Quantum Steenrod operations, p-curvature, and representation theory

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$$\mathrm{St} : H^*(X;\Bbbk) \to H^*_{\mathbb{Z}/p}(X^p;\Bbbk) \to H^*_{\mathbb{Z}/p}(X;\Bbbk) \cong H^*(X;\Bbbk)[\![t,\theta]\!],$$

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Let (X, ω) be a non-negatively monotone symplectic manifold. Analogously to the construction of quantum product on $QH^*(X;R)$, Fukaya ('97) defined a *quantum deformation* of the Steenrod operations:

$$Q\mathrm{St} : QH^*(X;\Bbbk) \to QH^*(X;\Bbbk)[[t,\theta]],$$

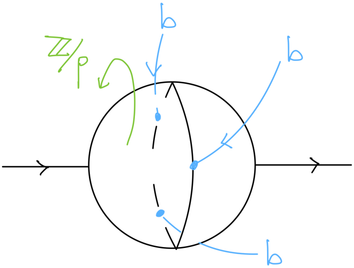
for $QH^*(X;\Bbbk) := H^*(X;\Bbbk)[[q^A : A \in H_2^{\omega \geq 0}(X;\mathbb{Z})]]$.

Quantum Steenrod operations and properties

It is more convenient to consider this as a set of operators $Q\Sigma_b$ for $b \in H^*(X;\Bbbk)$:

$$Q\Sigma_b : QH^*(X;\Bbbk)[\![t,\theta]\!] \to QH^*(X;\Bbbk)[\![t,\theta]\!] \tag{1}$$

defined from counts of parametrized $\mathbb{P}^1$ with $\mathbb{Z}/p$-symmetry, i.e. t, θ are identified with the equivariant parameters for discrete loop rotation.



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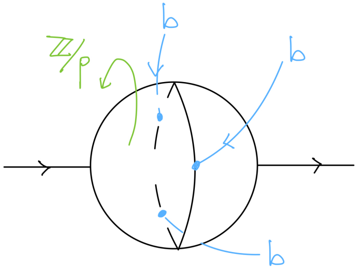

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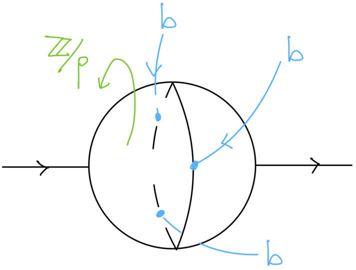

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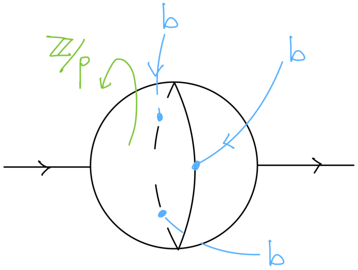

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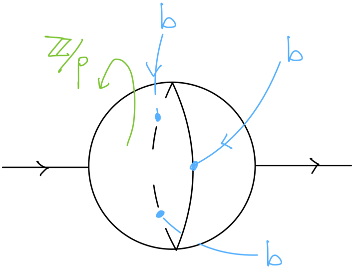

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- $Q\Sigma_b|_{t,\theta=0}(-) = \overbrace{b * \cdots * b}^{p} * (-)$, $\quad Q\Sigma_b \circ Q\Sigma_{b'} = (-1)^{\frac{p(p-1)}{2}|b||b'|} Q\Sigma_{b*b'}$.

Covariant constancy

A key property of $Q\Sigma_b$ is their compatibility with the *quantum connection*. These are operators indexed by $a \in H^2(X;\mathbb{Z})$ given by

$$\nabla_a = t\partial_a + a\, * : QH^*(X;\Bbbk)[\![t,\theta]\!] \to QH^*(X;\Bbbk)[\![t,\theta]\!]$$

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Theorem (Seidel–Wilkins '22)

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This is a *differential relation* satisfied by $Q\Sigma_b$.

Main question

Covariant constancy cannot *determine* $Q\Sigma_b$: it doesn't tell anything about coefficients of q^{pA}. That is, degrees supporting p-fold multiple covered curves are the most interesting part of $Q\Sigma_b$.

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Question

Can one compute $Q\Sigma_b$ in the range that supports p-fold multiple covers? More philosophically, what is the **role** *of quantum Steenrod operations in genus zero enumerative geometry?*

The answer arised through studying a rich class of examples coming from representation theory, known as *symplectic resolutions*.

Main result: QSt $= p$-curvature

We consider *symplectic resolutions* as targets X. For now, we just say that these are smooth non-compact Calabi–Yau manifolds equipped with Hamiltonian actions of a torus T.

Example

$X = T^*_{\mathrm{hol}}(\mathbb{P}^1)$ (with its Kähler form), together with two commuting S^1-actions, one induced by rotation of the base $\mathbb{P}^1$ and one given by rotation of the cotangent fibers.

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Theorem (L. '23)

Let X be a (conical) symplectic resolution with isolated T-fixed points and semisimple quantum cohomology. Then for $b \in H^2(X; \mathbb{Z})$,

$$Q\Sigma_b^T = (\nabla_b^T)^p - t^{p-1}\nabla_b^T. \tag{2}$$

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The right hand side is the p-*curvature* of the quantum connection ∇_b^T of X.

p-curvature

The p-curvature is a fundamental invariant one can define for any connection in characteristic p. Usual curvature $[\nabla_b, \nabla_{b'}] - \nabla_{[b,b']}$ measures the failure of ∇ to preserve the Lie bracket; p-curvature measures the failure of ∇ to preserve pth powers.

For the quantum connection, this should take the form

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This observation shows that p-curvature also satisfies the properties of $Q\Sigma_b$:

- $F_b|_{q^A=0}(-) = \mathrm{St}(b) \smile (-), \quad F_b|_{t,\theta=0}(-) = \overbrace{b * \cdots * b}^{p} * (-).$

Proof strategy

Result follows from these observations and a new compatibility relation:

Theorem (L. '24)

Operations $Q\Sigma_b^T$ *and* F_b^T *commute with the shift operators*

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The final theorem $Q\Sigma_b^T = F_b^T$ can be read in two ways:
(i) computation of quantum Steenrod operations in all degrees,
(ii) moduli description of p-curvature.

QSt $= p$-curvature as general philosophy

The result leads to the conjecture that quantum Steenrod $= p$-curvature is a more general phenomenon, which is subject of current investigation:

Theorem (Seidel–Pomerleano, forthcoming)

For X closed monotone, $Q\Sigma_{c_1(X)} = F_{c_1(X)}$.

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Theorem (Rezchikov, forthcoming)

For $X \subseteq \mathbb{P}^n$ a CY hypersurface, $Q\Sigma_H = F_H$ for $H \in H^2(X)$.

General case is related to the conjectural *Frobenius structure* on the p-adic quantum connection.

Symplectic resolutions and gauge theory

Let us further discuss the T-equivariant quantum Steenrod operations of symplectic resolutions, and their role in representation theory.

Definition

A *symplectic resolution* is a smooth holomorphic symplectic manifold (X, Ω) such that the affinization map $X \to \mathrm{Spec}\ H^0(X, \mathcal{O}_X)$ is a resolution of singularities (proper and birational).

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Example

Recall $T^*_{\mathrm{hol}}(\mathbb{P}^1)$; this is a blowup of the affine quadric cone (an A_1-singularity)

$$T^*\mathbb{P}^1 \to \{x^2 + yz = 0\} \subseteq \mathbb{C}^3 \cong \mathfrak{sl}_2^*.$$

These are often advertised as "Lie algebras of the 21st century."

3D mirror symmetry

A huge source of such symplectic resolutions come from *gauge theory*: fix a reductive group G and a complex G-representation N.
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Example

Braverman–Finkelberg–Nakajima construction $X_{\mathcal{C}} = H_\bullet^{G[\![z]\!]}(\mathcal{R}_{G,N})$ is the *Coulomb branch*.

One formulation of the 3D mirror symmetry program posits that the quantum connection of the Higgs branch can be identified with the D-module of twisted traces of the Coulomb branch.

3D mirror symmetry in positive characteristic

We extend the 3D mirror symmetry program to positive characteristic:

Theorem (Bai-L., forthcoming)

Let G be abelian (both $X_{\mathcal{H}}$, $X_{\mathcal{C}}$ are hypertoric varieties) and $\Bbbk = \mathbb{F}_p$. Then there is an isomorphism

$$\mathcal{D}^{\mathrm{tr}}(X_{\mathcal{C}}; \Bbbk) \cong QH^*_T(X_{\mathcal{H}}; \Bbbk)$$

compatible with the action of "Frobenius-constant" quantizations on $\mathcal{D}^{tr}(X_{\mathcal{C}}; \Bbbk)$ and the action of quantum Steenrod operators $Q\Sigma_b^T$ on $QH^*_T(X_{\mathcal{H}}; \Bbbk)$.

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The proof goes through quantum Steenrod $= p$-curvature on the Higgs side, and identifying the multiplication action of characteristic p quantizations on the Coulomb side with the p-curvature.

Thank you!

Frobenius-constant quantizations

Definition (Bezrukavnikov–Kaledin)

Suppose A is a quantization (i.e. $\hbar$-deformation) of a Poisson variety X in characteristic p. The data of an algebra map

$$s : \mathcal{O}(X)^{(1)} \to \mathcal{Z}(A)$$

such that $s(x) = x^p \pmod{\hbar}$ makes A a *Frobenius-constant quantization*.

Theorem (Lonergan, '17)

BFN Coulomb branches $X_{\mathcal{C}}$ admit a structure of a Frobenius-constant quantization, where $A_{\mathcal{C}}$ is given by $G[\![z]\!] \rtimes \mathbb{C}^\times$-equivariant BM-homology. The construction of the map s uses Steenrod operations!

D-module of twisted traces

Given Coulomb branch $X_{\mathcal{C}}$ in nice situations (in particular, for hypertoric varieties or Springer resolution), there is a universal deformation $\mathcal{X}_{\mathcal{C}}$ and its quantization $\mathcal{A}_{\mathcal{C}}$. Note that there is a Hamiltonian T-action on $\mathcal{X}_{\mathcal{C}}$ which induces a grading on $\mathcal{A}_C$ by the character lattice $X^\bullet(T)$.

Definition (Kamnitzer–McBreen–Proudfoot '18, Etingof–Stryker '19)

The $\mathcal{D}$-module of twisted traces is

$$\mathcal{D}^{\mathrm{tr}}(X_{\mathcal{C}}) = \mathcal{A}_0[q^\lambda]/\langle ab - q^\lambda ba : a \in \mathcal{A}_\lambda,\ b \in \mathcal{A}_{-\lambda}\rangle, \quad \lambda \in X^\bullet(T).$$

Given a Frobenius-constant quantization, $s(x) \in \mathcal{Z}(\mathcal{A}_0)$ for $x \in \mathcal{O}(X)_0$ acts on $\mathcal{D}^{\mathrm{tr}}(X_{\mathcal{C}})$ by multiplication.