S^1-equivariant relative symplectic cohomology and relative symplectic capacities

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Preliminaries

- The **Novikov field** Λ is defined by

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \mid c_i \in \mathbb{Q}, \lambda_i \in \mathbb{R} \text{ and } \lim_{i \to \infty} \lambda_i = \infty \right\}$$

  where T is a formal variable.
- There is a **valuation map** $val : \Lambda \to \mathbb{R} \cup \{\infty\}$ given by

$$val(x) = \begin{cases} \min\{\lambda_i \mid c_i \neq 0\} & \text{if } x = \sum_{i=0}^{\infty} c_i T^{\lambda_i} \neq 0 \\ \infty & \text{if } x = 0. \end{cases}$$

- For any $r \in \mathbb{R}$, define $\Lambda_{\geq r} = val^{-1}([r, \infty])$. In particular, we call

$$\Lambda_{\geq 0} = \left\{ \sum_{i=0}^{\infty} c_i T^{\lambda_i} \in \Lambda \mid \lambda_i \geq 0 \right\}$$

  the **Novikov ring**.

Preliminaries

- A compact symplectic manifold (K, ω) is said to have **contact type boundary** if there exists a Liouville vector field X defined on a neighborhood of ∂K satisfying $\mathcal{L}_X \omega = \omega$ and X is transverse to ∂K.
- Let $\lambda = \iota_X \omega$. Then $\alpha = \lambda|_{\partial K}$ is a canonical contact form on ∂K.
- The **symplectic completion** $\widehat{K}$ of K is the symplectic manifold $\widehat{K} = (K \amalg (\partial K \times [0, \infty)))/\sim$ with its symplectic form

$$\widehat{\omega} = \begin{cases} \omega & \text{on } K \\ d(e^{\rho}\alpha) & \text{on } \partial K \times [0, \infty). \end{cases}$$

  where ρ is a coordinate on $[0, \infty)$. The equivalence relation $\sim$ is given by the diffeomorphism $\partial K \times [0, \infty) \to U, \ (p, \rho) \mapsto \phi_{\rho}^{X}(p)$ where ϕ_{ρ}^{X} is the flow of the Liouville vector field X and U is a neighborhood of ∂K in K.

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Definition

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Let (M, ω) be a closed symplectic manifold and let $K \subset M$ be a compact domain with contact type boundary. A **contact type K-admissible Hamiltonian function** is a smooth function $H : S^1 \times M \to \mathbb{R}$ satisfying the following conditions.

- H is negative and C^2-small on $S^1 \times K$. Moreover, $H > -\epsilon$ on $S^1 \times K$ where $\epsilon > 0$ is the half of minimal period of Reeb orbits of ∂K.
- There exists $\eta \geq 0$ such that $H(t, p, \rho)$ is C^2-close to $h_1(e^\rho)$ on $S^1 \times \left(\partial K \times [0, \frac{1}{3}\eta]\right)$ for some convex and increasing function h_1.

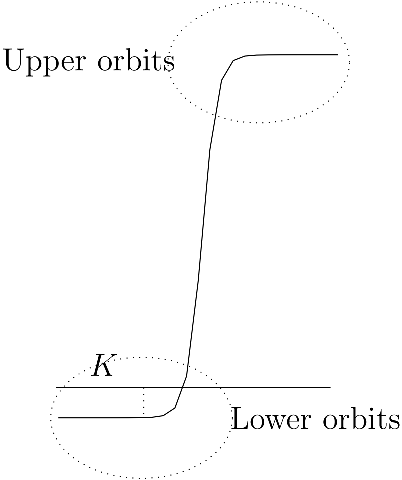
Definition

Definition (continued)

- $H(t, p, \rho) = \beta e^{\rho} + \beta'$ on $\partial K \times [\frac{1}{3}\eta, \frac{2}{3}\eta]$ where $\beta \notin \mathrm{Spec}(\partial K, \alpha)$ and $\beta' \in \mathbb{R}$.
- $H(t, p, \rho)$ is C^2-close to $h_2(e^{\rho})$ on $S^1 \times \left(\partial K \times [\frac{2}{3}\eta, \eta]\right)$ for some concave and increasing function h_2.
- H is C^2-close to a constant function on $S^1 \times (M - K \cup (\partial K \times [0, \eta]))$.

We denote the set of all contact type K-admissible Hamiltonian functions by $\mathcal{H}_K^{\mathrm{Cont}}$.

Definition



Definition

- Let $CF^{S^1}(H) = \Lambda_{\geq 0}[u] \otimes_{\Lambda_{\geq 0}} CF(H)$ be the S^1-equivariant Floer complex of H where u is a formal variable of degree 2. The S^1-equivariant Floer differential d^{S^1} of $CF^{S^1}(H)$ has the form

$$d^{S^1}(u^k \otimes x) = \sum_{i=0}^{k} u^{k-i} \otimes \psi_i(x).$$

- Choose a cofinal sequence $\{H_n\}$ of $\mathcal{H}_K^{\mathrm{Cont}}$, that is,

$$H_1 \leq H_2 \leq H_3 \leq \cdots \text{ and } \lim_{n\to\infty} H_n(t,x) = \begin{cases} 0 & \text{if } x \in K \\ \infty & \text{if } x \in M - K. \end{cases}$$

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- $SH_M^{S^1}(K) = H\left(\varinjlim_{n\to\infty} CF^{S^1}(H_n)\right)$?

Definition

Let A be a module over $\Lambda_{\geq 0}$. Then the **completion** $\widehat{A}$ of A is defined by

$$\widehat{A} = \varprojlim_{r \to 0} A \otimes \Lambda_{\geq 0} / \Lambda_{\geq r}.$$

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Let (M, ω) be a closed symplectic manifold and let $K \subset M$ be a compact domain with contact type boundary. Then the S^1**-equivariant relative symplectic cohomology** $SH_M^{S^1}(K)$ of K in M is defined by

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$$SH_M^{S^1}(K) = H\left(\widehat{\varinjlim_{n \to \infty} CF^{S^1}(H_n)}\right).$$

By completing the complex, we can ignore the upper orbits.

Property

Question. Does $SH_M^{S^1}(K)$ really depend on K?

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Example (Varolgunes)

Let (S^2, ω) be the 2-dimensional sphere equipped with an area form ω with total area 1. Let $D_\Delta \subset S^2$ be a smooth disk of area Δ.

- $SH^{S^1}(D_\Delta; \Lambda) = 0$ regardless of the size of D_Δ.
- $SH_{S^2}^{S^1}(D_\Delta; \Lambda) = \begin{cases} 0 & \text{if } \Delta < \frac{1}{2} \\ \Lambda[u] \oplus \Lambda[u] & \text{if } \Delta \geq \frac{1}{2} \end{cases}$ where u is a formal variable of degree 2.

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Theorem (Varolgunes)

Let (M, ω) be a closed symplectic manifold and let $K \subset M$ be a compact subset. If K is displaceable, then $SH_M(K; \Lambda) = 0$.

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Theorem

Let (M, ω) be a closed symplectic manifold and let $K \subset M$ be a compact domain with contact type boundary. Then there exists a spectral sequence $E_r^{p,q}(M, K)$ converging to $SH_M^{S^1}(K)$ such that its second page is given by

$$E_2^{p,q}(M, K) \cong H^p(BS^1; \Lambda_{\geq 0}) \otimes SH_M^q(K).$$

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Corollary

Let (M, ω) be a closed symplectic manifold and let $K \subset M$ be a compact domain with contact type boundary. If K is displaceable, then $SH_M^{S^1}(K; \Lambda) = 0$.

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Relative symplectic capacity

- From this point on, we assume that (M, ω) is *symplectically aspherical*, that is, $\omega|_{\pi_2(M)} = 0$ and $c_1(TM)|_{\pi_2(M)} = 0$.
- We say that ∂K is **index-bounded** if, for each $\ell \in \mathbb{Z}$, the set of periods of contractible Reeb orbits of $(\partial K, \alpha)$ of Conley-Zehnder index ℓ is bounded.
- Additionally, we add the index-boundedness of ∂K to our assumption list.
- Let $SH_M^{S^1,>L}(K)$ be the action filtration of $SH_M^{S^1}(K)$ generated by Hamiltonian orbits with action greater than L.
- Let $SH_M^{S^1,-}(K)$ be the cohomology generated by nonconstant Hamiltonian orbits and $SH_M^{S^1,-,>L}(K)$ be its action filtration.

Relative symplectic capacity

Definition

Let (M, ω) be a symplectic manifold and let $K \subset M$ be a subset. A **relative symplectic capacity** c assigns to each triple (M, K, ω) a number $c(M, K, \omega) \in [0, \infty]$ satisfying

- (Monotonicity) if there exists a symplectic embedding $\phi : (M, \omega) \hookrightarrow (M', \omega')$ such that $\mathrm{int}(\phi(K)) \subset K'$, then $c(M, K, \omega) \leq c(M', K', \omega')$, and
- (Conformality) if $r > 0$, then $c(M, K, r\omega) = rc(M, K, \omega)$.

We will usually drop the symplectic form ω in $c(M, K, \omega)$ if it is clear from the context.

Relative symplectic capacity

- Floer, Hofer and Wysocki introduced the symplectic (co)homology capacity, denoted by $c^{SH}(K)$, using symplectic cohomology.
- Gutt and Hutchings introduced the Gutt-Hutchings capacity, denoted by $c_k^{GH}(K)$, for each $k = 1, 2, 3, \cdots$ using S^1-equivariant symplectic cohomology.

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Theorem

- *There exists a relative Gutt-Hutchings capacity $c_k^{GH}(M, K)$ for each $k = 1, 2, 3, \cdots$.*
- *There exists a relative symplectic (co)homology capacity $c^{SH}(M, K)$.*

Relative symplectic capacity

To define $c^{SH}(M, K)$, we need the following exact triangle.

$$\begin{array}{ccc} H(K, \partial K; \Lambda) & \xrightarrow{j^L} & SH_M^{>L}(K; \Lambda) \\ & \nwarrow & \downarrow \\ & & SH_M^{-,>L}(K; \Lambda) \end{array}$$

Definition

Define the relative symplectic (co)homology capacity $c^{SH}(M, K)$ by

$$c^{SH}(M, K) = -\sup\left\{L < 0 \mid j^L(1_K) = 0\right\}$$

where 1_K is the unit of $H(K, \partial K; \Lambda)$.

Relative symplectic capacity

To define $c_1^{GH}(M, K)$, we need the S^1-equivariant version of the previous exact triangle.

$$\begin{array}{ccc} H(K, \partial K; \Lambda) \otimes H(BS^1; \Lambda) & \xrightarrow{j^{S^1, L}} & SH_M^{S^1, >L}(K; \Lambda) \\ & \nwarrow & \downarrow \\ & & SH_M^{S^1, -, >L}(K; \Lambda) \end{array}$$

Definition

Define the first relative Gutt-Hutchings capacity $c_1^{GH}(M, K)$ by

$$c_1^{GH}(M, K) = -\sup\left\{L < 0 \mid j^{S^1, L}(1_K \otimes 1) = 0\right\}.$$

Relative symplectic capacity

- Gutt and Ramos proved that $c_k^{GH}(K) = c_k^{EH}(K)$ on every star-shaped domain $K \subset \mathbb{R}^{2n}$ where $c_k^{EH}(K)$ is the k-th Ekeland-Hofer capacity.
- Abbondandolo and Kang proved that $c^{SH}(K) = c_1^{EH}(K)$ on every convex domain $K \subset \mathbb{R}^{2n}$.

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Question. Is $c_1^{GH}(M, K) = c^{SH}(M, K)$?

Answer. Sometimes. Generally, $c_1^{GH}(M, K) \leq c^{SH}(M, K)$. To prove the other inequality, we need some convexity assumption.

Relative symplectic capacity

Definition

Let (Σ, ξ, α) be a contact manifold of dimension $2n-1$ where ξ is a contact structure and α is a contact form. Assume that the first Chern class $c_1(\xi)$ vanishes. A contact form α is called **dynamically convex** if every contractible periodic Reeb orbit γ of α satisfies $CZ(\gamma) \geq n+1$.

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Theorem

If the canonical contact form α on ∂K is dynamically convex, then

$$c_1^{GH}(M, K) = c^{SH}(M, K).$$

Relative symplectic capacity

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Theorem

If $c^{SH}(M, K) = \infty$, then K is heavy and hence not displaceable.

Thank you!