Regularity and persistence in non-Weinstein Liouville geometry via hyperbolic dynamics

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Symplectic Zoominar

Assumptions:

- W: compact oriented connected 4-manifold.
- Everything smooth unless stated otherwise.

Definition

A 1-form α on W is a Liouville form if

$$d\alpha \wedge d\alpha > 0.$$

A dynamical perspective is provided by

Definition

There exists a unique vector field Y, called the Liouville vector field, satisfying

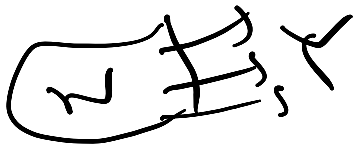
$$\iota_Y d\alpha = \alpha.$$

Boundary condition

- **Stoke's theorem:**

$$0 < \int_W d\alpha \wedge d\alpha = \int_{\partial W} \alpha \wedge d\alpha \Longrightarrow \partial W \neq \emptyset.$$

- Boundary condition?
- Cartan's formula $\Longrightarrow \mathcal{L}_Y d\alpha = d\alpha$.



Definition

The pair (W, α) is called a Liouville domain, if Y is positively transverse to ∂W.

- Geometric interpretation:

$$Y \pitchfork \partial W \text{ positively} \Longleftrightarrow \alpha \wedge d\alpha|_{\partial W} = \frac{1}{2}\iota_Y(d\alpha \wedge d\alpha) > 0.$$

$$\Longleftrightarrow \alpha|_{\partial W} : \textbf{positive contact form}.$$

Weinstein dichotomy

- When Y is **gradient-like**, i.e. there exists $f : W \to \mathbb{R}$ such that

$$Y \cdot f \geq \epsilon(|Y|^2 + |df|^2),$$

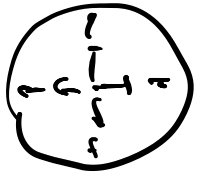
Morse theory $\longrightarrow$ **Symplectic handle decomposition**

- $\Longrightarrow$ **topological type** of W: $\leq 2 \Longrightarrow \partial W$: connected.
- In this case,

$$Skel(Y) := \{\text{points not flowing out under the flow of } Y\}$$

is CW-complex with 0,1,2 cells.

- Any Liouville domain with such Liouville flow (up to homotopy) is called Weinstein.
- Example: (1) $(\mathbb{D}^4_{(x_1,y_1,x_2,y_2)}, \Sigma_{i=1}^2 \frac{1}{2}(x_i dy_i - y_i dx_i))$.
(2) $(T_1^* S, \alpha_{can})$.
(3) Attaching symplectic handles. (4) Stein manifolds.



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Non-Weinstein examples

- Non-Weinstein Liouville geometry is far less understood!

Question

Are there examples of non-Weinstein Liouville geometry?

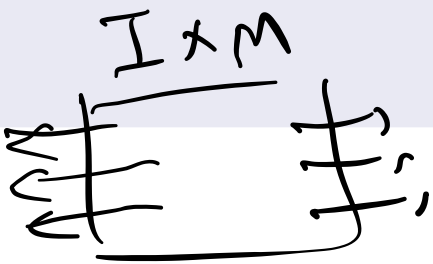
- (McDuff 91) (Geiges 95) (Mitsumatsu 95)

Theorem (Mitsumatsu 95)

If M is a 3-manifold admitting an Anosov flow, there exists a Liouville form α such that

$$([-1,1] \times M, \alpha)$$

is a (necessarily non-Weinstein) Liouville domain.



Outline

- Quick introduction to Anosov flows.
- Mitsumatsu's construction and the Liouville geometry of Anosov flows
- Dynamical rigidity and consequences
- Geometric rigidity and consequences
- Skeleton C^1-persistence (a characterization)

Part II: Introduction to Anosov flows

- M: closed oriented connected 3-manifold.
- X: a vector field on M. X^t: the flow generated by X

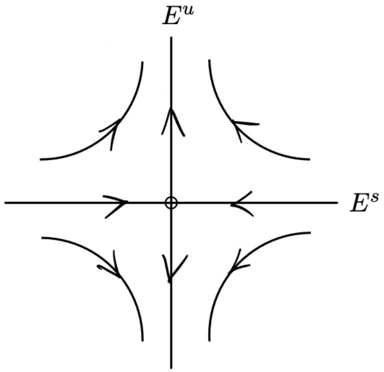
Definition

The flow X^t is Anosov, if there exists a continuous splitting $TM = E^s \oplus E^u \oplus \langle X \rangle$, such that the splitting is invariant under X^t and

$$||X^t_*(v)|| \geq e^{Ct}||v|| \text{ for any } v \in E^u,$$

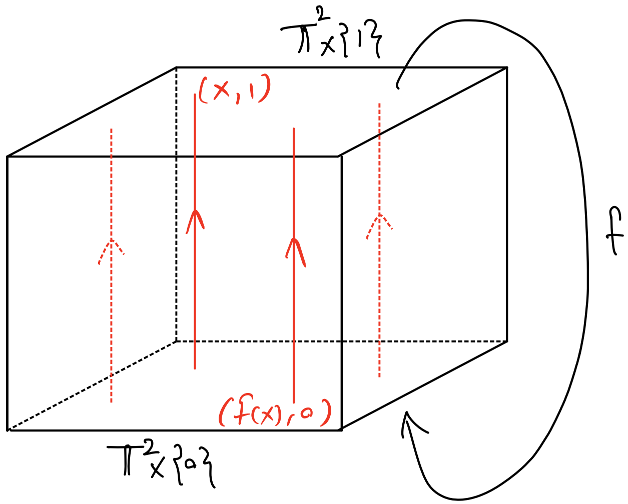
$$||X^t_*(u)|| \leq e^{-Ct}||u|| \text{ for any } u \in E^s,$$

where $C > 0$, and $||.||$ is induced from some Riemannian metric on TM.



The line bundle $E^s(E^u)$ is called the strong stable (unstable) line bundle.

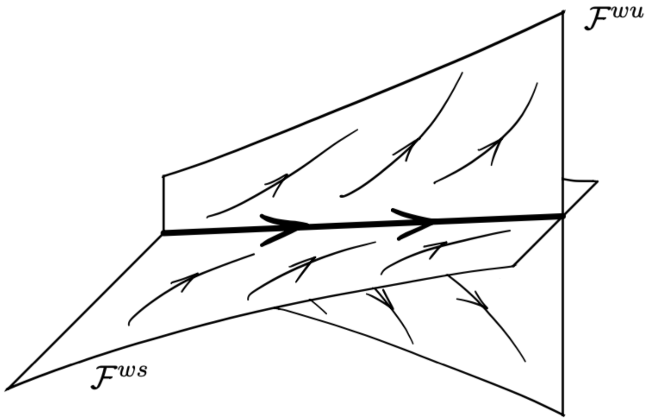
- Suspension flows
- Consider an area preserving hyperbolic diffeomorphism $f : \mathbb{T}^2 \simeq \mathbb{R}^2/\mathbb{Z}^2 \to \mathbb{T}^2$.
- e.g. $f = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \in SL(2, \mathbb{Z})$ with real eigenvalues.
- Let $M := \mathbb{T}^2 \times [0, 1]/(x, 1) \sim (f(x), 0)$.
- $X_f^t(x, s) = (x, s + t)$ is an Anosov flow.



- Geodesic flows
- Σ: hyperbolic surface. The geodesic flow on the unit tangent space $UT\Sigma$.

Part II: Introduction to Anosov flows

- The foliation theory has been the main tool in the study of Anosov flows.
- The plane fields $E^{wu} := E^u \oplus \langle X \rangle$ and $E^{ws} := E^s \oplus \langle X \rangle$, called the weak unstable/stable bundles, are tangent to foliations.
- Local picture:



Ghys ? ⟺ classical examples

- A priori only Hölder continuous.
- (Hirsch-Pugh-Shub 70) weak bundles are C^{1+}.
- (Hasselblatt 93) Lower bounds for regularity of weak bundles in terms of the expansion data (bunching constants).

Convention:

From now on, we are assuming E^s (and E^u) are orientable.

- A local model based on contact geometry has higher regularity, is truly local and reflects the stability features of Anosov flows!

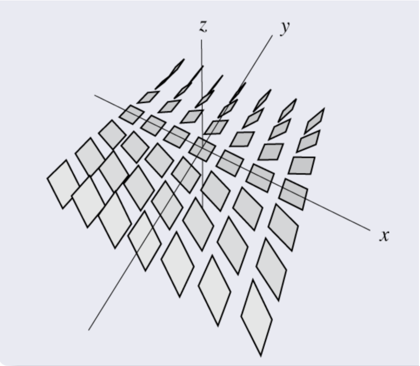
Definition

We call a 1-form α a positive (negative) contact form on M, if $\alpha \wedge d\alpha > 0$ (< 0).

$\xi = \ker \alpha$

Examples:

- The 1-form $\alpha_{std} = dz - y\,dx$ is a (positive) contact structure on $\mathbb{R}^3$ [1].



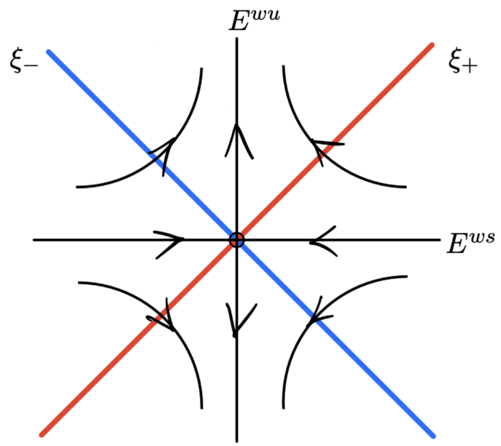
- [Darboux]: All contact structures locally look the same.

[1] Picture from Wkipedia: Standard contact structure on $\mathbb{R}^3$

Part III: Mitsumatsu's construction and the Liouville geometry of Anosov flows

Proposition (Mitsumatsu, Eliashberg-Thurston 95)

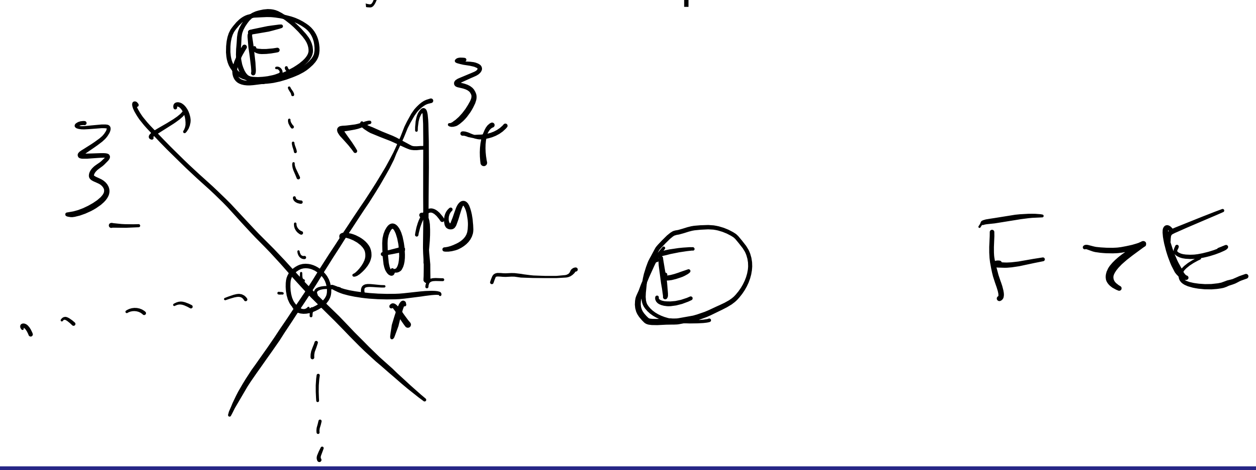
Suppose X generates an Anosov 3-flow. Then, $X \subset \xi_- \pitchfork \xi_+$, where $\xi_\pm$ is a positive/negative contact structure.



We call (ξ_-, ξ_+) a (supporting) bi-contact structure.

Part III: Mitsumatsu's construction and the Liouville geometry of Anosov flows

- This bi-contact condition has dynamical interpretation!



Definition

X is projectively Anosov, if it preserves a continuous splitting $TM/\langle X\rangle = E \oplus F$, such that $(C > 0)$

$$||X_*^t(v)||/||X_*^t(u)|| \geq e^{Ct}||v||/||u||$$

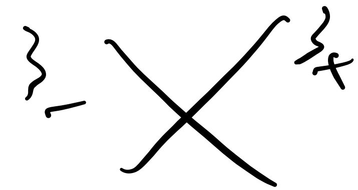
for any $v \in F$ and $u \in E$.

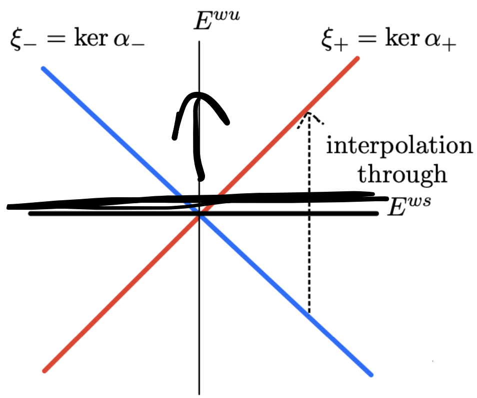
Mitsumatsu, Eliashberg-Thurston 95

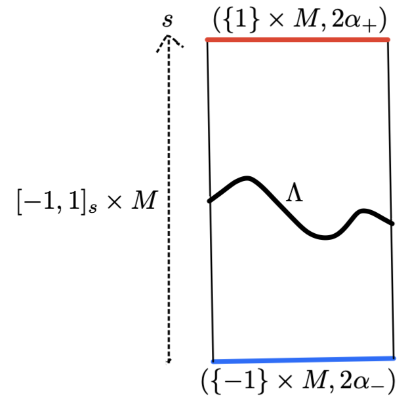
$$X \text{ projectively Anosov } \iff X \subset \xi_- \pitchfork \xi_+.$$

Part III: Symplectic geometry of Anosov flows and Mitsumatsu's construction

- Consider $\alpha = (1-s)\alpha_- + (1+s)\alpha_+$ on $W = [-1,1]_s \times M$.







- Consider the graph $\Lambda := \{(s_x, x) | \ker[(1-s_x)\alpha_- + (1+s_x)\alpha_+] = E^{ws}\}$.
- The Liouville condition of α:

$$\frac{1}{2}\iota_X \iota_{\partial_s}(d\alpha \wedge d\alpha) = \mathcal{L}_X\alpha \wedge \mathcal{L}_{\partial_s}\alpha$$

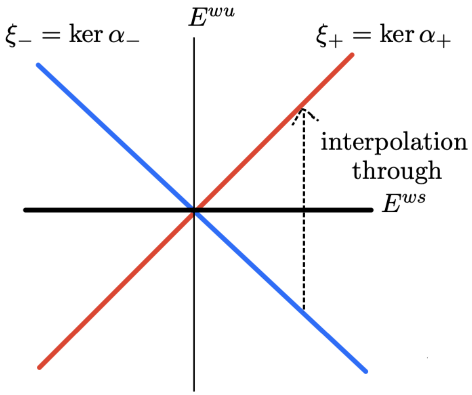
- At Λ:

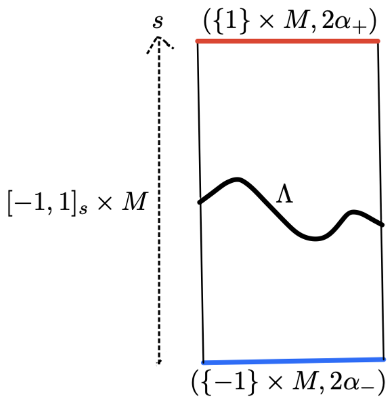
$$\frac{1}{2}\iota_X \iota_{\partial_s}(d\alpha \wedge d\alpha)|_\Lambda = \ldots = \mathcal{L}_X\alpha_u \wedge (\alpha_+ - \alpha_-),$$

where $\alpha_u = i_\Lambda^*\alpha$ and $(\alpha_+ - \alpha_-)$ is non-vanishing on E^s.

Part III: Symplectic geometry of Anosov flows and Mitsumatsu's construction

- Consider $\alpha = (1-s)\alpha_- + (1+s)\alpha_+$ on $W = [-1,1]_s \times M$.



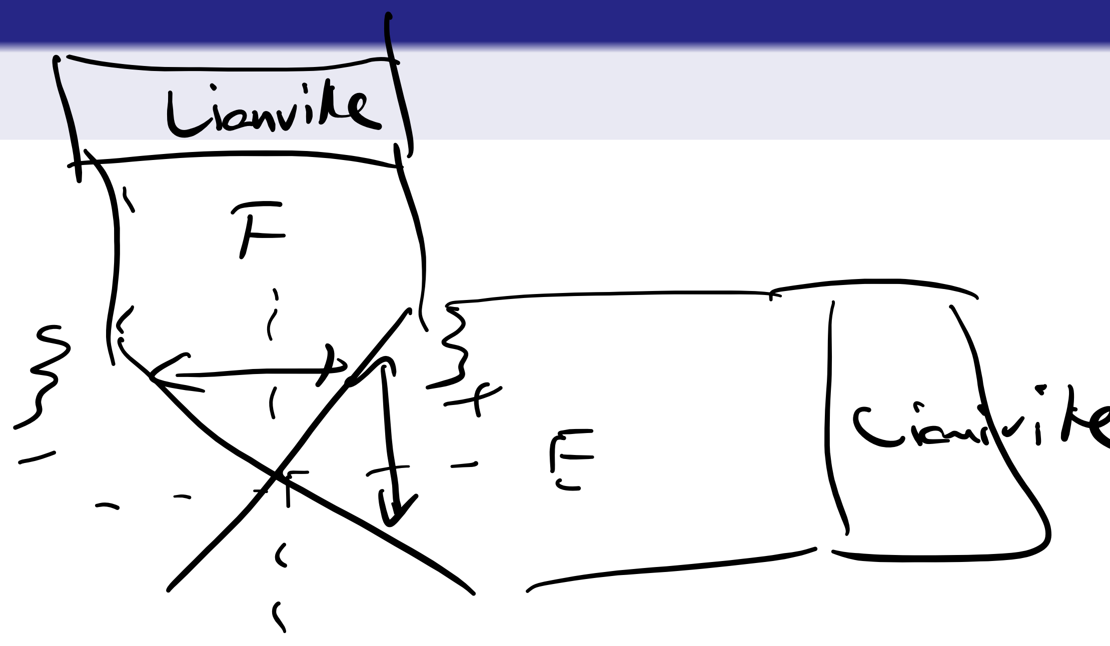


i.e. Liouville condition at $\Lambda \iff$ absolute expansion of the norm induced by α_u on E^{wu}.

- Conversely (Mitsumatsu 95), expanding α_u can be perturbed to contact forms with the Liouville property.
- Such pair $(\alpha_-, \alpha_+)_l$ is called a (linear) Liouville pair.

Theorem (H. 20)

X is Anosov, if and only if,

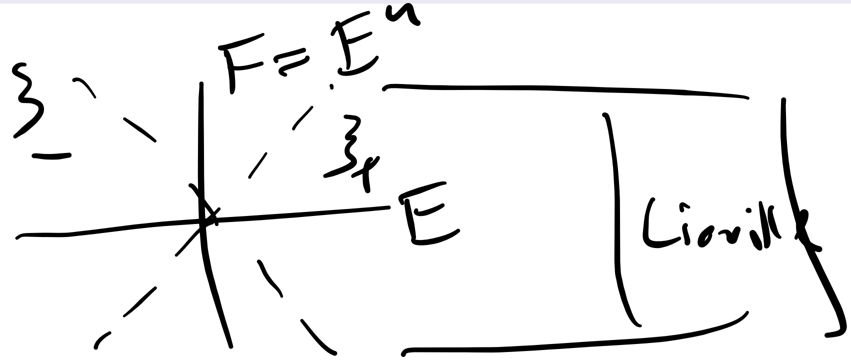


Non-singluat partially hyperbolic flows

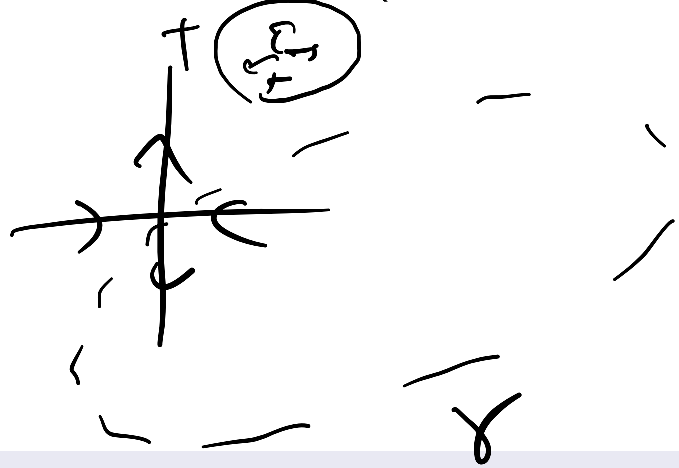
- What if we have only one Liouville condition?

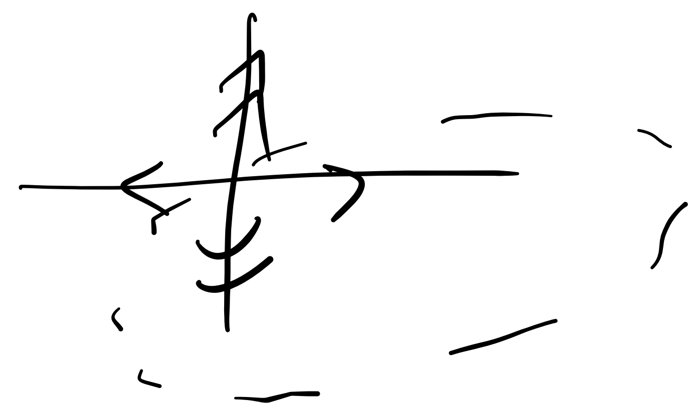
Theorem (H. 24)

X is partially hyperbolic, if and only if,

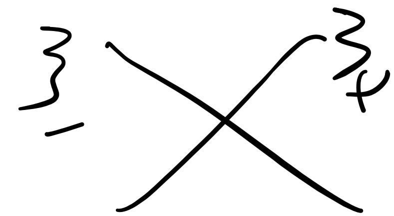


- Examples via DA (derived from Anosov) deformation.





Invariant plane fields are not necessarily C^1 anymore.



- A generalized (non-compact) framework:
- Everything is encoded in the interpolation!
- On $\mathbb{R}_s \times M$, consider the Liouville forms of the type

$$\alpha = \lambda_- \alpha_- + \lambda_+ \alpha_+,$$

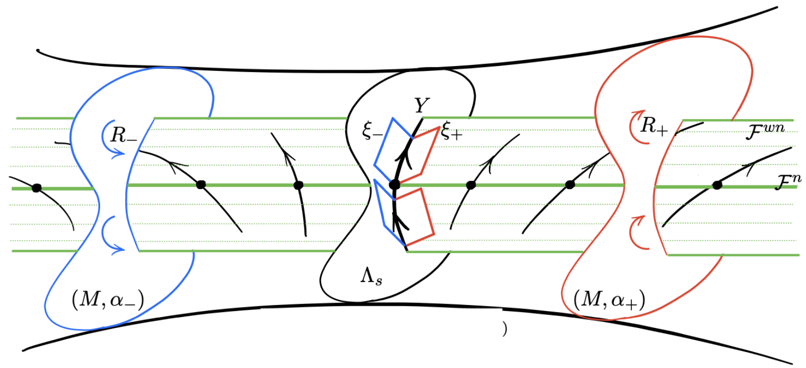
where $\lambda_\pm : \mathbb{R}_s \times M \to \mathbb{R}_{>0}$.

- Note $\alpha = \lambda_- [\alpha_- + \frac{\lambda_+}{\lambda_-} \alpha_+]$.
- Interpolation of plane fields $\Longleftrightarrow \partial_s \cdot \frac{\lambda_+}{\lambda_-} > 0$.
- + right ∞-condition: Liouville interpolation system (LIS): $(\alpha_-, \alpha_+)_{(\lambda_-,\lambda_+)}$
- e.g. exponential model: $\alpha = e^{-s}\alpha_- + e^{s}\alpha_+$.
- The space of such objects is homotopy equivalent to the space of Anosov flows (Massoni 22, H. 24).
- New Floer theoretic invariants by (Cieliebak-Lazarev-Massoni-Moreno 22).

Lemma

$E^{wn} := \langle \partial_s, X \rangle$ is tangent to a trivial exact Lagrangian foliation, i.e. $\alpha|_{E^{wn}} \equiv 0$ (called weak normal foliation), $Y \pitchfork \partial_s$ and $Y \subset E^{wn}$.

- X: Anosov, $(\alpha_-, \alpha_+)_{(\lambda_-, \lambda_+)}$: supporting LIS, Y is the Liouville v.f.



$\mathcal{L}_Y \alpha = \alpha$

Theorem (H. 24)

(1) $Skel(Y) = \{(s_x, x) \in \mathbb{R}_s \times M | \ker[\lambda_-(s_x,x)\alpha_- + \lambda_+(s_x,x)\alpha_+] = E^{ws}\}$, *implying that Skel*$(Y)$ *is* C^k *if and only if* E^{ws} *is* C^k. *In particular, Skel*(Y) *is always* C^{1+}.
(2) $Y|_{Skel(Y)}$ *is a synchronization of* X *(reparametrization unique up to smooth conjugacy).*
(3) Y *is normally repelling at Skel*(Y). *Therefore, Skel*(Y) *is* C^1*-persistent.*

Corollary

The Liouville v.f. is unique, up to C^1*-conjugacy, independent of all choices!*

Corollary

Dynamical rigidity uses Liouville geometry to translate the regularity of invariant plane fields to the regularity of graphs (much easier problem)!

Consequences:

- Recover Hasselblatt's bunching constants for Anosov flows (lower bounds for the regularity of invariant plane fields).
- Extend Hasselblatt's result to the partially hyperbolic case.
- Parametric version of Hasselblatt's lower bounds:
- In the Anosov case: the weak invariant plane fields C^1-depend on C^2-deformations of an Anosov flow!

Part V: Geometric rigidity and consequences

$\iota_Y d\alpha = \alpha$

- Suppose Y and α are Liouville v.f. and form induced from a LIS.
- Recall $Y \subset E^{wn} = \langle \partial_s, X \rangle$ and α.
- We can observe

$$Y = fX + g\partial_s \iff \alpha = f\mathcal{L}_X\alpha + g\mathcal{L}_{\partial_s}\alpha.$$

- The Moser technique works better than usual, if we fix X!!
- $\Longrightarrow$ We can recover the Liouville form strictly under deformation.

Theorem

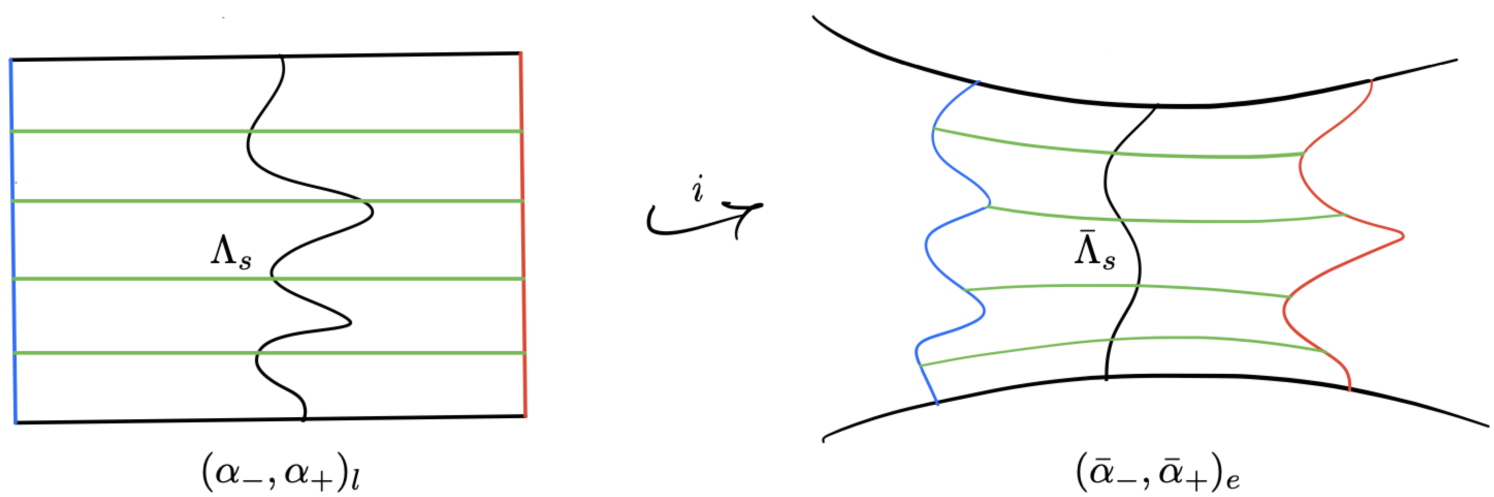
$$\left\{ \begin{array}{c} \textit{Positive reparametrization class of} \\ \textit{partially hyperbolic flows} \\ \textit{up to conjugacy} \end{array} \right\} \overset{\textit{1-to-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Liouville forms induced from some LIS on } \mathbb{R} \times M \\ \textit{up to strict Liouville equivalence} \end{array} \right\}.$$

Corollary

Fixing (reparametrization class of) X, the Liouville flow is unique up to smooth conjugacy.

Corollary

Any supporting linear Liouville pair can be strictly embedded into any supporting exponential pair.



Part VI: Skeleton persistence (a characterization)

C^1-persistence $\Longleftrightarrow$ normal hyp.

Theorem

Suppose (W^4, α) is Liouville manifold with an oriented C^1-persistent 3-dimensional skeleton Λ and $\ker \alpha \pitchfork \Lambda$. Then,
(1) the Liouville v.f. $Y|_\Lambda$ is a synchronized Anosov vector field;
(2) (W^4, α) is C^1-strictly Liouville equivalent to a Liouville form induced from a LIS supporting $Y|_\Lambda$ (Mitsumatsu's construction).

Corollary

$$\left\{ \begin{array}{c} \text{Positive reparametrization classes of} \\ \text{Anosov flows} \\ \text{up to conjugacy} \end{array} \right\} \overset{\text{1-to-1}}{\longleftrightarrow} \left\{ \begin{array}{c} \text{Liouville forms on } \mathbb{R} \times M \text{ with } C^1\text{-persistent} \\ \text{3-dimensional skeleton } \Lambda \text{ with } \ker \alpha \pitchfork T\Lambda \\ \text{up to strict Liouville equivalence} \end{array} \right\}.$$

Thank you!

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