Le Laboratoire de Mathématiques Jean Leray

Exact Lagrangians in cotangent bundles with locally conformally symplectic structure

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$\mathfrak{lcs}$ manifold :

$$\phi := \text{transition map}$$

$$\phi^* \omega_{\mathbb{R}^{2n}} = c\omega_{\mathbb{R}^{2n}}, \quad c > 0$$

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$\mathfrak{lcs}$ manifold :

$$(M, \omega, \beta)$$

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exact $\mathfrak{lcs}$ manifold :

(M, λ, β)

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↳ $\forall U, \ \beta = -dg_U$

$\lambda \in \Omega^1(M)$ such that,

for any small enough open set $U \subset M$,

$\exists g_U \in C^\infty(U) \ / \ d(e^{g_U}\lambda_{|U})$ is symplectic

Introduction

Definition (β-exact Lagrangian)

Let $L \subset M$ be a n-dimensional submanifold n and $i : L \to M$ be the inclusion. If for each open subset $U \subset M$ that is small enough, there is some $f_U \in C^\infty(i^{-1}(U))$ such that $e^{g_U \circ i} i^* \lambda = df_U$, then L is called a β-exact Lagrangian.

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Fact : L is a β-exact Lagrangian if and only if there is some $f \in C^\infty(L)$ such that $i^*\lambda = df - f\ i^*\beta =: d_\beta f$

Introduction

Example

1. Let M be a manifold, λ be the canonical Liouville form on T^*M, $\beta \in \Omega^1(M)$ be closed and $\pi : T^*M \to M$ be the canonical projection, then $(T^*M, \lambda, \pi^*\beta)$ is an exact $\mathfrak{lcs}$ manifold.

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2. Let (M, α) be a contact manifold, then $(M \times \mathbb{S}^1_\theta, \alpha, d\theta)$ is an exact $\mathfrak{lcs}$ manifold, for θ the coordinate on $\mathbb{S}^1_\theta$.

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$$\downarrow$$

$(\mathbb{S}^3 \times \mathbb{S}^1_\theta, \alpha, d\theta)$ is an exact $\mathfrak{lcs}$ manifold!

Introduction

Example

Let $\Lambda \subset (M, \alpha)$ be a Legendrian, then $\Lambda \times \mathbb{S}^1_\theta \subset M \times \mathbb{S}^1_\theta$ is a $d\theta$-exact Lagrangian submanifold.

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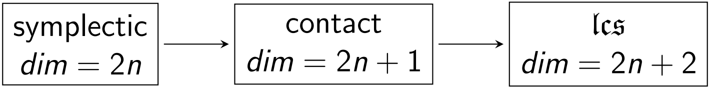

Introduction

Conventions

1. The exact Lagrangians of symplectic geometry will be called 0-exact Lagrangians.
2. M and L will always be closed connected manifolds of dimension n, β will be a closed 1-form on M and λ will be the canonical Liouville form on T^*M.
3. the various pullbacks of β will also be called β.

Nearby Lagrangians

Conjecture (nearby Lagrangians)

Let L be a 0-exact Lagrangian of $(T^*M, \lambda, 0)$, then L is the image of the 0-section M by an Hamiltonian isotopy.

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Weinstein : a Lagrangian in a symplectic manifold has a tubular neighborhood that "looks like" its cotangent bundle

P.S.: there is a $\mathfrak{lcs}$ version of Weinstein's neighborhood theorem. (see Otiman and Stanciu [2017])

Nearby Lagrangians

Conjecture

Let L be a 0-exact Lagrangian of $(T^*M, \lambda, 0)$, then L is the image of the 0-section M by an Hamiltonian isotopy.

Is it true ?

Let L be a β-exact Lagrangian of (T^*M, λ, β), then L is the image of the 0-section M by an Hamiltonian isotopy (of "$\mathfrak{lcs}$" type).

Nearby Lagrangians

Theorem (Abouzaid and Kragh [2018])

Let L be a 0-exact Lagrangian of $(T^*M, \lambda, 0)$ and $\pi : T^*M \to M$ be the canonical projection. Then $\pi_{|L} : L \to M$ is a simple homotopy equivalence.

Nearby Lagrangians

Corollary

Let L be a 0*-exact Lagrangian of* $(T^*M, \lambda, 0)$ *and* $\pi : T^*M \to M$ *be the canonical projection. Then*

$$(\pi_{|L})_* : H_*(L) \xrightarrow{\simeq} H_*(M).$$

Nearby Lagrangians

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Let L be a 0-exact Lagrangian of $(T^*M, \lambda, 0)$ and $\pi : T^*M \to M$ be the canonical projection. Then

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Is it true ?

Let L be a β-exact Lagrangian of (T^*M, λ, β) and $\pi : T^*M \to M$ be the canonical projection. Then

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Nearby Lagrangians

Corollary

Let L be a 0-exact Lagrangian of $(T^*M, \lambda, 0)$ and $\pi : T^*M \to M$ be the canonical projection. Then

$$(\pi_{|L})_* : H_*(L) \xrightarrow{\sim} H_*(M).$$

Is it true ?

Let L be a β-exact Lagrangian of (T^*M, λ, β) and $\pi : T^*M \to M$ be the canonical projection. Then

$$(\pi_{|L})_* : HN_*(L, i^*\beta) \xrightarrow{\sim} HN_*(M, \beta).$$

Note: HN_* stands for the Morse-Novikov homology

Nearby Lagrangians

Why Morse-Novikov ?

It has been successfully used to prove $\mathfrak{lcs}$ versions of classical symplectic theorems.

(e.g. see Chantraine and Murphy [2016] for the proof of an adaptation of the Laudenbach-Sikorav theorem)

Nearby Lagrangians

Proposition 1 (C.)

There is a manifold M and a β-exact Lagrangian L in (T^*M, λ, β) such that $\pi_{|L}$ induces neither an isomorphism of singular homologies, nor an isomorphism of Morse-Novikov homologies.

Nearby Lagrangians

Goals

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding and $\pi : T^*M \to M$ be the canonical projection.

1. Under which conditions is $(\pi_{|L})_* : H_*(L) \to H_*(M)$ an isomorphism?
2. Under which conditions is $(\pi_{|L})_* : HN_*(L, i^*\beta) \to HN_*(M, \beta)$ an isomorphism?

Liouville chords

Lemma (C.)

Let α be the canonical contact form on J^1M. Then

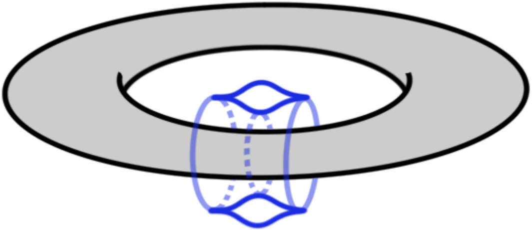
$$g : \left(T^*(M \times \mathbb{S}^1), \lambda_{M\times\mathbb{S}^1}, d\theta\right) \to \left(J^1M \times \mathbb{S}^1, \alpha, d\theta\right)$$
$$(q, p, \theta, z) \mapsto (q, -p, \theta, z)$$

is a "Liouville diffeomorphism (of $\mathfrak{lcs}$ type)"

Liouville chords

$$j: \quad \mathbb{T}^2 \to T^*\mathbb{T}^2 \simeq \mathbb{T}^2 \times \mathbb{R}^2$$

$$(\theta, \phi) \mapsto (\cos(\theta), \phi, -3\sin(\theta)\cos(\theta), -\sin(\theta)^3)$$



$$j^*\lambda = -3\sin(\theta)\cos(\theta)d\cos(\theta) - \sin(\theta)^3 d\phi$$

$$= d\sin(\theta)^3 - \sin(\theta)^3 d\phi$$

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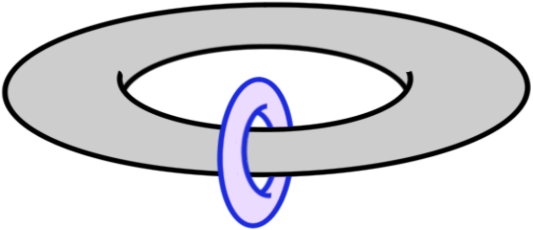

$$j^*\lambda = -3\sin(\theta)\cos(\theta) d\cos(\theta) - \sin(\theta)^3 d\phi$$
$$= d\sin(\theta)^3 - \sin(\theta)^3 d\phi$$

Introduction

Reeb chords
$dim = 2n - 1$

$\longrightarrow$

essential Liouville chords
$dim = 2n$

Liouville chords

Definition (essential Liouville chords)

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding and $f \in C^\infty(i(L), \mathbb{R}_{>0})$ such that $i^*(d_\beta f) = i^*\lambda$. Assume that β is not exact.

Given $t > 0$ and $(q, tp), (q, p) \in T_q^*M \cap i(L)$ such that

$$\frac{\ln(f(q,tp)) - \ln(f(q,p))}{\ln(t)} \geq 1,$$

the segment from (q, p) to (q, tp) will be called essential Liouville chord.

Liouville chords

Proposition 2 (C.)

Let $i : L \to (T^*M, \lambda, \beta)$ *be a* β*-exact Lagrangian embedding. If* β *is not exact, then there is exactly one* $f \in C^\infty(i(L))$ *such that* $i^*\lambda = i^*(d_\beta f)$.

Liouville chords

Proposition 2 (C.)

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding. If β is not exact, then there is exactly one $f \in C^\infty(i(L))$ such that $i^*\lambda = i^*(d_\beta f)$.

Theorem 3 (C.)

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding. If β is not exact, then the pullback of β to L is not exact.

Liouville chords

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Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding. If β is not exact, then there is exactly one $f \in C^\infty(i(L))$ such that $i^*\lambda = i^*(d_\beta f)$.

Theorem 3 (C.)

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding. If β is not exact, then the pullback of β to L is not exact.

Fact : If β is not exact, then $d_\beta h = 0 \iff h = 0$.

Liouville chords

Theorem 4 (C.)

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding such that $i^*\lambda = d_\beta f$ for some $f \in C^\infty(L, \mathbb{R}_{>0})$.
If L has no essential Liouville chord, then $\pi_{|L}$ is a simple homotopy equivalence.

Liouville chords

Theorem 4 (C.)

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding such that $i^*\lambda = d_\beta f$ for some $f \in C^\infty(L, \mathbb{R}_{>0})$.
If L has no essential Liouville chord, then $\pi_{|L}$ is a simple homotopy equivalence.

Let $L \subset (J^1_{>0}M = T^*M \times \mathbb{R}_{>0}, \alpha)$ be a Legendrian, with α the canonical contact form, and take the lift of L
$L \times \mathbb{S}^1 \subset (T^*(M \times \mathbb{S}^1), \lambda_{M\times\mathbb{S}^1}, d\theta)$. The projection of the essential Liouville chords in $T^*M \times (\mathbb{R}_{>0})_s$ are the Reeb chords of L for the contact form $\frac{\alpha}{s}$

Liouville chords

Goals

Let $i : L \to (T^*M, \lambda, \beta)$ be a β-exact Lagrangian embedding and $\pi : T^*M \to M$ be the canonical projection.

1. Under which conditions is $(\pi_{|L})_* : H_*(L) \to H_*(M)$ an isomorphism?

no essential Liouville chord

$\Longrightarrow$

$\pi_{|L}$ is an homology equivalence

References

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