E.H capacities from $SH^{S^1,+}$

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<u>Def</u>: A symp. capacity is a fct c which assigns a number to each symp mfd (X,ω) $\quad c(X,\omega)\in[0,+\infty]$ s.t.
1) If $(X,\omega)\hookrightarrow(X',\omega') \to c(X,\omega)\le c(X',\omega')$
2) If $r\in\mathbb{R}_{>0} \to c(X,r\omega)=r\,c(X,\omega)$

<u>Ex</u>: 0, $+\infty$, $(\mathrm{Vol})^{1/n}$

- $c_{Gr}(X,\omega) = \sup\{r \mid \exists\, (B^{2n}(r),\omega_0)\hookrightarrow(X,\omega)\}$
- c_{HZ}, c_{Vit}, c_L, $c_\square$
- in dim 4 $\quad c_k^{ECH}$, c_k^{EH}, c_k^{GH}

<u>Q</u>: Axiomatic characterization?

X convex $\subset\mathbb{R}^{2n}$

$$c_{HZ}(X)=c_{Vit}(X)=c_1^{EH}(X)=c_1^{GH}(X)=T_{min}$$

All those capacities are "ball-normalized"

i.e. $\quad c(B^{2n}(1),\omega_0)=\pi=c(\underset{D^2(1)\times\mathbb{R}^{2n-2}}{Z^{2n}(1)},\omega_0)$

<u>Conjecture</u>: All ball-normalized capacities agree on centrally symmetric

convex domains

This would imply Mahler conjecture

* $\lim_{k\to\infty} \frac{(c_k^{ECH}(X))^2}{k} = 4\,\mathrm{Vol}(X)$

* $\lim_{k\to\infty} \frac{c_k^{GH}(X)}{k} = c_L(X) = c_{\square}(X)$

<u>Main thm</u> For any star-shaped domain X in $\mathbb{R}^{2n}$ $\forall k \in \mathbb{N}_{>0}$

$$c_k^{EH}(X) = c_k^{GH}(X)$$

<u>c_k^{GH}</u>

Let $\lambda_0 = \frac{1}{2} \sum (x_i dy_i - y_i dx_i)$

∂X smooth hypersurface Σ

Then $\alpha := \lambda_0|_{\Sigma}$ is a contact form

i.e. $(\alpha \wedge (d\alpha)^{n-1} \neq 0)$

Reeb v.f. R_α $\quad \iota(R_\alpha)\, d\alpha = 0$ & $\alpha(R_\alpha) = 1$

A per. Reeb orbit is a map $\gamma : S^1 \longrightarrow \mathbb{R}^{2n}$ s.t.

$\dot{\gamma}(t) = (R_\alpha)_{\gamma(t)}$

Period of γ : T = symp action = $\int_\gamma \alpha$

γ is non-deg if 1 is not an eigenvalue of the Poincaré return map

If γ is non-deg $\rightsquigarrow$ CZ index
Assume all γ are non deg (true for generic X)

We can define $CH(X)$. It is the homology of a chain complex freely generated over $\mathbb{Q}$ by the "good" per. Reeb orbits.
It is $\mathbb{R}$-filtered by the symp. action
$CH^L(X)$

$$CH_*(X) = \begin{cases} \mathbb{Q} & \text{if } * \in n+1+2\mathbb{N} \\ 0 & \text{otherwise} \end{cases}$$

<u>Def</u>: $c_k^{GH}(X)$ is the smallest L s.t. a degree $n-1+2k$ generator of CH can be represented by a lin. combination of Reeb orbits all of which have action $\leq L$

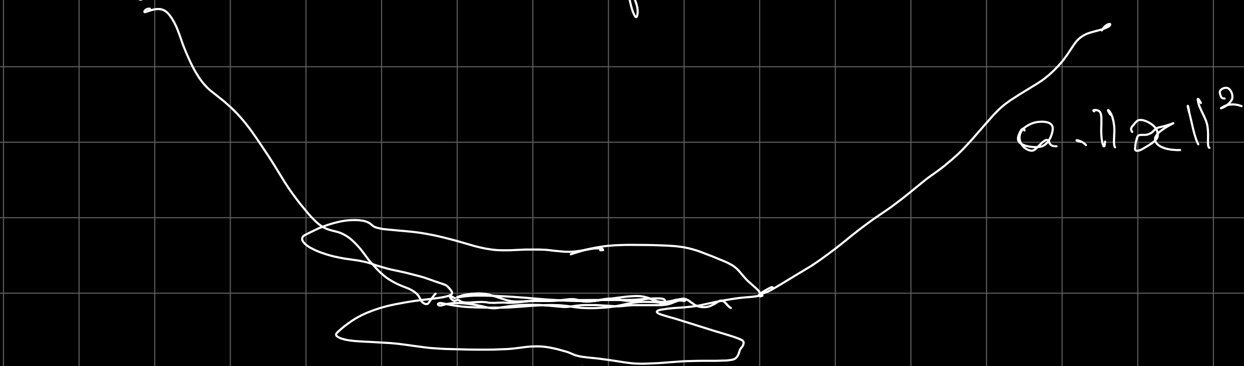
$\underline{\underline{c_k^{EH}}}$ let $E = H^{1/2}(S^1, \mathbb{R}^{2n})$

$$= \{ \gamma \in L^2(S^1, \mathbb{R}^{2n}) \mid \text{Fourier series } \gamma = \sum_{k \in \mathbb{Z}} \gamma_k e^{2\pi J_0 k t} \text{ s.t. } \|\gamma\|^2_{H^{1/2}} = |\gamma_0|^2 + \sum_{k \in \mathbb{Z}} |k| |\gamma_k|^2 < \infty \}$$

$E^+ = \{ \gamma \in E \mid k \leq 0 \Rightarrow \gamma_k = 0 \}$
$E^0 = \{ \quad — \quad k \neq 0 \quad — \quad \}$
$E^- \quad \{ \quad — \quad k \geq 0 \quad — \quad \}$

$$\mathcal{A}(\gamma) = \frac{1}{2}\left(\|\gamma^+\|^2 - \|\gamma^-\|^2 \right)$$

Let $H : \mathbb{R}^{2n} \longrightarrow \mathbb{R}$ Hamiltonian quadratic outside a compact set



$\mathcal{A}_H(\gamma) = \mathcal{A}(\gamma) - \int_0^1 H(\gamma(\tau))\, d\tau$ Ham. action functional

For $Y \subset E$ S^1-inv subspace
Classifying map $f : Y \times_{S^1} ES^1 \longrightarrow BS^1 = \mathbb{CP}^\infty$

$\rightsquigarrow f^* : H^*(\mathbb{CP}^\infty) \longrightarrow H^*(Y \times_{S^1} ES^1) = H^*_{S^1}(Y)$

Def: $\operatorname{ind}_{FR}(Y) = \max\{ k \in \mathbb{Z} \mid f^* u^{k-1} \neq 0 \}$

where u is a generator of the ring $H^*(\mathbb{CP}^\infty)$

E.A $\Gamma \leq \mathrm{Homeo}(E)$
$h \in \Gamma$ iff it is of the form

$$h(x) = e^{\gamma^+(x)} x^+ + x^0 + e^{\gamma^-(x)} x^- + K(x)$$

$\gamma^+, \gamma^- : E \longrightarrow \mathbb{R}$ continuous, S^1-inv & mapping bounded sets to bounded sets

$K : E \longrightarrow E$ cont. S^1-equiv mapping bounded sets to precompact sets

$S^+ =$ unit sphere in E^+

Def: $\operatorname{ind}_{EH}(Y) = \inf_{h \in \Gamma} \operatorname{ind}_{FR}(h(Y) \cap S^+)$

Def: For $k \in \mathbb{N}_0$ & H as above

$$c_k^{EH}(H) := \inf \{ \sup A_H(Y) \mid Y \text{ is } S^1\text{-inv } \& \ \operatorname{ind}_{EH}(Y) \geq k \}$$

Def: $c_k^{EH}(X) = \inf_{H \in \mathcal{H}} c_k^{EH}(H)$

Prop $H \in \mathcal{H}$

$$c_k^{EH}(H) = \inf \{ c \mid \operatorname{ind}_{EH}(\{A_H \leq c\}) \geq k \}$$

Def: $\kappa_c^{(H)} = \max \{ j \mid \exists \sigma \in \mathrm{Im}\left(FH^{S^1,+,c}_{2j+n-1}(H) \to FH^{S^1,+}(H)\right)$

s.t. $U^j \sigma \neq 0 \}$

Prop 2: $c_k^{GH}(H) = \inf \{ c \mid \kappa_c \geq k \}$

Main technical statement

$$CM^{S^1,c}(\mathcal{A}_H) \cong CF^{S^1,c}(H)$$

Csq: $\kappa_c(H) = \mathrm{ind}_{FR}(\{0 \leq \mathcal{A}_H \leq c\})$

Lemma 1: $\mathrm{ind}_{EH}(\{\mathcal{A}_H \leq c\}) \leq \mathrm{ind}_{FR}(\{0 \leq \mathcal{A}_H \leq c\})$

Lemma 2: $\mathrm{ind}_{EH}(\{\mathcal{A}_H \leq c\}) \geq \kappa_c(H)$

↑ Prop [HZ...] If K fin subspace of E^+

$\Rightarrow \mathrm{ind}_{EH}(K \oplus E^0 \oplus E^-) = \frac{1}{2} \dim K$

$\Gamma = \{ \phi_t^{-\nabla \mathcal{A}_H}, t \in \mathbb{R} \}$