Using h-cobordisms to detect non-trivial homotopy groups in spaces of Legendrian

Symplectic Geometry Zoominar

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December 18, 2024

Work in progress - joint with Y. Eliashberg

Talk overview

- The h-cobordism space $\mathcal{H}(M)$ and partition space $\mathrm{Par}(M)$.
- Type 1 maps: How to move a fold to define maps
$$\Phi : \mathcal{H}(M) \to \mathcal{L}eg(C) \qquad \text{(space of Legendrians)}.$$
- Type 2 maps: How to move a conormal to define maps
$$\Psi : \mathcal{H}(M) \to \mathcal{L}eg(C).$$
- Generating functions detect non-triviality of some type 1 maps.
- Gen. hyper surfaces detect non-triviality of some type 2 maps. (if time permits).
- Why they are not the same! (if time permits).

Theorem (Eliashberg, K)

In many cases both maps are non-trivial on homotopy groups, and independently so.

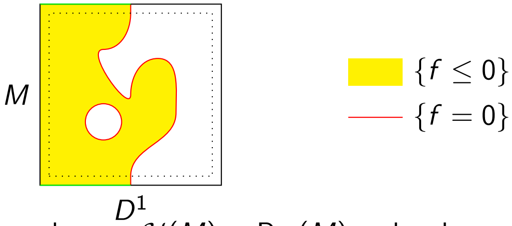
Definition of h-cobordism space and partitions

Let M be a compact manifold potentially with ∂ and corners.

Define the *partition* space $\mathrm{Par}(M)$ to consist of smooth functions (C^∞ topology) $f : M \times D^1 \to D^1$ with $D^1 = [-1, 1]$ such that

- $f(x, t) = t$ close to $\partial(M \times D^1)$.
- f has 0 as a regular value.

We consider $\{f \le 0\}$ a cobordism from M to $\{f = 0\}$. We consider the projection to D^1 the basepoint in $\mathrm{Par}(M)$.

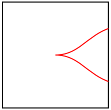


We define the subspace $\mathcal{H}(M) \subset \mathrm{Par}(M)$ to be those cobordisms where $\{f \le 0\}$ is an h-cobordism from M to $\{f = 0\}$.

Type 1 (moving a fold): Definition

Let $\Lambda \subset C$ be a Legendrian in a contact manifold. Let $F \subset \Lambda$ be a co-oriented codimension 1 smooth submanifold (possibly with ∂..).

Locally in C around a tubular neighborhood of $F \times D^1 \subset \Lambda$ we may identify $\Lambda \subset C$ with

$F \times D^1 \subset J^1(F \times D^1)$ Front: $F \times$  $z = \pm\frac{2}{3}\sqrt{t}^3$

t

So the front projection looks like a standard fold over $F \times \{0\}$ double covering $F \times (0, 1]$. We define $\mathrm{Par}(F) \xrightarrow{\Phi_F} \mathcal{L}eg(C)$ by

$$\Phi_F(f) = \{z = \pm\tfrac{2}{3}\sqrt{f(x,t)}^3\}$$

inside the neighborhood and extended constantly by Λ outside of the neighborhood. Embedded because Reeb chord length is $\frac{4}{3}\sqrt{f}^3$.

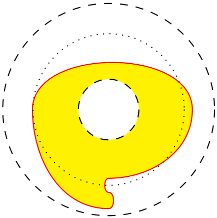
Type 1: Example of Whitney sphere

Consider the Whitney unknot: $\Lambda \subset J^1(\mathbb{R}^n) = \mathbb{R}^{2n+1}$. Its front projection is given by:

$$\left\{ z = \pm \tfrac{2}{3}\sqrt{1 - \|x\|^2}^{\,3} \right\}$$

With fold $F = S^{n-1}$. The image of $\Phi_F : \mathcal{H}(S^{n-1}) \to \mathcal{L}eg(\mathbb{R}^n)$ on $f \in \mathcal{H}(S^{n-1})$ thus has front projection

$$f \mapsto \left\{ z = \pm \tfrac{2}{3}\sqrt{f(\hat{x}, 1 - \|x\|^2)}^{\,3} \right\} \qquad (f(\hat{x}, t) = t \text{ for } t \leq -1)$$



corresponds to

Type 2 (moving a conormal): Definition

For a smooth manifold X the unit co-sphere $S^*X \xrightarrow{p} X$ is a contact manifold. Let $\Lambda \subset C$ be a Legendrian and let $c : S^*(M \times D^1) \to C$ be a codim 0 contact embedding such that:

- $\Lambda \cap \operatorname{Im} c = c(S^*_{M\times D^1} M \times \{0\})$

Here $S^*_X Y$ is the directed conormal of $Y \subset X$ when Y is a cooriented codim 1 submanifold.

Now we define the map $\operatorname{Par}(M) \xrightarrow{\Psi_M} \mathcal{L}eg(C)$ by

$$\Psi_M(f) = c(S^*_{M\times D^1}\{f = 0\})$$

inside the image of c extended by Λ elsewhere.

Type 2: Example: Zero-section

Consider $Q \subset Q \times \mathbb{R}$ (Q closed) and its oriented conormal $S^*_{Q\times\mathbb{R}}Q \subset \mathcal{L}eg(S^*(Q \times \mathbb{R}))$. This defines a type 2 map

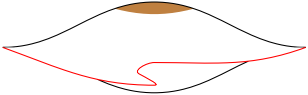
$$\Psi_Q : \mathcal{H}(Q) \to \mathcal{L}eg(S^*(Q \times \mathbb{R})).$$

Note that the projection of each produced Legendrian $\Psi_Q(f)$ to $Q \times \mathbb{R}$ is in fact embedded as it is $\{f = 0\} \subset Q \times \mathbb{R}$.

Combined type 1 and type 2 example

Consider that $J^1\mathbb{R}^n \subset S^*(\mathbb{R}^n \times \mathbb{R})$. Any Legendrian $\Lambda \subset J^1\mathbb{R}^n$ is mapped to a Legendrian whose projection to $\mathbb{R}^n \times \mathbb{R}$ is the front projection of Λ.

It follows that $\Lambda_f = \Phi_{S^{n-1}}(f) \subset J^1\mathbb{R}^n \subset S^*(\mathbb{R}^n \times \mathbb{R})$ from the first example has a small disc at the top



that agrees with the standard Whitney sphere. Hence we can apply a map of type 2 with $M = D^n$ on each Λ_f getting a combined map

$$\Phi_{S^{n-1}} \# \Psi_{D^n} : \mathcal{H}(S^{n-1}) \times \mathcal{H}(D^n) \to \mathcal{L}eg(S^*(Q \times \mathbb{R})).$$

Type 1: Lift to generating functions

A generating function (family) is a smooth $G : Q \times \mathbb{R}^k \to \mathbb{R}$ s.t.

- The set $\Sigma_G = \{dG_{|\mathbb{R}^k} = 0\}$ is transversely cut out.

It generates an immersed Legendrian $\Sigma_G \to J^1(Q)$ whose front is

$$(x, v) \mapsto (x, G(x, v)).$$

It is called quadratic at infinity if

- $F(x, v) = q(v)$ at infinity where q is a n.d.q.f.

It is called linear at infinity if

- $F(x, v) = -v_1$ at infinity.

Idea: Controlled "parametrized Morse theory" aka Cerf theory.
Example: The Whitney sphere $\Lambda \subset J^1\mathbb{R}^n$ is generated by

$$G(x, v) = \frac{1}{3}v^3 + (\|x\|^2 - 1)v.$$

which is "linear" at infinity ($k = 1$).

Type 1: Lift to generating functions

Assume we have a generating function $G : Q \times \mathbb{R} \to \mathbb{R}$, either quadratic or linear at ∞, generating a $\Lambda \subset J^1 Q$.

Assume also that our fold $F \subset \Lambda$ (over Q) is such that in some normal neighborhood of the projection $F \times D^1 \subset Q$ we have

$$G((x,t),v) = \frac{1}{3}v^3 + tv$$

with $(x,t) \in F \times D^1 \subset Q$ and (x,t,v) close to $F \subset \Lambda \cong \Sigma_G$. This is what generates the standard fold - so we can again simply define

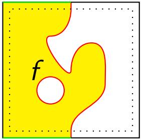
$$G_f((x,t),v) = \frac{1}{3}v^3 + f(x,t)v$$

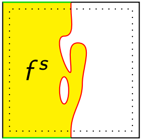
depending on $f \in \mathcal{H}(F)$ (with some bumping off). This lifts Φ_F to

$$\Phi_F^g : \mathcal{H}(F) \to \mathcal{L}eg^g(J^1 Q) \qquad \text{(Legendrians with such g.f.)}$$

Stabilizations of h-cobordisms

A positive stabilization $\sigma^+ : \mathrm{Par}(M) \to \mathrm{Par}(M \times D^1)$ is defined using the following process:

M 

$\mapsto$ 

$\mapsto$ Adding v^2 for $v \in D^1$ (the new variable), bumping of appropriately.

A specific formula:

$$(x, v, t) \mapsto \varphi(v)(\underbrace{s^{-1}f(x, st) - t}_{\text{small}}) + t + \underbrace{\psi(x, v, t)}_{0 \text{ close to boundary.}} v^2.$$

$$\varphi = 1 \text{ for } v \approx 0 \qquad \psi = 1 \text{ when } \varphi(v) > 0.$$

The homotopy type of $\{f \leq 0\}$ is unchanged. You may think of this as fattening the cobordism a bit (and making it standard close to the boundary). *E*.g. an Annulus turns into a solid torus.

Stabilizations

When f is a Morse function we can use Morse theory to give $\{f \leq 0\}/M$ a based CW structure with one non-trivial cell per critical point. Then $\sigma^+(f)$ has the same critical points with the same Morse indices and builds the same homotopy type.

However, there is also $\sigma^- : \mathrm{Par}(M) \to \mathrm{Par}(M \times D^1)$, which is called a negative stabilization. This is defined in the same way but using $-v^2$ instead of v^2. However, the homotopy type of $\{f \leq 0\}$ changes for this.

Considering the Morse theory it is relatively easy to see that the new homotopy type is the reduced suspension of $\{f \leq 0\}/M$ (the Morse indices are all increased by 1). Alt: positive stabilization of $\{f \geq 0\}$.

We define $\sigma = \sigma^+ \circ \sigma^- : \mathrm{Par}(M) \to \mathrm{Par}(M \times (D^1)^2)$ and

$$\mathcal{H}_\infty(M) \subset \mathrm{Par}_\infty(M) = \operatorname*{colim}_{k \to \infty} \mathrm{Par}(M \times (D^1)^{2k})$$

using these maps.

Stable range and computations

There are in fact many non-trivial homotopy groups in $\mathcal{H}_\infty(M)$. To utilize these the two following theorems are most relevant for us.

Theorem (Igusa)

Either stabilization $\mathcal{H}(M) \to \mathcal{H}(M \times D^1)$ is $\frac{\dim M - 7}{2}$ connected.

Let $i : N \subset M$ be a codimension 0 submanifold. Let $\mathcal{H}(N) \to \mathcal{H}(M)$ be the map that extends the function by the projection to D^1 (extension by "0"). This commutes with stabilizations and thus induces maps $i_* : \mathcal{H}_\infty(N) \to \mathcal{H}_\infty(M)$.

Theorem (Waldhausen)

The map i_* is $k-2$ connected if i is k connected.

We can for any smooth map $i : N \to M$ more generally lift to an embedding $\tilde{i} : N \to M \times D^{2k}$ and get an induced map

$$\mathcal{H}(N) \to \mathcal{H}_\infty(M). \quad \text{(stable range)}$$

Type 1: The difference function

For a generating function $G : Q \times \mathbb{R}^k \to \mathbb{R}$ we can construct the difference function:

$$DG : Q \times \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$$

by $DG(x, v, w) = G(x, v) - G(x, w)$ (bumped off). Its critical points are in 1-1 with Reeb orbits. Classically this defines generating function homology:

$$GH_*(G) = MC_*(DG_\varepsilon) \qquad \text{(Morse homology)}$$

where $\varepsilon > 0$ is very small and $g_\varepsilon = g_{|\{g \geq \varepsilon\}}$. However, there is much richer structure in realizing that $\{\varepsilon \leq DG \leq \varepsilon^{-1}\}$ is a cobordism! Its one end $M_k = \{DG = \varepsilon^{-1}\}$ does not depend on G (within the fixed family quadratic/linear), which makes it (almost) possible to define a map

$$\mathcal{L}eg^g(J^1(Q)) \to \mathrm{Par}(M_k).$$

This is not useful as an invariant of Legendrian isotopies as on needs to *stabilize* the generating functions.

A result for type 1

Stabilizing generating functions is similar and compatible. So we get a map

$$D_1 : \mathcal{L}eg^g_\infty(J^1Q) \to \mathrm{Par}_\infty(Q). \qquad ("\, Q \simeq M_\infty")$$

It is known that

$$\mathcal{L}eg^g_\infty(J^1Q) \to \mathcal{L}eg(J^1Q)$$

is a Serre fibration. We essentially prove the follow two statements (in some cases):

- The composition $\mathcal{H}(F) \xrightarrow{\Phi^g_F} \mathcal{L}eg^g_\infty(J^1Q) \xrightarrow{D_1} \mathrm{Par}_\infty(Q)$ is (essentially) the map induced by $i : F \to Q$.
- The composition from the fiber of the above fibration with DC into to $\mathrm{Par}_\infty(Q)$ is null homotopic.

This implies that Φ_F is highly non-trivial in stable range.

Type 2: Lift to generating hyper surfaces

A generating hyper surface is a function $G : Q \times \mathbb{R} \times \mathbb{R}^{2k} \to \mathbb{R}$ s.t.

- The surface $S_G = \{G = 0\}$ is transversely cut out.
- The set $\Sigma_G = S_G \cap \{dG_{|\mathbb{R}^{2k}} = 0\}$ is transversely cut out. Hence its dimension is $n = \dim Q$.

It generates an immersed Legendrian $\Sigma_S \to S^*(\mathbb{Q} \times \mathbb{R})$ given by symplectic reduction of the Legendrian given by the conormal $S^*_B S$.

Example: if $G : Q \times \mathbb{R}^{2k} \to \mathbb{R}$ is a g.f.q.i. then $(x, s, v) \mapsto s - G(x, v)$ is a g.h.s.q.i. it generates the same as G under the inclusion $J^1 Q \subset S^*(Q \times \mathbb{R})$.

It is called quadratic at infinity if

- $F(x, s, v) = s - q(v)$ at infinity where q is the std q.f.

It is called linear at infinity if

- $F(x, s, v) = s - v_1$ at infinity.

A result for type 2

Example: any hyper surface in $Q \times \mathbb{R}$ generates its own (directed) conormal. I.e. we have lift $\Psi_Q^S : \mathcal{H}(Q) \to \mathcal{L}eg^S(Q \times \mathbb{R})$.

There is a much easier map $D_2 : \mathcal{L}eg_\infty^S(S^*(Q \times \mathbb{R})) \to \mathcal{H}_\infty(Q)$.

Again there is a fibration:

$$\mathcal{L}eg_\infty^S(S^*(Q \times \mathbb{R})) \to \mathcal{L}eg(S^*(Q \times \mathbb{R})).$$

We essentially prove (use) the follow two statements:

- The composition $\mathcal{H}(Q) \xrightarrow{\Psi_Q^S} \mathcal{L}eg_\infty^S(J^1 Q) \xrightarrow{D_2} \mathcal{H}_\infty(Q)$ is the stabilization map.
- The composition from the fiber of the above fibration with DC into to $\mathcal{H}_\infty(Q)$ is *rationally* null homotopic (70%).

This implies that Ψ_Q is highly non-trivial in stable range.

Why they are different!

Consider again the inclusion

$$J^1\mathbb{R}^n \subset S^*(\mathbb{R}^n \times \mathbb{R}) \cong J^1 S^n.$$

This composition is isotopic to the inclusion $J^1\mathbb{R}^n \subset J^1 S^n$. Our detection of type 1 shows that including the example of the Whitney sphere into $J^1 S^n$ is often non-trivial.

Hence they are non-trivial in the middle term. Hence the combined map

$$\Phi \# \Psi : \mathcal{H}(S^{n-1}) \times \mathcal{H}(D^n) \to \mathcal{L}eg(S^*(\mathbb{R}^n \times \mathbb{R}))$$

is very non-trivial in the first factor.

However, our detection map for type 2 is actually zero on the part from Φ. (It is graphical over $Q \times \mathbb{R}^{2k}$)

This implies that the non-trivial images of Φ and Ψ are complementary!

Thanks

Thank you!