Classification of some open toric domains

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Outline

1. Statement of the main result
2. Barcode invariants of open domains
3. Computation of barcodes for convex toric domains
4. (Unnecessarily hard?) proof of the main result

Open domains

General question

Classify (families of examples of) bounded open sets in $\mathbb{R}^{2n}$ up to symplectomorphism.

There has been much study of when there exist symplectic embeddings of one open set into another, but seeming less study of when a symplectomorphism between open sets exists.

Detecting Reeb orbits on the boundary

Consider bounded open sets $X \subset \mathbb{R}^{2n}$ such that ∂X is smooth and transverse to the radial vector field, with nondegenerate Reeb flow. For such an open set, let $\mathcal{P}(\partial X)$ denote the set of Reeb orbits on ∂X, and if γ is a Reeb orbit, let $\mathcal{A}(\gamma) > 0$ and $\mathrm{CZ}(\gamma) \in \mathbb{Z}$ denote its symplectic action and Conley-Zehnder index.

Theorem (special case of Cieliebak-Floer-Hofer-Wysocki, 1996)

If X and X' are two such open sets, and if there is a symplectomorphism $X \stackrel{\simeq}{\to} X'$, then we have an equality of multisets

$$\{(\mathcal{A}(\gamma), \mathrm{CZ}(\gamma)) \mid \gamma \in \mathcal{P}(\partial X)\} = \{(\mathcal{A}(\gamma), \mathrm{CZ}(\gamma)) \mid \gamma \in \mathcal{P}(\partial X')\}.$$

The proof detects Reeb orbits on the boundary using symplectic homology in a narrow action window.

Open toric domains in $\mathbb{R}^4$

Let $\Omega \subset \mathbb{R}^2_{\geq 0}$ be a bounded open set. We define the open toric domain

$$X_\Omega = \{z \in \mathbb{C}^2 \mid \pi(|z_1|^2, |z_2|^2) \in \Omega\}.$$

Definition

An open convex toric domain is an open toric domain X_Ω as above such that the set

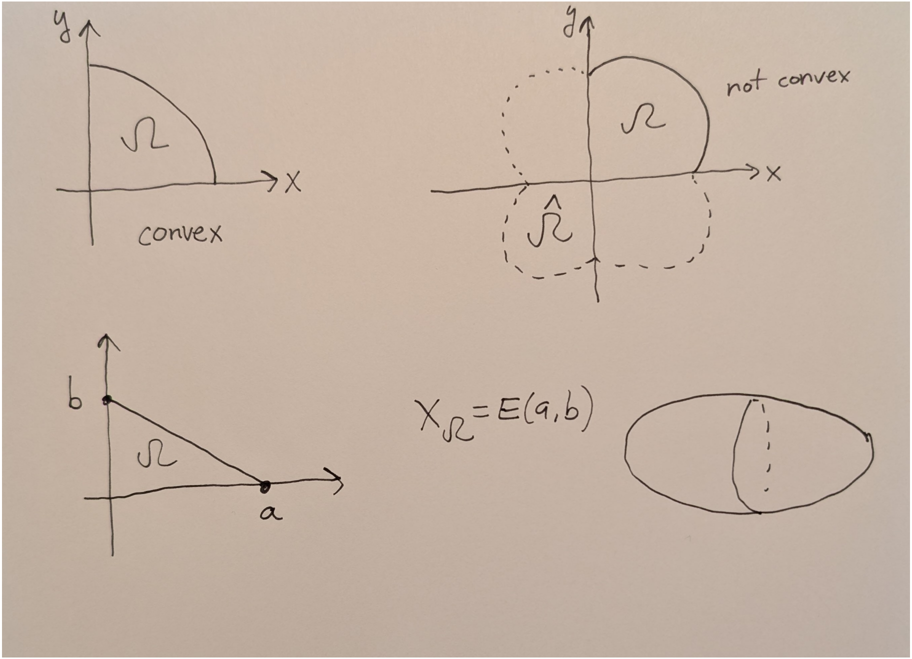
$$\widehat{\Omega} = \{(x, y) \in \mathbb{R}^2 \mid (|x|, |y|) \in \Omega\}$$

is convex.

Example

If $a, b > 0$ and Ω is the triangle $x \geq 0, y \geq 0, \frac{x}{a} + \frac{y}{b} < 1$, then X_Ω is the open ellipsoid

$$E(a, b) = \{z \in \mathbb{C}^2 \mid \frac{\pi|z_1|^2}{a} + \frac{\pi|z_2|^2}{b} < 1\}.$$


y
x
Ω
convex
y
x
Ω
not convex
Ω̂
b
Ω
a
X_Ω = E(a,b)

Theorem

Let $X_\Omega, X_{\Omega'} \subset \mathbb{R}^4$ be (generic) open convex toric domains. Suppose there exists a symplectomorphism $X_\Omega \xrightarrow{\simeq} X_{\Omega'}$. Then either $\Omega = \Omega'$, or $\Omega = \phi(\Omega')$, where $\phi(x, y) = (y, x)$.

Remark

Xiudi Tang and Jun Zhang have work in progress proving a similar result.

Remark

Moatty proved in his 1994 thesis that the above result holds if the curves $\partial\Omega$ and $\partial\Omega'$ are real analytic. More precisely, certain symplectic capacities of X_Ω (similar to the "Gutt-Hutchings capacities") determine the germ of $\partial\Omega$ where it intersects the line $y = x$.

Question

Does this result generalize to higher dimensions?

Symplectic action

Definition

Let $X_\Omega \subset \mathbb{R}^4$ be an open convex toric domain. If a, b are nonnegative integers, not both zero, define the action

$$\mathcal{A}_{a,b}(\Omega) = \max\{ax + by \mid (x, y) \in \overline{\Omega}\} > 0.$$

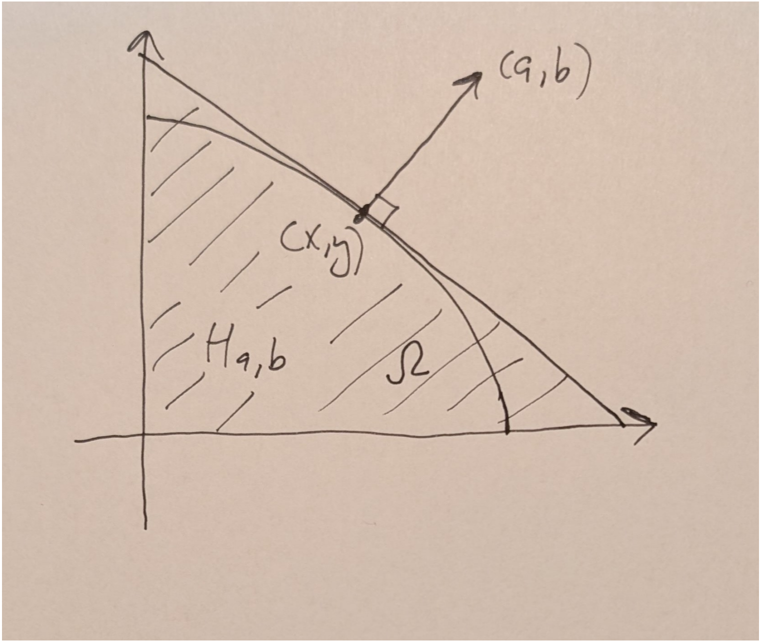
The maximum is realized by a point $(x, y) \in \partial\Omega$ for which an outward normal vector is parallel to (a, b). If we define

$$H_{a,b} = \{(x, y) \in \mathbb{R}^2_{\geq 0} \mid ax + by < \mathcal{A}_{a,b}(\Omega)\},$$

then by convexity,

$$\Omega = \bigcap_{(a,b)\in\mathbb{Z}^2_{\geq 0}\setminus\{(0,0)\}} H_{a,b}.$$

Thus the function $(a, b) \mapsto \mathcal{A}_{a,b}(\Omega)$ determines Ω.


(a,b)
(x,y)
$H_{a,b}$
Ω

Reeb orbits on the boundary of a smooth toric domain

Let X_Ω be an open convex toric domain. Let $\partial_+\Omega$ denote $\partial X \setminus$ axes, and suppose that $\partial_+\Omega$ is strictly convex and not perpendicular to the axes. Suppose that $\partial_+\Omega$ is smooth up to the boundary, so that ∂X_Ω is smooth. Then the simple Reeb orbits on ∂X_Ω are as follows:

- The circle $\pi|z_1|^2 = \mathcal{A}_{1,0}(\Omega), z_2 = 0$ is an elliptic Reeb orbit $e_{1,0}$ with $\mathcal{A}(e_{1,0}) = \mathcal{A}_{1,0}(\Omega)$ and $\mathrm{CZ}(e_{1,0}) = 3$.
- The circle $z_1 = 0, \pi|z_2|^2 = \mathcal{A}_{0,1}(\Omega)$ is an elliptic Reeb orbit $e_{0,1}$ with $\mathcal{A}(e_{0,1}) = \mathcal{A}_{0,1}(\Omega)$ and $\mathrm{CZ}(e_{0,1}) = 3$.
- If $(x, y) \in \partial_+\Omega$, and if an outward normal vector to Ω at (x, y) is parallel to (a, b) where a, b are relatively prime positive integers, then the torus $\pi|z_1|^2 = x, \pi|z_2|^2 = y$ is foliated by Reeb orbits with symplectic action $\mathcal{A}_{a,b}(\Omega)$. This circle of Reeb orbits can be perturbed to an elliptic orbit $e_{a,b}$ and a hyperbolic orbit $h_{a,b}$ with $\mathrm{CZ}(e_{a,b}) = (a + b) + 1$ and $\mathrm{CZ}(h_{a,b}) = 2(a + b)$.

- It follows from a modification of the Cieliebak-Floer-Hofer-Wysocki argument that if X_Ω is an open convex toric domain, then for each positive integer k, the multiset

$$\{\mathcal{A}_{a,b} \mid a+b=k\}$$

is a symplectomorphism invariant of open convex toric domains.
- The genericity condition we need in the main theorem is that the numbers in the above multiset are distinct.
- To prove the classification result, we just need a symplectomorphism-invariant way to identify which number in this set corresponds to a given pair (a,b).
- We remark that for a given k, the capacity k^{th} GH-capacity

$$c_k^{\mathrm{GH}}(X_\Omega) = \min\{\mathcal{A}_{a,b} \mid a+b=k\}.$$

Foundations

Cylindrical contact homology

Let $X \subset \mathbb{R}^4$ be an open domain with smooth star-shaped boundary on which the Reeb flow is nondegenerate.

- Cylindrical contact homology is the homology of a chain complex over $\mathbb{Q}$ generated by "good" Reeb orbits in ∂X, with grading $\mathrm{CZ} - 1$. (Originally defined by Eliashberg-Givental-Hofer.)
- The differential ∂ counts J-holomorphic cylinders in $\mathbb{R} \times \partial X$ interpolating between good Reeb orbits, with appropriate signs and combinatorial factors.
- In joint work with Jo Nelson, we proved that for dynamically convex contact three-manifolds (which includes nondegenerate perturbations of boundaries of convex toric domains), the differential ∂ is well-defined and satisfies $\partial^2 = 0$ for generic J. (A paper in progress will complete the construction of cobordism maps.)

Positive S^1-equivariant symplectic homology (à la Bourgeois-Oancea)

- Homology of a chain complex over $\mathbb{Q}$ with generators $U^k\check{\gamma}$ of grading $\mathrm{CZ}(\gamma) + 2k - 1$ and $U^k\hat{\gamma}$ of grading $\mathrm{CZ}(\gamma) + 2k$ for every Reeb orbit γ (good or bad) and every nonnegative integer k.
- Differential counts Hamiltonian Floer trajectories coupled to gradient flow lines of a Morse function on BS^1.
- By a spectral sequence argument, can be computed as the homology of a chain complex with one generator for each good Reeb orbit.
- Defined for star-shaped domains in all dimensions, and expected to agree with cylindrical contact homology in the dynamically convex case.

Filtered cylindrical contact homology

- For a dynamically convex domain $X \subset \mathbb{R}^4$, we have

$$CH_*(X) \simeq \begin{cases} \mathbb{Q}, & * = 2, 4, \ldots, \\ 0, & \text{else.} \end{cases}$$

- To get more interesting information, we need to use *filtered* cylindrical contact homology, keeping track of actions of Reeb orbits.
- Can also do this with positive S^1-equivariant symplectic homology, but we will use cylindrical contact homology for simplicity.

- If $X \subset \mathbb{R}^4$ is star-shaped, smooth, nondegenerate, and dynamically convex, and $L < 0$, the filtered cylindrical contact homology of ∂X, denoted by $CH^{\leq L}(X)$, is the homology of the subcomplex generated by good Reeb orbits with action $\leq L$. (One can also substitute positive S^1-equivariant symplectic homology and drop the star-shaped hypothesis.)
- If $L \leq L'$, then inclusion of chain complexes induces a "persistence morphism" $CH^{L',L}(X) : CH^{\leq L}(X) \to CH^{\leq L'}(X)$.
- The $\mathbb{Q}$-vector spaces and $CH^{\leq L}(X)$ and the maps $CH^{L',L}(X)$ constitute a $\mathbb{Z}$-graded persistence module, which has an associated barcode.
- If $\varphi : X \to X'$ is a symplectic embedding with $\overline{\varphi(X)} \subset X'$, then there is a cobordism map, aka transfer morphism

$$CH(\phi) : CH(X') \longrightarrow CH(X)$$

which is a morphism of persistence modules.

We want to extend the above persistence module to a symplectorphism invariant of arbitrary open sets $X \subset \mathbb{R}^4$, assuming that they can be exhausted by (symplectomorphic images of) dynamically convex domains as above.


Making auxiliary choices and proving independence of choices
Taking an inverse limit
imgflip.com

The inverse limit construction

- If $X \subset \mathbb{R}^4$ is a bounded open set as above, we define

$$\mathring{CH}^L(X) = \varprojlim CH^L(W)$$

  where the inverse limit is over nondegenerate dynamically convex domains W together with a symplectic embedding $\varphi : W \to X$ with $\overline{\varphi(W)} \subset X$.
- An element of this inverse limit assigns to every pair (W, φ) an element of $CH^L(W)$, such that if (W', φ') is another such pair and if $\overline{\varphi'(W')} \subset \varphi(W)$, then the choices are compatible via the transfer morphism associated to $\varphi^{-1} \circ \varphi' : W' \to W$.
- There are maps $\mathring{CH}^{L',L}(X) : \mathring{CH}^L(X) \to \mathring{CH}^{L'}(X)$ and transfer morphisms as before.
- If X is nondegenerate and dynamically convex, then there is a canonical isomorphism $\mathring{CH}^L(X) = CH^L(X)$.

Symplectomorphism invariance

- The collection of $\mathbb{Q}$-vector spaces $\mathring{CH}^L(X)$, together with the maps

$$\mathring{CH}^{L',L}(X) : \mathring{CH}^L(X) \longrightarrow \mathring{CH}^{L'}(X),$$

  is not necessarily a persistence module, as finite dimensionality may fail.
- Nonetheless, the inverse limit construction enables a clean proof that this structure is a symplectomorphism invariant of open domains as above. (See the preprint arXiv:2402.07003 for details for the analogous construction using equivariant symplectic homology.)
- When X is a convex toric domain, we will see that we do in fact obtain a persistence module.

Cylindrical contact homology of a convex toric domain

If $X_\Omega \subset \mathbb{R}^4$ is a convex toric domain, we define a "model" chain complex $C_*^{model}(\Omega)$ over $\mathbb{Q}$ as follows.

- Generators: $e_{a,b}$ for nonnegative integers a, b, not both zero, and $h_{a,b}$ for positive integers a, b.
- Grading: $|e_{a,b}| = 2(a+b)$ and $|h_{a,b}| = 2(a+b) - 1$.
- Action: $\mathcal{A}(e_{a,b}) = \mathcal{A}(h_{a,b}) = \mathcal{A}_{a,b}(\Omega)$.
- Differential: $\partial e_{a,b} = 0$ and $\partial h_{a,b} = e_{a-1,b} + e_{a,b-1}$.

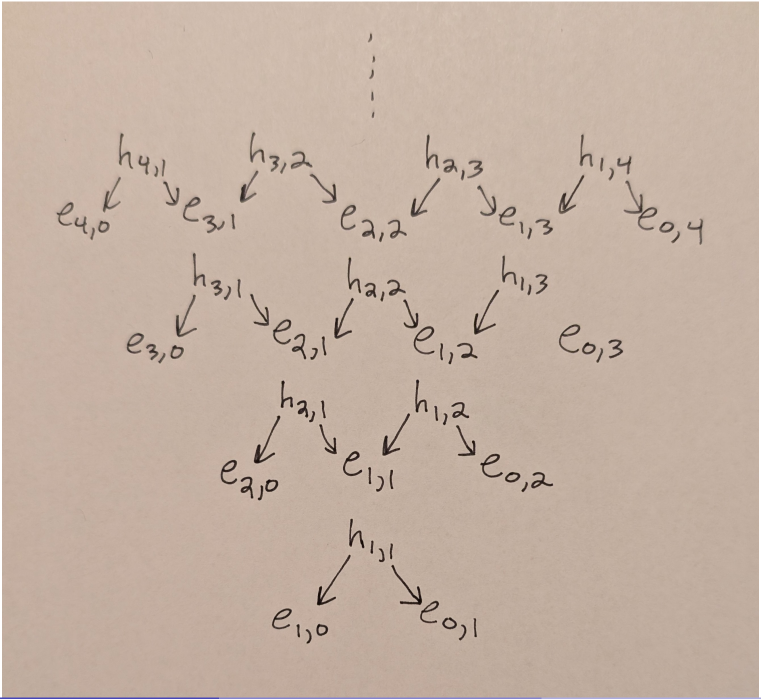
This has an associated persistence module where $H_*^{model,\leq L}(\Omega)$ is the homology of the subcomplex spanned by generators with $\mathcal{A} \leq L$.

Theorem

There is an isomorphism of persistence modules

$$\mathring{CH}_*(X_\Omega) \simeq H_*^{model}(\Omega).$$

Likewise for positive S^1-equivariant symplectic homology (consistent with a conjecture of Irie for more general toric domains).


$h_{4,1}$ $h_{3,2}$ $h_{2,3}$ $h_{1,4}$
$e_{4,0}$ $e_{3,1}$ $e_{2,2}$ $e_{1,3}$ $e_{0,4}$
$h_{3,1}$ $h_{2,2}$ $h_{1,3}$
$e_{3,0}$ $e_{2,1}$ $e_{1,2}$ $e_{0,3}$
$h_{2,1}$ $h_{1,2}$
$e_{2,0}$ $e_{1,1}$ $e_{0,2}$
$h_{1,1}$
$e_{1,0}$ $e_{0,1}$

Outline of the proof

- For a given grading k, we can perturb X_Ω to a domain with nondegenerate boundary Y whose Reeb orbits up to grading k are described as above, with actions slightly perturbed.
- We need to show that for a suitable almost complex structure J on $\mathbb{R} \times Y$, the differential is given as above.
- In fact, the differential is given as above for any generic J as needed to define the differential ∂.

Constraints from intersection positivity

- If $a, b > 0$, then we have the following linking numbers between Reeb orbits in Y:

$$\ell(e_{a,b}, e_{1,0}) = \ell(h_{a,b}, e_{1,0}) = b,$$
$$\ell(e_{a,b}, e_{0,1}) = \ell(h_{a,b}, e_{0,1}) = a,$$
$$\ell(e_{1,0}, e_{0,1}) = 1.$$

- If $\gamma, \gamma' \neq e_{1,0}$ and if $\langle \partial\gamma, \gamma' \rangle \neq 0$, then intersection positivity with $\mathbb{R} \times e_{1,0}$ implies that $\ell(\gamma, e_{1,0}) \geq \ell(\gamma', e_{1,0})$. In particular the b subscript of γ is greater than or equal to the b subscript of γ'.
- Likewise, if $\gamma, \gamma' \neq e_{0,1}$ and if $\langle \partial\gamma, \gamma' \rangle \neq 0$, then the a subscript of γ is greater than or equal to the a subscript of γ'.
- Together with the grading formula, it follows that the only possible differentials are when $a, b > 0$ from $e_{a,b}$ to $h_{a,b}$, or from $h_{a,b}$ to $e_{a-1,b}$ or $e_{a,b-1}$.

- By standard Morse-Bott theory, there are two holomorphic cylinders from $e_{a,b}$ to $h_{a,b}$, which cancel in the differential.
- So we just need to show that $\langle \partial h_{a,b}, e_{a-1,b}\rangle \neq 0$ and $\langle \partial h_{a,b}, e_{a,b-1}\rangle \neq 0$.
- We know that the homology in degree $2(a+b-1)$ is $\mathbb{Q}$, so it suffices to show that each $e_{a,b}$ represents a nonzero homology class. (Then there need to be enough differentials to show that the different $e_{a,b}$ for the same value of $a+b$ are homologous.)
- A similar intersection positivity argument using cobordism maps shows that ∂ does not depend on J or Ω.
- Given a, b we can find Ω such that $\mathcal{A}_{a,b}(\Omega) < \mathcal{A}_{a',b'}(\Omega)$ when $a+b = a'+b'$ and $a \neq a'$.
- If $e_{a,b}$ were nullhomologous, then the "k^{th} CH capacity" of X_Ω would be too big.

Remark

It might be possible to directly construct holomorphic cylinders from $h_{a,b}$ to $e_{a-1,b}$ and $e_{a,b-1}$ using "tropical" methods, cf. Taubes.

Computation of the barcode

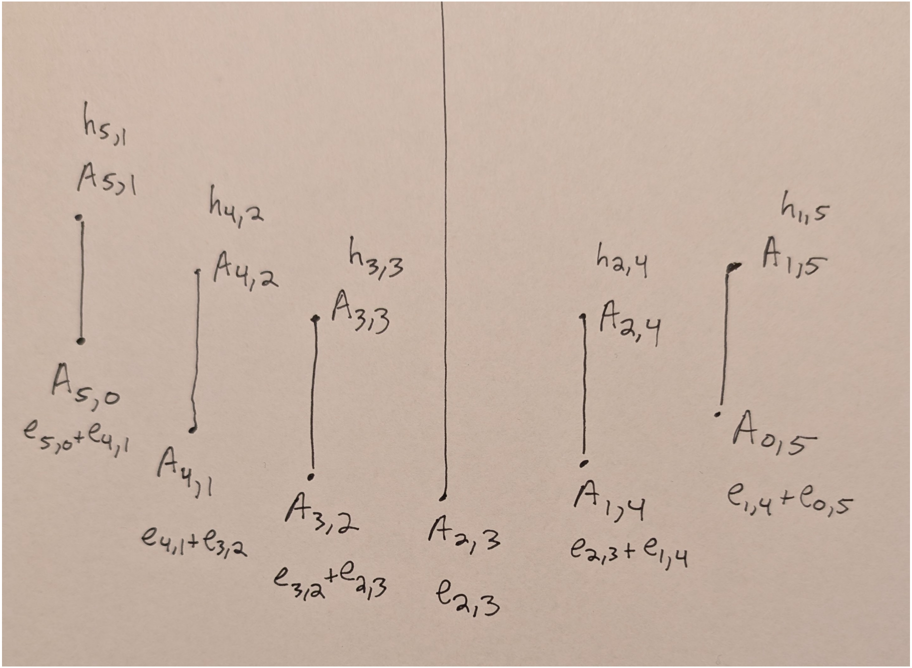
Now that we know the differential, how do we compute the barcode?

Lemma

Let $X_\Omega \subset \mathbb{R}^4$ be a convex toric domain. Then for a fixed positive integer k, the function $\{0, \ldots, k\} \to \mathbb{R}$ sending $a \mapsto \mathcal{A}_{a,k-a}(\Omega)$ is convex.

In particular, the function decreases until it reaches its minimum, then increases. Consequently, the part of the barcode between grading $2k+1$ and $2k$ is described as follows:

- There is an infinite bar with endpoint $\mathcal{A}_{a,b}(\Omega)$ where $\mathcal{A}_{a,b}(\Omega)$ is miminal as on the previous slide.
- There are k finite bars pairing up the numbers $\mathcal{A}_{1,k}(\Omega), \ldots, \mathcal{A}_{1,k}(\Omega)$ with the numbers $\mathcal{A}_{k,0}(\Omega), \ldots, \mathcal{A}_{0,k}(\Omega)$ (with $\mathcal{A}_{a,b}(\Omega)$ above omitted) respectively *in order*.



$h_{5,1}$
$A_{5,1}$
$A_{5,0}$
$e_{5,0}+e_{4,1}$
$h_{4,2}$
$A_{4,2}$
$A_{4,1}$
$e_{4,1}+e_{3,2}$
$h_{3,3}$
$A_{3,3}$
$A_{3,2}$
$e_{3,2}+e_{2,3}$
$A_{2,3}$
$e_{2,3}$
$h_{2,4}$
$A_{2,4}$
$A_{1,4}$
$e_{2,3}+e_{1,4}$
$h_{1,5}$
$A_{1,5}$
$A_{0,5}$
$e_{1,4}+e_{0,5}$

Proof of the classification theorem

- Let $X_\Omega \subset \mathbb{R}^4$ be a generic open convex toric domain. We need to show that symplectic invariants of X_Ω recover the function sending $(a, b) \mapsto \mathcal{A}_{a,b}(\Omega)$, up to a global switching of (a, b) with (b, a).
- We know the unordered pair of numbers $\{\mathcal{A}_{1,0}(\Omega), \mathcal{A}_{0,1}(\Omega)\}$, as these are the only grading 2 endpoints of bars in the barcode $\mathring{C}H(X_\Omega)$. So it is enough to show that the ordered pair of numbers $(\mathcal{A}_{1,0}(\Omega), \mathcal{A}_{0,1}(\Omega))$ together with symplectic invariants of X_Ω determine the function sending $(a, b) \mapsto \mathcal{A}_{a,b}(\Omega)$.
- We use induction on $k = a + b$.
- Suppose that $k \geq 1$ and that we know the function sending (a, b) with $a + b = k$ to $\mathcal{A}_{a,b}(\Omega)$ for $a + b = k$. Recall our genericity hypothesis that this function is injective.
- The barcode pairs up these values (aside from the minimum) with the numbers $\mathcal{A}_{a,b}(\Omega)$ with $a, b > 0$ and $a + b = k + 1$, in order.
- Thus we know the function sending (a, b) with $a + b = k + 1$ and $a, b > 0$ to $\mathcal{A}_{a,b}(\Omega)$.
- Also $\mathcal{A}_{k+1,0}(\Omega) = (k + 1)\mathcal{A}_{1,0}(\Omega)$ and $\mathcal{A}_{0,k+1}(\Omega) = (k + 1)\mathcal{A}_{0,1}(\Omega)$.

Further questions

- Can one remove the genericity hypothesis in the main theorem?
- Can one prove a similar classification for arbitrary (generic) open star-shaped toric domains in $\mathbb{R}^4$?
- Does one learn anything more from the ECH barcode?
- Can one classify (generic) open (convex or concave) toric domains in $\mathbb{R}^{2n}$?