Open enumerative mirror symmetry for lines in the mirror quintic

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The quintic and its mirror

- Let $X \subset \mathbb{CP}^4$ denote a smooth quintic hypersurface equipped with the pullback of the Fubini–Study form.
- The *Dwork family* of quintic threefolds is the family over $\mathbb{C}^*$ with fibers

$$X_z = \left\{ \sum_{j=1}^{5} x_j^5 - \frac{z^{1/5}}{5} \prod_{j=1}^{5} x_j = 0 \right\} \subset \mathbb{CP}^4 \; ; \; z \in \mathbb{C}^* \text{ and } |z| < 1$$

- There is an action of $(\mathbb{Z}/5)^3$ on X_z inherited from the action of $(\mathbb{Z}/5)^5$ on $\mathbb{CP}^4$.
- The *mirror quintic family* $X^\vee$ is the family of Calabi–Yau 3-folds with fibers given by crepant resolutions

$$X_z^\vee = \widetilde{X_z/(\mathbb{Z}/5)^3}$$

Enumerative mirror symmetry

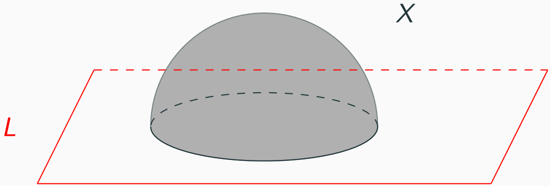
Theorem (Candelas–de la Ossa–Green–Parkes 1991, Givental 1995, Lian–Liu–Yau 1997)

$$\text{\textit{Genus zero Gromov–Witten invariants of } } X \quad \updownarrow \quad \text{\textit{Period integrals on }} X^\vee$$

Theorem (Gantra–Perutz–Sheridan 2015)
Assuming the existence of a **negative cyclic open-closed map** *on the Fukaya category, the predictions of Candelas–de la Ossa–Green–Parkes are implied by homological mirror symmetry for the quintic.*

Open Gromov–Witten invariants

- Let $L \subset X$ be a graded Lagrangian submanifold in a Calabi–Yau threefold. The (genus zero) *open Gromov–Witten invariants* count pseudoholomorphic disks in X with boundary on L.



- To obtain well-defined counts, L should be nullhomologous and should come equipped with a choice of bounding cochain and local system (cf. Fukaya, Solomon–Tukachinsky, H.).

Open Gromov–Witten invariants

Under mirror symmetry, we should have (cf. Witten '95, Ooguri–Vafa '00, Aganagic–Vafa '00):

$$\text{Open Gromov–Witten invariants of } X \\ \updownarrow \\ \text{Relative period integrals on } X^\vee$$

Example (Walcher 2007, Pandharipande–Solomon–Walcher 2008)
Let X be a quintic threefold defined over $\mathbb{R}$, and $X_{\mathbb{R}} \cong \mathbb{RP}^3 \subset X$ be the set of real points. The open Gromov–Witten invariants of $X_{\mathbb{R}}$ were predicted using mirror symmetry (W '07) and calculated using equivariant localization (PSW '08).

Main theorem

Theorem (H.)
Assume the existence of a negative cyclic open-closed map on the Fukaya category. There is an immersed Lagrangian submanifold $\widetilde{L}^5_{\mathrm{null}}$ *in the quintic threefold and a rank one* $\mathbb{C}$*-local system* ∇^{vG} *whose open Gromov–Witten potential is a formal power series of the form*

$$\Psi(\widetilde{L}^5_{\mathrm{null}}, \nabla^{\mathrm{vG}}) = \sum_{d=1}^{\infty} \widetilde{n}_d Q^d$$

where $\widetilde{n}_d \in \mathbb{Q}(\sqrt{-3})$ *are explicitly determined by relative period integrals computed by Walcher (2012) via homological mirror symmetry.*

Main theorem

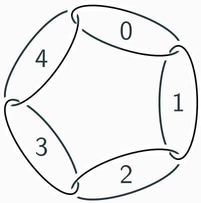
- The first few values of $\widetilde{n}_d$ are

$$\widetilde{n}_1 = 560000\sqrt{-3}$$
$$\widetilde{n}_2 = \frac{44592400000}{3}\sqrt{-3}$$
$$\widetilde{n}_3 = \frac{20063791178000000}{27}\sqrt{-3}$$
$$\widetilde{n}_4 = \frac{1320551611743490000000}{27}\sqrt{-3}$$

- The holonomy representation of ∇^{vG} is valued in $\mathbb{Q}(\sqrt{-3})$.

The minimally-twisted five-component chain link

- Let L' denote the complement of the minimally-twisted five-component chain link in S^3:



- There is a covering space $\widetilde{L'} \to L'$ with deck group $(\mathbb{Z}/5)^3$.
- The subgroup $\pi_1(\widetilde{L'}) \subset \pi_1(L')$ is generated by the meridians of the 0th link complement and by fifth powers of the meridians of the other components.
- $\widetilde{L'}$ is a cusped hyperbolic 3-manifold with *invariant trace field* $\mathbb{Q}(\sqrt{-3})$.

Invariant trace fields and fields of definition

The immersion $\widetilde{L}^5_{\mathrm{null}}$ is obtained from two copies of a Lagrangian immersion $\widetilde{L}^5_{\mathrm{im}} \to X$, whose domain is $\widetilde{L}^5_{\mathrm{im}} \cong \widetilde{L}' \cup_{\partial \widetilde{L}'} \widetilde{L}'$.

Conjecture (Jockers–Morrison–Walcher)
There is a hyperbolic Lagrangian submanifold of the quintic threefold with invariant trace field $\mathbb{Q}(\sqrt{-3})$.

Theorem (H.)
If L is a closed Lagrangian in a closed symplectic manifold (M, ω) with a bounding cochain b defined over a field $\mathbb{K}$ of characteristic 0, then (L, b) has $\mathbb{K}$-valued open Gromov–Witten invariants.

van Geemen lines

- The *van Geemen* lines in the Dwork quintic X_z are cut out by

$$x_1 + \omega x_2 + \omega^2 x_3 = 0$$
$$x_4 = \frac{a}{3}(x_1 + x_2 + x_3)$$
$$x_5 = \frac{b}{3}(x_1 + x_2 + x_3)$$

  where $a, b, \omega \in \mathbb{C}$ are constants satisfying $1 + \omega + \omega^2 = 0$, $a^5 + b^5 = 27$, and $ab = 6z^{1/5}$.
- (van Geemen) The orbit of C_z^ω under the action of $(\mathbb{Z}/5)^3 \times S_5$ contains $5000 = 125 \times 40$ lines, implying that X_z (and $X_z^\vee$) contains infinitely many lines.
- (Candelas–de la Ossa–van Geemen–van Straten) The families of non-isolated lines in X_z are curves of genus 626. The families of lines in the mirror quintic $X_z^\vee$ are curves of genus 6.

Relative period integrals

- The van Geemen lines descend to lines C_z^ω in the mirror quintic $X_z^\vee$ which do not depend on $a, b \in \mathbb{C}$.
- The power series $\Psi(\widetilde{L}^5_{\mathrm{null}}, \nabla^{\mathrm{vG}})$ is given, up to a change of variables $z \leftrightarrow Q$ and an additive constant in $\mathbb{C}$, by

$$2 \int_{\Gamma_z} \Omega_z$$

  where

  - Γ_z is a smooth singular 3-chain with boundary

$$\partial \Gamma_z = C_z^\omega - C_z^{\omega^2}$$

    and;
  - Ω_z is a volume form on $X_z^\vee$.

Mirrors to lines

Theorem (H.)
The Lagrangian immersion $\widetilde{L}^5_{\mathrm{im}} \to X$ supports a 1-dimensional family of objects in the Fukaya category which can be identified with a (punctured) genus 6 curve. Each of these objects is mirror to the pushforward of a rank 2 vector bundle on $\mathbb{P}^1$.

- Gives an *A*-model analogue of the results of Candelas–de la Ossa–van Geemen–van Straten.
- The supports of the mirror sheaves are determined by computing the Floer cohomology of $\widetilde{L}^5_{\mathrm{im}}$ with a Lagrangian torus.
- Computing $HF^*(\widetilde{L}^5_{\mathrm{im}})$ shows that it is mirror to the pushforward of a vector bundle.
- The computation of open Gromov–Witten invariants follows from this theorem, together with a comparison of weak proper Calabi–Yau structures on the Fukaya category and derived category (GPS 2015).

Thanks!