Systolic inequalities for $\mathbb{S}^1$-invariant contact forms in dimension three

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- Systolic inequalities in metric geometry
- Relation with symplectic capacities of convex domains

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Theorem (Abbondandolo, Bramham, Hryniewicz, Salomão / Sağlam)

Let (M, ξ) a contact manifold, and $\epsilon > 0$. There exists a contact form α on M with $\ker \alpha = \xi$, $\mathrm{Vol}(\alpha) < \epsilon$ and $\mathrm{sys}(\alpha) \geq 1$.

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Theorem (Álvarez–Paiva, Balacheff / Abbondandolo, Benedetti)

A contact form is a local maximizer of $\alpha \mapsto \frac{\mathrm{sys}(\alpha)^{n+1}}{\mathrm{Vol}(\alpha)}$ if and only if it is Zoll.

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Framework:

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- Lutz classifies $\mathbb{S}^1$-invariant contact structures in dimension three:

  $\Omega(g, e)$ has infinitely many components, with a combinatorial description.

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Theorem A (V. 2024)

There is a constant $C > 0$ such that for all $g \in \mathbb{N}, e \neq 0$,

$$\forall \alpha \in \Omega(g, e), \quad \mathrm{sys}(\alpha)^2 \leq C\mathrm{Vol}(\alpha).$$

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- On trivial bundles: unclear.
- We get a sharp inequality under more restrictive assumptions.

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Theorem B (V. 2024)

- If $e < 0$ or $e \in \{1; 2\}$ then

$$\forall \alpha \in \Omega^{\text{tight}}(0, e), \qquad \text{sys}(\alpha)^2 \leq \frac{1}{|e|}\text{Vol}(\alpha)$$

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  and equality if and only if α is Zoll.

- if $e > 2$ then

$$\forall \alpha \in \Omega^{\text{tight}}(0, e), \qquad \text{sys}(\alpha)^2 < \frac{1}{2}\text{Vol}(\alpha)$$

  and $\frac{1}{2}$ is optimal.

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  The only e compatible with that property is 0.

Thank you!!