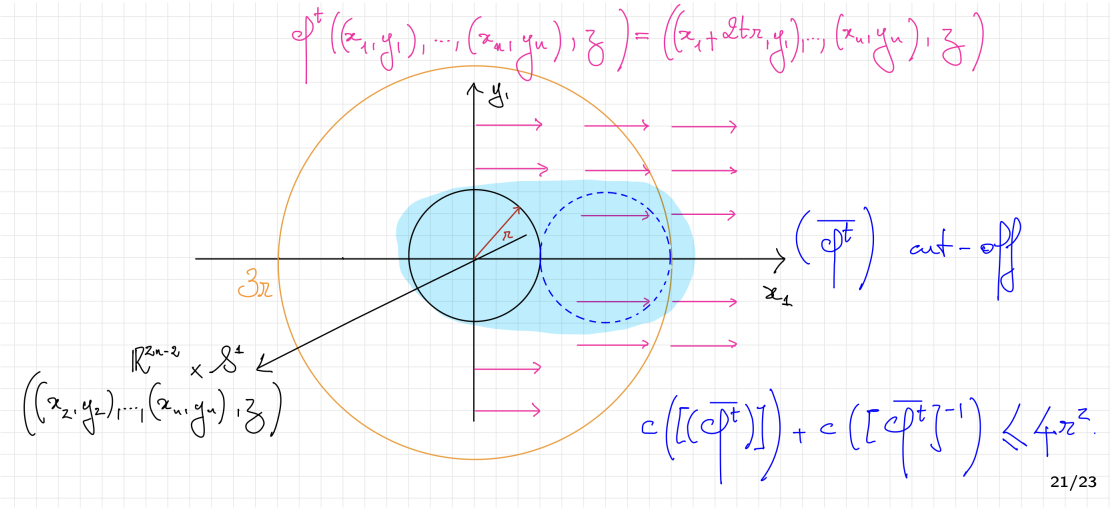
Together with triangular inequality

$$c_k(\widetilde{\varphi}) = c_k(\widetilde{\psi}^{-1}\widetilde{\psi}\widetilde{\varphi}) \leq c_k(\widetilde{\psi}^{-1}) + \lceil c_k(\widetilde{\psi}\widetilde{\varphi}) \rceil_{\frac{2\pi}{k}} = c_k(\widetilde{\psi}^{-1}) + \lceil c_k(\widetilde{\psi}) \rceil_{\frac{2\pi}{k}}.$$



□

# $C_k(B_k(r)) \leq \pi r^2$ for $r \in (0, \frac{R}{\sqrt{3}})$



## $C_k(B_k(r)) \geq \min\{\pi r^2, 2\pi\}$

For any $\varepsilon > 0$ there exists $(\varphi_t) \subset \mathrm{Cont}_0^c(B_1(r), \alpha_1)$ s.t. for all $t \leq 1$

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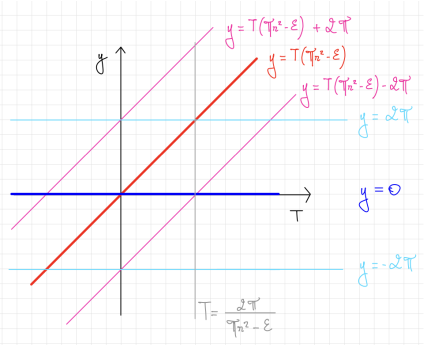
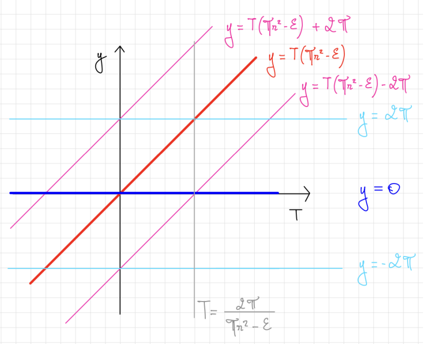
$$\mathrm{Spec}^{\alpha_1}(\widetilde{\varphi}_t) = \{t(\pi r^2 - \varepsilon), 0\} + 2\pi\mathbb{Z} \text{ and } \mathrm{id} \preceq \widetilde{\varphi}_t.$$



# $C_k(B_k(r)) \geq \min\{\pi r^2, 2\pi\}$

For any $\varepsilon > 0$ there exists $(\varphi_t) \subset \mathrm{Cont}_0^c(B_1(r), \alpha_1)$ s.t. for all $t \leq 1$

$$\mathrm{Spec}^{\alpha_1}(\widetilde{\varphi}_t) = \{t(\pi r^2 - \varepsilon), 0\} + 2\pi\mathbb{Z} \text{ and } \mathrm{id} \preceq \widetilde{\varphi}_t.$$




Consider the corresponding path $(\varphi_t^k) \subset \mathrm{Cont}_0^c(B_k(r), \xi_k)$.

1. If $(M_1, \xi_1)$ is str. ord. $\Rightarrow c_k([(\widetilde{\varphi}_t^k)]) \geq c_1([(\widetilde{\varphi}_t)]) = \min\{\pi r^2 - \varepsilon, 2\pi\}$
2. If $(M_1, \xi_1) = (\mathbb{S}^{2n+1}, \xi_1) \Rightarrow c_k([(\widetilde{\varphi}_t^k)]) = \min\{\pi r^2 - \varepsilon, 2\pi\}$.

# Thank you for your attention!