The shape invariant and Lagrangian intersections

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Work in $\mathbb{R}^4 \cong \mathbb{C}^2$ with coordinates (z, w) and its standard symplectic form $\omega = \frac{i}{2}(dz \wedge d\bar{z} + dw \wedge d\overline{w})$.

We define the product Lagrangian tori

$$L(a, b) = \left\{\pi|z|^2 = a,\ \pi|w|^2 = b\right\},$$

and also ellipsoids and polydisks

$$E(a, b) = \left\{\frac{\pi|z|^2}{a} + \frac{\pi|w|^2}{b} < 1\right\};\ P(a, b) = \left\{\pi|z|^2 < a, \pi|w|^2 < b\right\}.$$

The ball $B(a) = E(a, a)$ and the cylinder $Z(a) = P(a, \infty)$.

More generally, a toric domain is a subset $X_\Omega := \mu^{-1}\Omega \subset \mathbb{C}^2$ where

$$\mu : \mathbb{C}^2 \to \mathbb{R}^2_{\geq 0}, \quad (z, w) \mapsto (\pi|z|^2, \pi|w|^2)$$

and $\Omega \subset \mathbb{R}^2_{\geq 0}$.

Motivating question: Given Ω_1 and Ω_2, does there exist Hamiltonian diffeomorphism of $\mathbb{C}^2$ with $X_{\Omega_1} \hookrightarrow X_{\Omega_2}$?

In particular, the Hamiltonian condition excludes embeddings $\{\pi|z|^2 \in (1,2)\} \hookrightarrow Z(1)$.

We'll try to study this using the (Hamiltonian) shape invariant.

Definition

Let $X \subset \mathbb{C}^2$.

$$\mathrm{Sh}_H(X) = \{(r,s) \,|\, r \leq s,\ L(r,s) \hookrightarrow X\}.$$

Here $L(r,s) \hookrightarrow X$ means there exists a Hamiltonian diffeomorphism of $\mathbb{C}^2$ mapping $L(r,s)$ into X.

The definition is taken from a paper with Jun Zhang; a more general definition was introduced by Eliashberg in the 1990s.

Some immediate properties are

$$\Omega^+ := \{(r,s) \,|\, r \le s,\ (r,s) \in \Omega \,\mathrm{or}\, (s,r) \in \Omega\} \subset \mathrm{Sh}_H(X_\Omega);$$

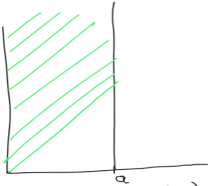
$$\mathrm{Sh}_H(X_{\lambda\Omega}) = \lambda \mathrm{Sh}_H(X_\Omega) \ \mathrm{for\,all}\ \lambda > 0;$$

$$X_{\Omega_1} \hookrightarrow X_{\Omega_2} \ \mathrm{implies}\ \mathrm{Sh}_H(X_{\Omega_1}) \subset \mathrm{Sh}_H(X_{\Omega_2}).$$

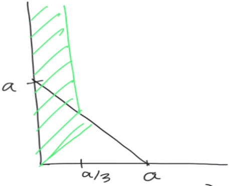
Shape is a ‘set-valued symplectic capacity’.

There are computations for cylinders $Z(R)$, balls $B(R)$ and polydisks $P(a,b)$ (joint work with E. Opshtein) and also for ellipsoids $E(a,b)$ with $b/a \in \mathbb{Z}_{\ge 2}$ (joint with J. Zhang).

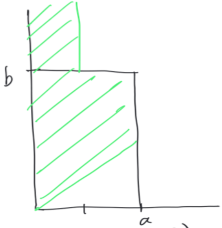
Some shapes



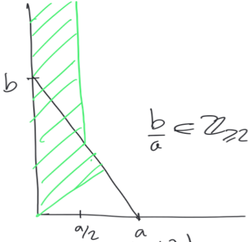
$Sh_H(Z(a))$



$Sh_H(B(a))$



$Sh_H(P(a,b))$



$Sh_H(E(a,b))$

The strips here come from an explicit embedding.

Proposition

For all $r < s$ there exists a Hamiltonian diffeomorphism with

$$L(r,s) \hookrightarrow X_{(1+\varepsilon)Q_r},$$

where Q_r is the quadrilateral with vertices $(0,0)$, $(2r,0)$, $(r,2r)$, $(0,2r)$.

We observe that the shape is far from a complete obstruction to symplectic embeddings. This is due to the existence of vertical strips, and also because $\mathrm{Sh}_H(X_\Omega)$ only sees Ω^+.

For example,

$$\mathrm{Sh}_H(B(c)) \subset \mathrm{Sh}_H(Z(R))$$

if and only if $c \leq 2R$.

Proposition

Suppose there exist a smooth family of Hamiltonian diffeomorphisms with

$$L(r,s) \hookrightarrow Z(R)$$

for all $(r,s) \in \mu(B(c))$*. Then* $c \leq R$*.*

(Compare Entov-Ganor-Membrez, Shelukhin-Tonkonog-Vianna, H.-Zhang.)

It turns out that even a smooth family of Lagrangian embeddings is not sufficient to guarantee a symplectic embedding. However we do have the

Lemma

Suppose there exist a smooth family of Hamiltonian diffeomorphisms with

$$L(r,s) \hookrightarrow X_{\Omega_2}$$

for all $(r,s) \in \Omega_1$*.*
Then there exists a 'stabilized embedding'

$$X_{\Omega_1} \times \mathbb{C} \hookrightarrow X_{\Omega_2} \times \mathbb{C}.$$

Explicit calculations for Lagrangian isotopies (joint with Jun Zhang) and for stabilized embeddings (Cristofaro-Gardiner, McDuff, Siegel) show that the converse holds in many examples.

Here is our main theorem.

Theorem

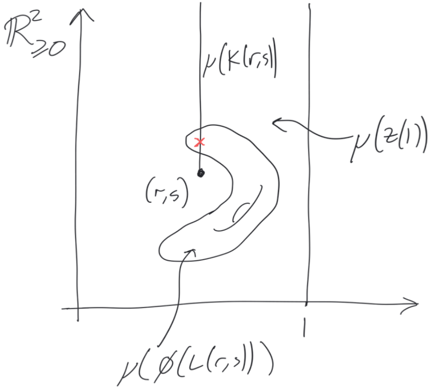
Suppose $1/2 \leq r \leq 1 \leq s$.
Let $\phi \in \mathrm{Ham}(\mathbb{C}^2)$ *satisfy* $\phi(L(r,s)) \subset Z(1)$.
Then $\phi(L(r,s))$ *must intersect* $K(r,s) = \cup_{t \geq s} L(r,t)$.

There's also a variation.

Theorem

Suppose $1/2 \leq r \leq 1 \leq s$.
Let $\phi \in \mathrm{Ham}(\mathbb{C}^2)$ *satisfy* $\phi(L(r,s)) \subset P(1,2s)$.
Then $\phi(L(r,s))$ *must intersect* $L(r,s)$.

Moment projections



Comparison with Lagrangian intersection results.

Suppose $b = (k+1)s - g$ with $0 \le g < s$.

Then there exist k disjoint circles $\Lambda_i \subset D(b) = \{\pi |w|^2 < b\}$ each bounding a disk of area s.

(A $(k+1)^{\text{st}}$ would need to overlap these disks in a region of area g.)

Let $L_i = \{\pi |z|^2 = r\} \times \Lambda_i \subset P(1, b)$.

For example, setting $\Lambda_1 = \{\pi |w|^2 = s\}$ we have $L_1 = L(r, s)$ and $L_i \subset K(r, s)$ for all i.

Theorem (Polterovich, Shelukhin (see also Mak, Smith))

Suppose $g > (k-1)r$*.*
Let $\phi \in \mathrm{Ham}(P(2r, b))$*.*
Then $\phi(L(r, s))$ *must intersect at least one of the* L_i*.*

However, following methods of Hicks and Mak we have

Proposition

Suppose $6g < k(1-r)$*. Then* $P(r, s)$ *is displaceable from* $\cup L_i$ *inside* $P(1, b)$*.*
(If $6g < kr$*, then can displace* $P(r, s)$ *inside* $P(2r, b)$*.)*

Corollary

Given any (infinite) family of disjoint circles Λ_i *in the* w *plane, each bounding area* s*, there exists a Hamiltonian diffeomorphism* $\phi \in \mathrm{Ham}(Z(1))$ *with* $\phi(L(r, s)) \subset Z(1) \setminus \cup L_i$*.*

Consequences for the shape.

Corollary

Suppose $f : [0,1] \to [1,\infty)$ *and*

$$\Omega = \{(r,s) \mid r \in [0,1),\ s < f(r)\}.$$

Then $\mathrm{Sh}_H(X_\Omega) = (\Omega^+ \cup \{r < 1/2\}) \cap \{r \le s\}$.

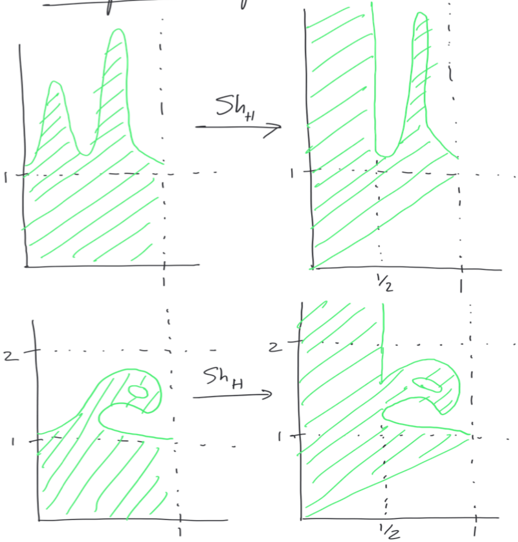
Corollary

Suppose

$$[0,1] \times [0,1] \subset \Omega \subset [0,1] \times [0,2].$$

Then $\mathrm{Sh}_H(X_\Omega) = (\Omega^+ \cup \{r < 1/2\}) \cap \{r \le s\}$.

Shape examples



Outline of the proof. (Following H.-Kerman, Boustany.)

Argue by contradiction and suppose we have $\mathbb{L} = \phi(L(r,s)) \subset Z(1)$ with $\mathbb{L} \cap K(r,s) = \emptyset$.

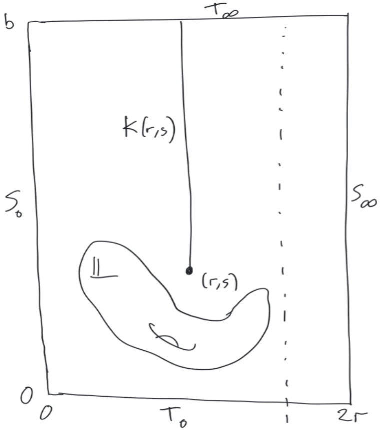
Then for b large we have $L(r,s), \mathbb{L} \subset P(1,b) \subset P(2r,b) \subset X := S^2(2r) \times S^2(b)$.

In $S^2(2r) \times S^2(b)$ we denote

$$S_0 = \{0\} \times S^2(b);\ S_\infty = \{\infty\} \times S^2(b);$$
$$T_0 = S^2(2r) \times \{0\};\ T_\infty = S^2(2r) \times \{\infty\}.$$

By a stretching argument, we may assume $\mathbb{L}$ is disjoint from S_0 and T_0 (as well as S_∞, T_∞ and the tori $L(r,t)$ for $s \leq t < b$).

More moment images



Next introduce almost complex structures J on $X = S^2(2r) \times S^2(b)$.

We'll always assume J is the standard product structure near $S_0 \cup S_\infty \cup T_0 \cup T_\infty$, and in fact in $X \setminus P(1, b)$.

There exists a foliation of X by J-holomorphic spheres in the class $[(1, 0)]$. These each intersect S_∞ in a single point, and so we get a projection map

$$p : X \to S_\infty.$$

Hence

$$p^{-1}(0) = T_0;\ p^{-1}(\infty) = T_\infty.$$

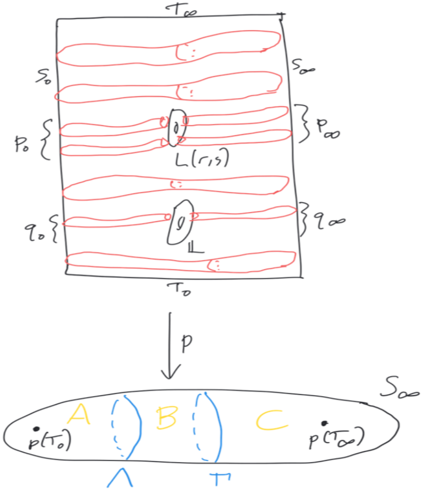
We also consider almost complex structures which are singular along $L(r, s) \cup \mathbb{L}$, that is, the almost complex structures on (the completion of) $X \setminus (L(r, s) \cup \mathbb{L})$ which result from stretching the neck.

Now we have a (finite energy) foliation of $X \setminus (L(r, s) \cup \mathbb{L})$ where generic leaves are still spheres, but there are 1 parameter families of 'broken curves' asymptotic to geodesics on the tori.

The projection map $p : X \setminus (L(r, s) \cup \mathbb{L}) \to S_\infty$ extends to X and generically the images $p(L(r, s)) = \Gamma$ and $p(\mathbb{L}) = \Lambda$ are circles, although a priori not embedded. (See the paper of Goodman, Ivrii, Dimitroglou-Rizell.)

In our situation we can say more.

Foliations by J-spheres



Lemma

1. *The broken leaves (preimages of points on $\Gamma \cup \Lambda$) are pairs of finite energy planes, each of area r, asymptotic to the same geodesic with opposite orientation.*
2. *The circles Γ and Λ are embedded and disjoint.*
3. *Γ separates Λ from $\infty \in S_\infty$.*

The proof relies on

1. For all J, Maslov 2 holomorphic planes in $\mathbb{C}^2$ asymptotic to $L(r, s)$ have area at least r.
2. Holomorphic curves in X which intersect S_∞ have area at least $2r - 1$.
3. Replacing $L(r, s)$ by $L(r, t)$ for t close to b, the corresponding Γ_t is a small circle around ∞.

The disk A lifts to Maslov 2 sections of p with boundary on $\mathbb{L}$; the disk $A \cup B$ lifts to Maslov 2 sections with boundary on $L(r, s)$.

Lemma

These disks have area s.

For the proof, the boundaries intersect the boundaries of broken curves once, and with the Maslov 2 condition we see that the area is s or $2r - s$.

We will see that there are holomorphic disks lifting A. If these have area $2r - s < r$ then they must intersect S_∞, but then by monotonicity the area is at least $2r - 1$, a contradiction.

Recall we denote the two families of broken planes asymptotic to $L(r,s)$ by $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$, and the two families asymptotic to $\mathbb{L}$ by $\mathfrak{q}_0$ and $\mathfrak{q}_\infty$.

We know planes in $\mathfrak{p}_0$ and $\mathfrak{p}_\infty$ intersect S_0 and S_∞ respectively, and also by definition the planes in $\mathfrak{q}_\infty$ intersect S_∞.

Homologically nontrivial case.

Suppose the planes in $\mathfrak{q}_0$ intersect S_0. Then $L(r,s)$ and $\mathbb{L}$ are homologically nontrivial in $U := X \setminus (S_0 \cup S_\infty \cup T_0 \cup T_\infty)$.

In fact, we can identify U with an open subset of T^*T^2, with $L(r,s)$ the zero section. Then our area estimates imply $\mathbb{L}$ is an exact Lagrangian torus. But this is a contradiction, as a theorem of Gromov says exact Lagrangians intersect the zero section.

Homologically trivial case.

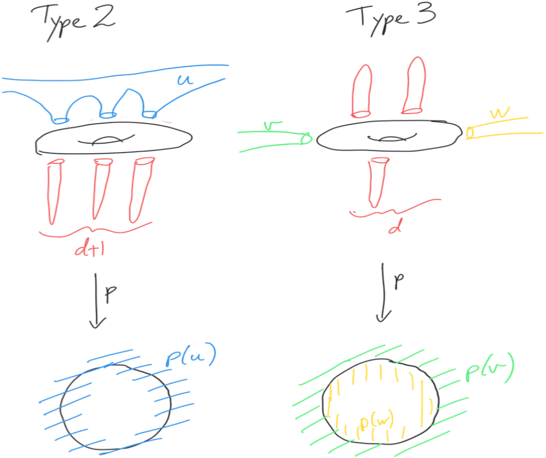
Hence we can assume $\mathbb{L}$ is homologically trivial in the complement of the axes, in other words the planes $\mathfrak{q}_0$ are disjoint from S_0.

We plan to substitute S_0 and S_∞ with high degree curves to act as new axes.

We study J holomorphic spheres in X in the class $[(d, 1)]$. Such curves exist, intersecting $2d + 1$ constraint points. We place $d + 1$ points on $L(r, s)$ and d points on $\mathbb{L}$ and stretch the neck along $L(r, s) \cup \mathbb{L}$.

There are two types of potential limits near $p^{-1}\Gamma$, and similarly two types near $p^{-1}\Lambda$.

Limiting buildings near $p^{-1}(T')$



Lemma

For d large, the limiting holomorphic building **F** *has Type 3 near both $p^{-1}\Gamma$ and $p^{-1}\Lambda$.*

For example, suppose **F** has Type 2 near $p^{-1}\Lambda$, and the d planes asymptotic to $\mathbb{L}$ lie in $\mathfrak{q}_0$. Removing the d or $d+1$ planes asymptotic to $L(r,s)$, the remaining components of our building intersect $S_0 \cup S_\infty$ at least

$$2d - (d+1) = d - 1$$

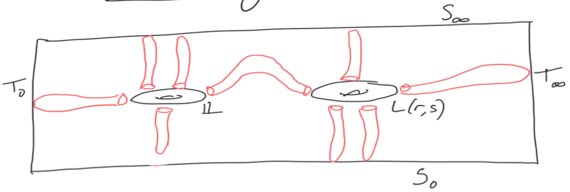
times, but have area at most

$$(2rd + b) - dr - dr = b.$$

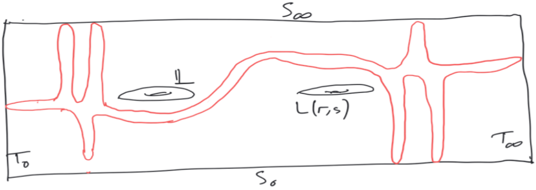
This contradicts monotonicity when d is large with respect to b.

We conclude that **F** has three constituent curves projecting injectively to A, B and C respectively, together with d planes coinciding with broken leaves asymptotic to $L(r,s)$ and $d-1$ planes coinciding with broken planes asymptotic to $\mathbb{L}$.

The building F



A deformation



There exists a similar holomorphic building **G**, but now containing $d - 1$ planes asymptotic to $L(r, s)$ and d planes asymptotic to $\mathbb{L}$.

Lemma

1. *The buildings* **F** *and* **G** *can be deformed to give smooth symplectic spheres F and G in $X \setminus (L(r, s) \cup \mathbb{L})$.*
2. *Exactly one of F or G intersects the disks in each of the families $\mathfrak{p}_0$, $\mathfrak{p}_\infty$, $\mathfrak{q}_0$ and $\mathfrak{q}_\infty$.*
3. *Moreover, F and G intersect d times positively and transversally in $p^{-1}A$ and d times in $p^{-1}B$.*
   These are the only intersections.

The spheres F and G have self intersection $2d$, but we can form spheres $\hat{F}$ and $\hat{G}$ of self intersection 0 in a new copy Y of $S^2 \times S^2$ by blowing up balls of capacity δ centered at each intersection point, then blowing down the proper transforms of the corresponding fibers of p (that is, leaves through the intersection points).

The manifold Y is symplectomorphic to $S^2 \times S^2$ via a symplectomorphism mapping T_0 and T_∞ to $S^2 \times \{0\}$ and $S^2 \times \{\infty\}$ respectively, and mapping $\hat{F}$ and $\hat{G}$ to $\{0\} \times S^2$ and $\{\infty\} \times S^2$ respectively.

(Hence the areas of the factors are $2r$ and $b + 2d(r - \delta)$.)

The key point is that $L(r, s)$ and $\mathbb{L}$ are disjoint from the new axes (as they were disjoint from F and G), but now both are homologically nontrivial, and they are relatively exact.

(They're homologically nontrivial because we have axes intersecting each of the families $\mathfrak{p}_0$, $\mathfrak{p}_\infty$, $\mathfrak{q}_0$ and $\mathfrak{q}_\infty$.

They're relatively exact because we don't blow up any fibers over B.)

We can finish as before by applying Goodman-Ivrii-Dimitroglou-Rizell and Gromov.

Thank-you!