A deformation of the Chekanov–Eliashberg dga using annuli

Milica Đukić

Uppsala University

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Introduction

We consider Legendrian knots in $\mathbb{R}^3$ with the standard contact structure $\xi = \ker(dz - ydx)$, i.e. embeddings $\Lambda : \mathbb{S}^1 \to \mathbb{R}^3$ tangent to ξ.

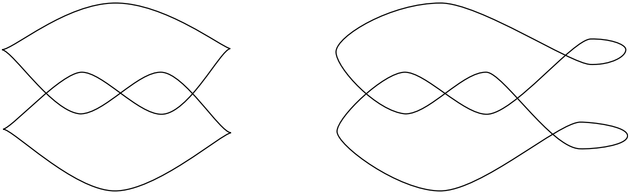


Figure: Front projection and Lagrangian projection of the right trefoil

Introduction

Legendrian isotopy between Legendrian knots Λ_0, Λ_1 is a path of Legendrian knots from Λ_0 to Λ_1.

The main question is how to distinguish Legendrian knots up to Legendrian isotopy.

Legendrian knot invariants:

- classical invariants (Thurston-Bennequin number, rotation number)
- SFT invariants (Chekanov–Eliashberg dga, rational SFT, ...)

Introduction—disk SFT invariants

Chekanov–Eliashberg dga
$\mathcal{A}^{CE}$ tensor algebra generated by $q_i, i \in \{1, \dots, n\}$ for $\gamma_1, \dots, \gamma_n$ Reeb chords on Λ, grading by the Maslov index;

differential: $d_{CE} : \mathcal{A}^{CE} \to \mathcal{A}^{CE}$ counts index zero pseudoholomorphic disks with one positive puncture in the symplectization $(\mathbb{R}^4, \mathbb{R} \times \Lambda)$.

Rational SFT

$\mathcal{A}^{Ng}$ vector space generated by cyclic words in $q_i, p_i, i \in \{1, \dots, n\}$;

differential: $d_{Ng} = d_J + d_{str} : \mathcal{A}^{Ng} \to \mathcal{A}^{Ng}$, where d_J counts index zero pseudoholomorphic disks with arbitrarily many positive punctures in the symplectization, d_{str} inserts trivial strips.

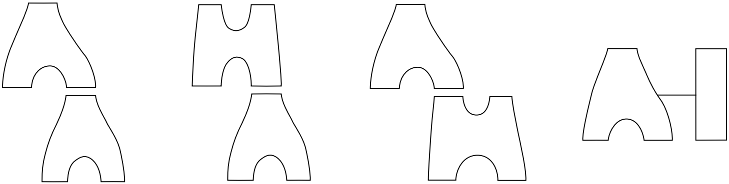


Figure: SFT breaking for disks

Main goal

We take the next step and define an invariant that also includes J-holomorphic annuli. We also describe a way to compute the invariant combinatorially from the Lagrangian projection.

Theorem

For every Legendrian knot Λ, there exists an algebra $\mathcal{A} = \mathcal{A}(\Lambda)$ with a second order dga structure $(d, \{\cdot, \cdot\})$ invariant under Legendrian isotopy up to II order stable tame equivalence. In particular, $(\mathcal{A}(\Lambda), d)$ is a chain complex such that for Legendrian isotopic knots Λ_0, Λ_1, we have

$$H_*(\mathcal{A}(\Lambda_0), d_0) \cong H_*(\mathcal{A}(\Lambda_1), d_1).$$

Moreover, the count of annuli (which is a part of the differential) can be replaced with zeros of (obstruction) sections with certain properties, which gives us a combinatorial way to compute the invariant from the Lagrangian projection of the knot.

Chain complex

Chain complex $(\mathcal{A}(\Lambda), d)$:

$\mathcal{A}(\Lambda) = \widetilde{\mathcal{A}} \oplus \hbar(\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{cyc}})$

$\widetilde{\mathcal{A}} = \mathcal{A}^{CE}$ tensor algebra (over $\mathbb{Q}$) generated by $t^{\pm}, q_i, i \in \{1, \ldots, n\}$ for $\gamma_i, i \in \{1, \ldots, n\}$ Reeb chords on Λ, with relation $t^+t^- = t^-t^+ = 1$; grading by the Maslov index, $|\hbar| = -1$;

The differential $d = d_{\mathbb{D}} + d_A + d_s : \mathcal{A}(\Lambda) \to \mathcal{A}(\Lambda)$ consists of three parts: $d_{\mathbb{D}}$ counts disks, d_A counts annuli, and d_s a string topological part.

Remarks:
- Taking $\hbar = 0$ gives us the standard Chekanov–Eliashberg dga;
- We can see elements in $\mathcal{A}(\Lambda)$ as strings (elements in $\widetilde{\mathcal{A}}$) and pairs of strings (elements in $\hbar(\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{cyc}})$) on $\mathbb{R} \times \Lambda$ with negative punctures at the corresponding chords (up to homotopy relative ends).

Algebraic structure—second order dga

$$\mathcal{A} = \mathcal{A}(\Lambda) = \widetilde{\mathcal{A}} \oplus \hbar(\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{cyc}}), \quad \widetilde{\mathcal{A}} = \mathcal{T}\mathbb{Q}\langle q_i, t^{\pm}\rangle/(t^+t^- = t^-t^+ = 1)$$

Definition (second order differential graded algebra)

A *second-order differential graded algebra* structure $(\mathcal{A}, d, \{\cdot,\cdot\})$ on $\mathcal{A}$ consists of an antibracket $\{\cdot,\cdot\}$ on $\widetilde{\mathcal{A}}$ and a degree -1 linear map $d : \mathcal{A} \to \mathcal{A}$ such that (here $d_0 := \pi_{\widetilde{\mathcal{A}}} \circ d \circ \iota_{\widetilde{\mathcal{A}}}$)

$$\begin{aligned}
&d(vw) = d(v)w + (-1)^{|v|} v d(w) + \hbar\, \pi_{\mathrm{cyc}}\{\pi_{\widetilde{\mathcal{A}}} v, \pi_{\widetilde{\mathcal{A}}} w\},\\
&d(\hbar(v \otimes w)) = (-1)^{|w|+1}\hbar(d_0 v \otimes w) - \hbar(v \otimes d_0^{\mathrm{cyc}} w),\\
&(d_0 \otimes 1 + 1 \otimes d_0)\{v, w\} = \{d_0 v, w\} + (-1)^{|v|}\{v, d_0 w\} \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}},\\
&d^2 = 0.
\end{aligned}$$

A degree 0 bilinear map $\{\cdot,\cdot\} : \widetilde{\mathcal{A}} \times \widetilde{\mathcal{A}} \to \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}$ is called an *antibracket* if

$$\begin{aligned}
&\{v, w_1 w_2\} = \{v, w_1\} \cdot (w_2 \otimes 1) + (-1)^{|v||w_1|}(1 \otimes w_1) \cdot \{v, w_2\},\\
&\{v_1 v_2, w\} = (v_1 \otimes 1) \cdot \{v_2, w\} + (-1)^{|v_2||w|}\{v_1, w\} \cdot (1 \otimes v_2).
\end{aligned}$$

Differential—disk part

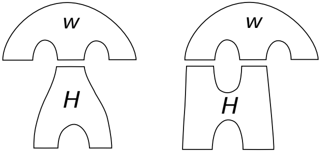
$d_{\mathbb{D}}(w) \in \mathcal{A}$ obtained by gluing positive punctures of an index zero pseudoholomorphic disk (with one or two positive punctures) to the string w in all possible ways.

More precisely,

$$d_{\mathbb{D}}(q_i) = \sum_{\substack{u_1 = p_i q_{i_1} \dots q_{i_k} \text{ } J\text{-hol.} \\ \text{disk}, \operatorname{ind}(u_1)=0}} \pm t^{a_0} q_{i_1} t^{a_1} \dots q_{i_k} t^{a_k},$$

$$\{q_i, q_j\}_{\mathbb{D}} = \sum_{\substack{u_2 = p_i q_{i_1} \dots q_{i_k} p_j q_{j_1} \dots q_{j_l} \\ J\text{-hol. disk}, \operatorname{ind}(u_2)=0}} \pm t^{a_0} q_{i_1} \dots q_{i_k} t^{a_k} \otimes t^{b_0} q_{j_1} \dots q_{j_l} t^{b_l}.$$

$t^{\pm}$ — intersections of the boundary with $\mathbb{R} \times \{T\}$, $T \in \Lambda$ a fixed base point.



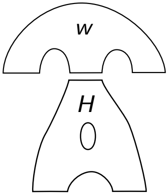
Differential—annulus part

$d_A(w) \in \mathcal{A}$ obtained by gluing the positive puncture of an index zero pseudoholomorphic annulus to w in all possible ways.

More precisely,

$$d_A(q_i) = \sum_{\substack{u_\hbar = p_i q_{i_1} \dots q_{i_k} \otimes q_{j_1} \dots q_{j_l} \\ J\text{-hol. annulus, } \mathrm{ind}(u_\hbar)=0}} t^{a_0} q_{i_1} \dots q_{i_k} t^{a_k} \otimes (q_{j_1} t^{b_1} \dots q_{j_l} t^{b_l})_{\mathrm{cyc}},$$

$$\{q_i, q_j\}_A = 0.$$



Differential—string topological part

$w : \mathbb{S}^1 \backslash \{t_1, \ldots, t_k\} \to \mathbb{R} \times \Lambda$ a (generic) string on $\mathbb{R} \times \Lambda$ with *generic asymptotic behavior*, together with a spanning disk $\widetilde{w}$ ($\partial\widetilde{w} = w$) holomorphic at the boundary;

$\mathcal{B}$ set of boundary self-intersections of $\widetilde{w}$, $\mathcal{C}$ set of interior intersections of $\widetilde{w}$ with the Lagrangian cylinder $\mathbb{R} \times \Lambda$;

$d_s(w) = \hbar \sum_{B \in \mathcal{B}} \nabla(w, B) + \hbar \sum_{C \in \mathcal{C}} \pm(w \otimes 1) \in \hbar(\widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{cyc}})$,
where $\nabla(w, B) \in \widetilde{\mathcal{A}} \otimes \widetilde{\mathcal{A}}^{\mathrm{cyc}}$ is the string pair obtained by resolving the string w at the intersection B — string coproduct.

$d_s(w)$ doesn't depend on the choice of the representative string and the choice of the spanning disk.

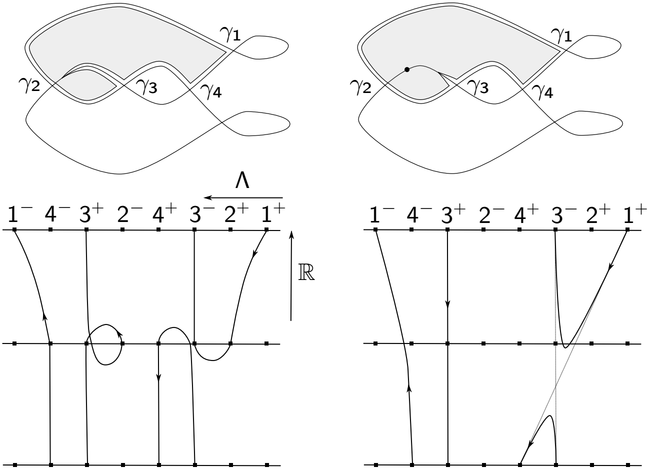


Figure: $d_s(p_1 p_3 q_3 q_4) = \pm\hbar(p_1 p_3 q_3 q_4 \otimes 1)$

More precisely,

$$
\begin{aligned}
d_s(q_i) &= \big(\mathrm{lk}(\Lambda, \mathrm{cap}_i) \pm \delta(i^-, i^+)\big)\,\hbar(q_i \otimes 1) - \delta(i^-, i^+)\hbar(1 \otimes q_i),\\
d_s(t^+) &= (\mathrm{tb}(\Lambda)+1)\,\hbar(t^+ \otimes 1),\\
d_s(t^-) &= -\,\mathrm{tb}(\Lambda)\hbar(t^- \otimes 1) - \hbar(1 \otimes t^-),
\end{aligned}
$$

where cap_i is the path from $i^- = \gamma_i(0)$ to $i^+ = \gamma_i(1)$ on Λ shifted off of Λ in a certain way, $\delta(i^-, i^+) \in \{0,1\}$ depending on the ordering of i^-, i^+ with resect to the base point $T \in \Lambda$.

Additionally,

$$
\begin{aligned}
d_s(q_i, q_j) =& \delta(j^+, i^+) q_j \otimes q_i + (-1)^{|q_i||q_j|}\delta(j^-, i^-) q_i \otimes q_j -\\
& -\delta(j^+, i^-) q_i q_j \otimes 1 - (-1)^{|q_i||q_j|}\delta(j^-, i^+) 1 \otimes q_j q_i, \ i \neq j\\
d_s(q_i, q_i) =& -\delta(i^+, i^-) q_i q_i \otimes 1 - (-1)^{|q_i|}\delta(i^-, i^+) 1 \otimes q_i q_i + \delta(i) q_i \otimes q_i,\\
d_s(q_i, t^+) =& \{q_i, t^+\}_d = t^+ \otimes q_i - q_i t^+ \otimes 1,\\
d_s(t^+, q_i) =& \{t^+, q_i\}_d = -t^+ q_i \otimes 1 + t^+ \otimes q_i \ldots
\end{aligned}
$$

$d^2 = 0$

$d = d_{\mathbb{D}} + d_A + d_s : \mathcal{A} \to \mathcal{A}$

Proposition

We have $d^2 = 0$.

Proof idea: Consider the boundary of the 1-dimensional moduli space of pseudoholomorphic disks and annuli on $\mathbb{R} \times \Lambda$.

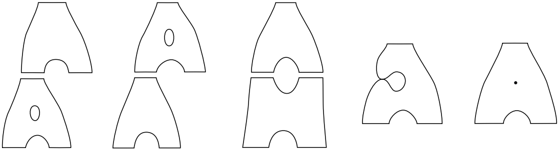


Figure: SFT breaking for annuli

□

Counting curves

Counting J-holomorphic curves is generally difficult.

We choose a special J given by

$$\begin{aligned} J\partial_x &= \partial_y + y\partial_r, \qquad & J\partial_y &= -\partial_x - y\partial_z, \\ J\partial_z &= -\partial_r, & J\partial_r &= \partial_z, \end{aligned} \tag{1}$$

(and take generic Λ).

It is well known that index zero J-holomorphic disks on $\mathbb{R} \times \Lambda$ are in bijection with immersed holomorphic polygons in $\mathbb{C}$ (Lagrangian projection) with boundary on $\pi_{xy}\Lambda$ and convex corners at the self-intersections of $\pi_{xy}(\Lambda)$.

Counting annuli

Lagrangian projections of index zero J-holomorphic annuli on $\mathbb{R} \times \Lambda$ belong to 1-parameter families of holomorphic annuli on $\pi_{xy}(\Lambda)$ with corners.

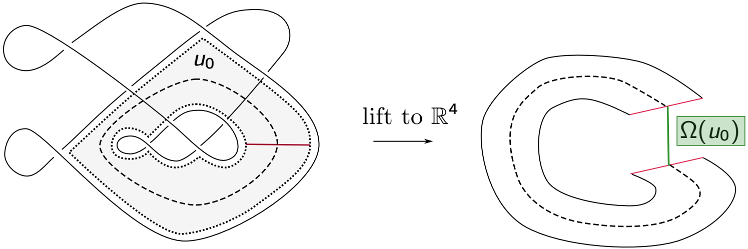


Figure: Rigid holomorphic annulus in the Lagrangian projection and its lift after cutting.

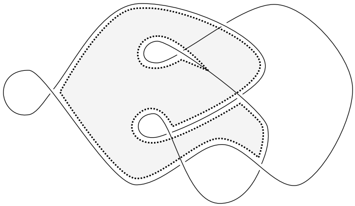


Figure: Projection of an index zero annulus on $\mathbb{R} \times \Lambda$.

Counting annuli

Denote by $\mathcal{M}_k^\pi$ the k-dimensional moduli space of holomorphic annuli on $\pi_{xy}\Lambda$.

Proposition

There exists a smooth section $\Omega : \mathcal{M}_k^\pi \to \mathbb{R}$ *such that*

1. $u \in \mathcal{M}_k^\pi$ *can be lifted to a* J*-holomorphic annulus on* $\mathbb{R} \times \Lambda$ *iff* $\Omega(u) = 0$,
2. $\Omega \pitchfork 0$, *for* Λ *generic,*
3. $$\lim_{\substack{u_n \to \widetilde{u} \\ \widetilde{u} \in \partial \mathcal{M}^\pi}} \Omega(u_n) = \begin{cases} \pm\infty, & \widetilde{u} \text{ non-split boundary} \\ \Omega(\widetilde{u}_o), & \begin{array}{l} \widetilde{u} \text{ split boundary,} \\ \widetilde{u}_o \text{ its annular part} \end{array} \end{cases}$$

Counting annuli

Example:

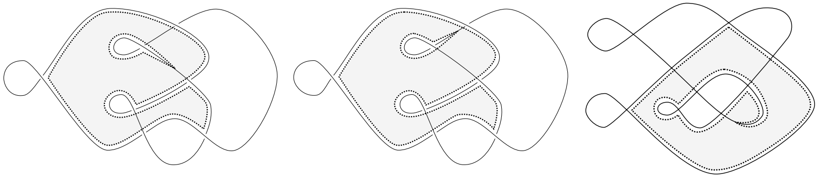


Figure: Non-split boundary with $\Omega \to +\infty$, non-split boundary with $\Omega \to -\infty$, split boundary of the moduli space $\mathcal{M}_{\mathbf{1}}^{\pi}$.

Counting annuli—combinatorial obstruction section

Conclusion: If we know $\Omega(\mathcal{M}_0^\pi)$, we can count index zero annuli on $\mathbb{R} \times \Lambda$.

Difficult to compute! However, the following result allows us to replace the count of annuli with a count of zeros of a section Ω^{vir} with similar properties (but with arbitrary $\Omega^{\mathrm{vir}}(\mathcal{M}_0^\pi)$).

Proposition

Let $\Omega^{vir} : \mathcal{M}_0^\pi \cup \mathcal{M}_1^\pi \to \mathbb{R}$ be a smooth section such that

1. $\Omega^{vir} \pitchfork 0, \;\; \Omega^{vir}(\mathcal{M}_0^\pi) \subset (\mathbb{R} \setminus \{0\})$,
2. $$\lim_{\substack{u_n \to \widetilde{u} \\ \widetilde{u} \in \partial \mathcal{M}_1^\pi}} \Omega^{vir}(u_n) = \begin{cases} \Omega(\widetilde{u}), & \widetilde{u} \text{ non-split boundary} \\ \Omega^{vir}(\widetilde{u}_o), & \begin{array}{l} \widetilde{u} \text{ split boundary,} \\ \widetilde{u}_o \in \mathcal{M}_0^\pi \text{ annular part} \end{array} \end{cases}$$

then the second order dga $(\mathcal{A}(\Lambda), d_{\Omega^{vir}})$ defined using the count of zeros of Ω^{vir} instead of $\Omega|_{\mathcal{M}_1^\pi}$ is isomorphic to $(\mathcal{A}(\Lambda), d)$.

The idea behind it is that there is a new type of Reidemeister move where the values of $\Omega|_{\mathcal{M}_0^\pi}$ can change sign.