Symplectic Orbifold Gromov-Witten Invariants (work in progress).

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- Our aim is to construct Gromov-Witten invariants of symplectic orbifolds.
- This was done over $\mathbb{Q}$ by Chen and Ruan in arXiv:0103156, but we wish to define more general counts (e.g. for K-theory).
- We also wish to present Moduli spaces of holomorphic curves in terms of Global-Kuranishi Charts.
- This is part of a larger project with Ritter in which we will attempt to prove a version of the Crepant resolution conjecture relating Gromov-Witten invariants of birational orbifolds.
- There is also ongoing work by Mak, Seyfaddini and Smith for global quotient orbifolds

Orbifolds

- We think of an *orbifold* X as a 'manifold', except that the charts are locally modelled on open subsets of $\mathbb{R}^n$ quotiented by a finite linear group action.
- So, locally, there is a coordinate chart $V \overset{\text{open}}{\subset} \mathbb{R}^n$ together with a linear group action of a finite group Γ on V and a map $V/\Gamma \to X$ which is a homeomorphism onto its image.

orbifolds

- Suppose G is a compact Lie group acting on a smooth manifold M with finite stabilizers.
- Then the quotient $X = [M/G]$ is naturally an orbifold.
- (Slice theorem): For each point $x \in M$, there is a G_x-equivariant submanifold $S_x \subset M$ containing x and a G-equivariant neighborhood $U_x \subset M$ of x so that so the following map is a G-equivariant diffeomorphism:

$$G \times_{G_x} S_x \to U_x.$$

orbifolds

- After shrinking the slice $S_x \subset M$, we can assume that S_x has a global coordinate system with G_x acting linearly.
- Then (S_x, G_x) is our induced orbifold chart centered at x.
- The set theoretic quotient M/G is called the underlying *coarse moduli space* which we will write as $\underline{X}$.
- Theorem (Pardon): Every smooth orbifold is a quotient $[M/G]$.

Morphisms of Orbifolds

- Let $[M_1/G_1]$ and $[M_2/G_2]$ be orbifolds.
- An *HS (Hilsum-Skandalis) morphism* between these orbifolds is a diagram:

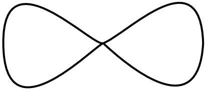
$$\begin{array}{ccc} P & \xrightarrow[G_2-\text{equiv}]{f} & M_2 \\ {\scriptstyle \pi}\Big\downarrow {\scriptstyle G_1-\text{equiv}} & & \\ M_1 & & \end{array}$$

  where P is a smooth manifold admitting a $G_1 \times G_2$-action and with π a principal G_2-bundle.
- Really, it is an equivalence class of such diagrams.
- Locally, there are charts (V_1, Γ_1), (V_2, Γ_2) and a map $\Gamma_1 \to \Gamma_2$ and a Γ_1-equivariant map $V_1 \to V_2$.

Symplectic Orbifold

- A *symplectic orbifold* is a smooth orbifold X together with a closed non-degenerate 2-form ω on it.
- We can define compatible almost complex structures J on such symplectic orbifolds.
- The spaces of such J's is contractible.
- Let us fix (X, ω, J) and $\beta \in H_2(\underline{X}; \mathbb{Z})$.

- We can define a *complex orbifold* to be an orbifold with an integrable almost complex structure.
- A *twisted nodal curve* Σ is a space of the form $\widetilde{\Sigma}/\sim$ where Σ is a one dimensional complex orbifold and where $\sim$ identifies a finite collection of distinct pairs of points $p \sim q$ so that the following *balancing condition* holds:
  - p admits an orbifold chart with coordinate z and where $\mathbb{Z}/k\mathbb{Z}$ acts by $(m, z) \to e^{2\pi i m/k} z$ and
  - q admits an orbifold chart with coordinate w where $\mathbb{Z}/k\mathbb{Z}$ acts by $(m, w) \to e^{-2\pi i m/k} w$.



- We call $\widetilde{\Sigma}$ the *normalization* of Σ and the points that we have identified are called the *nodes*.

- So, near a node, a twisted nodal curve looks like

$$\{xy = 0\}/(\mathbb{Z}/k\mathbb{Z}) \subset \mathbb{C}^2/(\mathbb{Z}/k\mathbb{Z})$$

  where the group action is $(g, (x, y)) \to (gx, g^{-1}y)$ where $g = e^{2i\pi m/k}$.
- The reason for the balancing condition is it allows the node to be smoothed. Locally

$$\{xy = t\}/(\mathbb{Z}/k\mathbb{Z}), \quad t \in \mathbb{C}$$

  is the smoothing of the nodal curve $t = 0$.
- A *marking* on a twisted nodal curve Σ is a collection of distinct points $p_1, \cdots, p_h$ on Σ disjoint from the nodes and containing all the points with nontrivial stabilizers.

- We call $\Sigma = (\Sigma, p_1, \cdots, p_h)$ a *twisted nodal curve with h marked points*.
- A *twisted nodal curve* $u : \Sigma \to X$ is an *HS*-morphism from the normalization $\widetilde{\Sigma}$ of Σ to X so that
  - the induced map of stabilizer groups $G_\sigma \to G_{u(\sigma)}$ is injective for each $\sigma \in \widetilde{\Sigma}$
  - and which descends to a continuous map $\underline{\Sigma} \to \underline{X}$ of coarse moduli spaces.
- The *genus* of u is the arithmetic genus of the underlying coarse moduli space of its domain Σ.

- A map $u : \Sigma \to X$ from a twisted nodal curve is *stable* if it has finitely many automorphisms:

$$\begin{array}{ccc} \Sigma & \xrightarrow{u} & X \\ {\scriptstyle\phi}\downarrow & \nearrow_{u} & \\ \Sigma & & \end{array}$$

- If the domain has marked points, the the automorphism ϕ must fix these marked points.
- We let $\mathcal{M}_{g,h,\beta}(X)$ be the moduli space of stable J-holomorphic maps from genus g twisted nodal curves to X representing β.

- **Example:** $M = pt$, $G = \mathbb{Z}/2$. So $\mathcal{M}_{g,h,0}$ is the moduli space of twisted nodal curves together with a $\mathbb{Z}/2$ principal bundle.
- See arXiv:0106211 Abramovich, Corti, Vistoli.

- We wish to put a fundamental class on $\mathcal{M}_{g,h,\beta}(X)$ so that we can integrate pullbacks of cohomology classes from the inertia stack against it to give Gromov-Witten invariants.
- A *global Kuranishi chart* is a tuple (G, T, E, s) where G is a Lie group acting semi-freely on a manifold T and E is a G-vector bundle over T with a G-equivariant section s.
- We call T the *thickening* and E the *obstruction bundle*.
- Such a global Kuranishi chart *models* $\mathcal{M}_{g,h,\beta}(X)$ if this moduli space is homeomorphic to $s^{-1}(0)/G$.

- The fundamental class is given by $[T/G] \cap s^*(Th(E))$ where $[T/G]$ is the fundamental class in G-equivariant homology and $Th(E)$ is the Thom class of the obstruction bundle in G-equivariant cohomology.
- Here, we need an appropriate orientation for $T - \mathfrak{g}$ and E.
- So, how do we construct such a global Kuranishi chart for $\mathcal{M}_{g,h,\beta}(X)$?

Genus Zero Manifold Case

- Let us start in the simpler setting where X is a smooth manifold and the genus is zero.
- The following construction is due to Abouzaid-M-Smith.
- We let $\mathcal{F}_{h,d}$ be the moduli space of genus zero degree d curves with h marked points mapping to $\mathbb{P}^d$ whose image is not contained in a hyperplane
- This is a smooth quasi-projective variety.
- We let $\mathcal{C}_{h,d} \to \mathcal{F}_{h,d}$ be the corresponding universal curve and $C^o_{h,d}$ the complement of its nodes.

Genus Zero Manifold Case

- Let $Y_{h,d} \to \mathcal{C}^o_{h,d} \times X$ be the vector bundle whose fiber over a point (p, x) is the space of anti-holomorphic maps from the tangent bundle of the fiber of $\mathcal{C}^o_{h,d}$ at p to $T_x X$.
- A *finite dimensional approximation scheme* is a sequence $(V_\mu, \lambda_\mu)_{\mu \in \mathbb{N}}$ of $PU(d+1)$-equivariant maps $\lambda_\mu : V_\mu \to C^\infty_c(Y_{h,d})$ from a $PU(d+1)$ representation V_μ so that the union of their images is dense, $V_\mu \subset V_{\mu+1}$ and $\lambda_{\mu+1}|_{V_\mu} = \lambda_\mu$ for each μ.
- We define the *pre-thickened moduli space* T^{pre} to be the space of tuples (u, ϕ, e) where $\phi \in \mathcal{F}$, $u : \mathcal{C}|_\phi \to X$ is a stable map and $e \in V_\mu$ so that

$$\overline{\partial}_J u(p) = \lambda_\mu(e)(p, u(x)), \quad \forall\ p \in \mathcal{C}^o|_\phi \times X$$

- The topology on this space is induced from the Hausdorff topology on graphs in $\mathcal{C} \times X$ as well as the topology on V_μ.

- A naive guess for the obstruction bundle is V_μ with the section sending (u, ϕ, e) to e since setting $e = 0$ gives J-holomorphic curves.
- This would be fine if our group G is $PGL_{\mathbb{C}}(d+1)$ - however this does not work since λ_μ cannot be made to be $PGL_{\mathbb{C}}(d+1)$-equivariant.
- So we need to reduce the group $PGL_{\mathbb{C}}(d+1)$ to $PU(d+1)$.
- First, choose a Hermitian line bundle $L \to X$ whose curvature form Ω_L tames J.

- A *framed curve* is a triple (u, Σ, F) where $u : \Sigma \to X$ is a smooth map representing β and $F = (f_0, \cdots, f_d)$ is a basis of $H^0(u^* L)$ where $d = c_1(L)(\beta) + 1$
- The basis F induces a map

$$\phi_F : \Sigma \to \mathbb{P}^d, \quad \phi_F(\sigma) = [\tau f_0(\sigma), \cdots, \tau f_d(\sigma)]$$

  where τ is a trivialization $\tau : L|_\sigma \cong \mathbb{C}$.
- Hence we have an identification $\psi_F : \Sigma \xrightarrow{\cong} \mathcal{C}|_{\phi_F}$.

- Let $\mathcal{H}_{d+1}$ be the space of $(d+1) \times (d+1)$ Hermitian matrices.
- We have an identification

$$\exp : \mathcal{H}_+ \xrightarrow{\cong} PGL_{d+1}(\mathbb{C})/PU(d+1).$$

- We define $A_F := \exp^{-1} B$ where B is the matrix with i,j entry

$$\int_\Sigma \langle f_i, f_j \rangle \Omega_L.$$

- We define the *thickening* T to be the space of isomorphism classes of tuples (u, Σ, F, e) so that $(u \circ \psi_F^{-1}, \phi_F, e) \in T^{pre}$.
- The group $G = PU(d+1)$ acts on T via postcomposition in $\mathbb{P}^N$.
- The obstruction bundle E has fiber $\mathcal{H}_+ \times V_\mu$.
- The section s sends (u, Σ, F, e) to (A_F, e).
- So, (G, T, E, s) is our Global Kuranishi chart.

- We wish to generalize this to higher genus with X an orbifold rather than a manifold.
- There are two problems.
- The first problem is that twisted nodal curves with at least one orbifold point don't map to $\mathbb{P}^d$.
- The second problem is that line bundles of a given degree on a higher genus curve aren't unique.

- Let us deal with the first problem.
- We will use work of Ross and Thomas.
- Instead of looking at moduli spaces of curve mapping to projective space, we use weighted projective space $\mathbb{P}(w_0, \cdots, w_d) = (\mathbb{C}^{d+1} - 0)/\sim$, $(z_0, \cdots, z_d) \sim (t^{w_0} z_0, \cdots, t^{w_d} z_d)$ for each $t \in \mathbb{C}^*$.

- Let Y be a complex compact orbifold with only cyclic quotient singularities
- In our case, we are only interested in one dimensional complex orbifolds corresponding to normalizations of twisted nodal curves.
- A line bundle L over Y is *locally ample* if for each $y \in Y$, the stabilizer of y acts faithfully on the fiber $L|_y$.
- It is *globally positive* if L^N is the pullback of an ample line bundle from the coarse moduli space $\underline{Y}$ where N is the least common multiple of all the stabilizers of all the points on Y.

- L is *orbi-ample* if it is locally ample and globally positive.
- Let $n_i := |H^0(L^i)|$ for each $i \in \mathbb{N}$.
- A *k-framing* of L is a tuple

$$(f_{ij})_{i=k\cdots,2k,j=0,\cdots,n_i}$$

where f_{ij}, $j = 1, \cdots, n_i$ is a basis for $H^0(L^i)$ for each $i = k, \cdots, 2k$.

- Define

$$\mathbb{P}_k(L) := \mathbb{P}(k, \cdots, k, k+1, \cdots, k+1, \cdots, 2k, \cdots, 2k)$$

  where there are exactly n_i copies of $k+i$ for each $i = 1, \cdots, N$.
- Define the map $\phi_F : Y \to \mathbb{P}_k(L)$ sending $y \in Y$ to $[\tau f_{ij}(s)]_{i=k\cdots,2k,j=0,\cdots,n_i}$ where τ is any trivialization $\tau : L|_s \xrightarrow{\cong} \mathbb{C}$.
- **Theorem.** (Ross, Thomas). ϕ_F is an embedding for k large if L is orbi-ample.

- Let (X, ω) be a symplectic orbifold with compatible almost complex structure J and let $\beta \in H_2(X; \mathbb{Z})$.
- Choose a locally ample orbi-vector bundle $W \to X$ (this exists by Pardon's result).
- Choose a Hermitian line bundle L on X which is a pullback from the coarse moduli space whose curvature for Ω_L tames J.

- Abramovich and Vistoli have constructed moduli spaces of twisted nodal curves mapping to smooth DM stacks (i.e. complex orbifolds).
- For any weighted projective spaces $\mathbb{P}$, define $\mathcal{F} := \mathcal{F}_{g,h,D}(\mathbb{P})$ to be the moduli space of stable twisted nodal curves u of degree at most D satisfying $H^1(u^*O(1)) = 0$ and u is automorphism free.
- This is a smooth quasi-projective variety with universal curve $\mathcal{C} := \mathcal{C}_{g,h,D}(\mathbb{P}) \to \mathcal{F}_{g,h,D}(\mathbb{P})$.

- Let $k \gg 1$.
- We define $\mathscr{F}_{\mathcal{F}}$ be the space of tuples (ϕ, u, R) where
  - $\phi \in \mathcal{F}$,
  - $u : \mathcal{C}|_\phi \to X$ is a twisted nodal curve and
  - $R = (R_{ij})_{i=k\cdots,2k, j=1,\cdots,n_i}$ is a k-framing of $W_u := K_{\mathcal{C}|_\phi} \otimes (u^* W \otimes L^i)$.
- The topology is the Hausdorff topology induced by graphs of R_{ij} on the coarse moduli space of $\mathcal{C} \times W_u^{\sum_{i=k}^{2k} n_i}$.

- Let $\mathcal{C}^o \subset \mathcal{C}$ be the complement of the nodes and marked points.
- Let $Y \to \mathcal{C}^o \times X$ be the vector bundle whose fiber over a point (p, x) is the space of anti-holomorphic maps from the tangent space at p of the fiber of $\mathcal{C}^o$ to $T_x X$.
- Choose a finite dimensional approximation scheme $(\lambda_\mu, W_\mu)_{\mu \in \mathbb{N}}$, $\lambda_\mu : W_\mu \to C_c^\infty(Y)$ for Y.

- **Definition:** The *pre-thickened moduli space* $\mathscr{T}^{pre}$ is the space of tuples $((\phi, u, R), e) \in \mathscr{F}_{\mathcal{F}} \times V_\mu$ satisfying:

$$\overline{\partial}_J u(p) = \lambda_\mu(e)(p, u(x)).$$

- For $k, \mu \gg 1$, we get have that $\mathscr{T}^{pre}$ is a topological manifold (it has a C^1_{loc} structure, when enables us to put a smooth structure on an enlargement of it).

- For each $((\phi, u, R), e) \in \mathscr{T}^{pre}$, define $L_u := K_{\mathcal{C}|_\phi}(p_1, \cdots, p_h) \otimes u^* L$ where $K_{\mathcal{C}|_\phi}$ is the canonical bundle.
- We define the *thickened moduli space* $\mathscr{T}$ to be the space of tuples (ϕ, u, R, e, F) where $(\phi, u, R, e) \in \mathscr{T}^{pre}$ and F is a k-framing of L_u.
- We now need to construct the obstruction bundle.

- Define $\mathbb{P}_{\mathscr{T}} := \mathbb{P}_k(L_u)$ for some (ϕ, u, R, e, F) in $\mathscr{T}$ (this does not depend on the point in $\mathscr{T}$ for $k \gg 1$ after shrinking).
- This is the weighted projective space that our framing F maps to. So, we get a natural map $\phi_{\mathscr{T}} : \mathcal{C}|_{\phi} \to \mathbb{P}_{\mathscr{T}}$ and hence a map $\mathscr{T} \to \mathcal{F}_{g,h,D}(\mathbb{P}_{\mathscr{T}})$, $D \gg 1$.
- Define $\mathcal{F}_{\mathbb{P}^2_{\mathscr{T}}}$ to be an appropriate moduli space of maps to $\mathbb{P}^2_{\mathscr{T}}$ and let $\Delta_{\mathscr{T}}$ be the normal bundle of the diagonal map $\mathcal{F}_{g,h,D}(\mathbb{P}_{\mathscr{T}}) \to \mathcal{F}_{g,h,D}(\mathbb{P}^2_{\mathscr{T}})$.
- We can pull back this diagonal bundle to $\mathscr{T}$. Call it $\widetilde{\Delta}_{\mathscr{T}}$.

- Our obstruction bundle $\mathscr{E}$ is then $\widetilde{\Delta}_{\mathscr{T}} \times \mathcal{H}_{\mathbb{P}_{\mathscr{T}}} \times \mathcal{H}_W \times V_\mu$.
- The first component tells us how far away the bundle $\phi_F^* O(1)$ is from L_u. In other words, how far apart is ϕ and ϕ_F (which is an element of $\mathcal{F}_{\mathbb{P}^2_{\mathscr{T}}}$ and hence, via a metric maps to $\widetilde{\Delta}_{\mathscr{T}}$).
- The second component $\mathcal{H}_{\mathbb{P}_{\mathscr{T}}}$ is $G_{\mathscr{T}}^{\mathbb{C}}/G_{\mathscr{T}}$ where $G_{\mathscr{T}}$ is the automorphism group of our weighted projective space $\mathbb{P}_{\mathscr{T}}$ and $G_{\mathscr{T}}$ is its maximal compact subgroup. It tells us how far our framing F is from being orthogonal.
- The third component $G_W^{\mathbb{C}}/G_W$ tells us how far R is from being orthogonal. It is defined analogously. Here G_W is a product of unitary groups.
- The last component tells us how far our map u is from being holomorphic.

- **Theorem** (in progress, M-Ritter). $(G_{\mathscr{T}} \times G_W, \mathscr{T}, \mathscr{E}, s)$ is a Global-Kuranishi chart for $\mathcal{M}_{g,h,\beta}(X)$. It is unique up to a series of standard operations and their inverses: stabilization, group enlargement and germ equivalence.
- The thickening also admits a C^1_{loc}-structure, which means up to stabilization, it admits a smooth structure.
- If we deform J, then we get cobordant global Kuranishi charts, which shows that our Gromov-Witten counts are in fact invariants of the symplectic form.