Two or infinity

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June 14, 2024

Section 1

Introduction

Setup

Notation:

- Y, a closed oriented three-manifold
- λ, a **contact form** on $Y : \lambda \wedge d\lambda > 0$
- R, the **Reeb vector field**: $d\lambda(R, \cdot) = 0, \lambda(R) = 1.$
- Periodic orbits of R are called *Reeb orbits*
- ξ, the **contact structure**: $\xi = Ker(\lambda).$

Questions, conjectures

Finite energy foliations of tight three-spheres and Hamiltonian dynamics

By H. Hofer, K. Wysocki, and E. Zehnder*

Abstract

Surfaces of sections are a classical tool in the study of 3-dimensional dynamical systems. Their use goes back to the work of Poincaré and Birkhoff. In the present paper we give a natural generalization of this concept by constructing a system of transversal sections in the complement of finitely many distinguished periodic solutions. Such a system is established for nondegenerate Reeb flows on the tight 3-sphere by means of pseudoholomorphic curves. The applications cover the nondegenerate geodesic flows on $T_1S^2 \equiv \mathbb{R}P^3$ via its double covering S^3, and also nondegenerate Hamiltonian systems in $\mathbb{R}^4$ restricted to sphere-like energy surfaces of contact type.

Questions, conjectures

An interesting conjecture from their paper:

Conjecture 1.13. A tight Reeb flow on S^3 has either precisely two or infinitely many geometrically distinct periodic orbits.

As already mentioned, the conjecture is true for dynamically convex contact forms, $f\lambda_0$ for f constituting an open subset of $C^\infty(S^3, (0,\infty))$, and also for every generic $f \in \Theta_2$, in view of Corollary 1.10.

(In this case, tight is equivalent to demanding that ξ is standard.)

Questions, conjectures

Some reasons I find this interesting:

- Why should the generic case be representative? (e.g. on T^2, there are functions with 3 critical points, but a Morse function has at least 4.)
- How do we develop tools to study Reeb flows, without requiring nondegeneracy? (e.g. usually assume this for defining Floer theory)

More questions, conjectures

TAUBES'S PROOF OF THE WEINSTEIN CONJECTURE IN DIMENSION THREE

MICHAEL HUTCHINGS

ABSTRACT. Does every smooth vector field on a closed three-manifold, for example the three-sphere, have a closed orbit? The answer is no, according to counterexamples by K. Kuperberg and others. On the other hand, there is a special class of vector fields, called Reeb vector fields, which are associated to contact forms. The three-dimensional case of the Weinstein conjecture asserts that every Reeb vector field on a closed oriented three-manifold has a closed orbit. This conjecture was recently proved by Taubes using Seiberg-Witten theory. We give an introduction to the Weinstein conjecture, the main ideas in Taubes's proof, and the bigger picture into which it fits.

More questions, conjectures

Another interesting question, from Hutchings' '09 article:

Question. If Y is a closed oriented connected 3-manifold other than a sphere or a lens space, then does every contact form on Y have infinitely many embedded Reeb orbits?

More questions, thoughts, conjectures

Геодезические в Финслеровой Геометрии

Д. В. Аносов

1. В финслеровой геометрии, как и в римановой, рассматривается гладкое многообразие M, для касательных векторов которого определено понятие длины (так что можно говорить о длине параметризованной кривой; последняя длина равна интегралу от длины вектора скорости). Отличие от римановой геометрии состоит в том, что выражение для длины может быть более общим. Именно, длина вектора $v \in T_x M$ даётся функцией $L(x, v)$, которая обращается в нуль лишь при $v = 0$, положительна при $v \neq 0$ и является положительно-однородной первой степени по v. На L налагаются два общих условия:

Условие достаточной гладкости вне нулевого сечения касательного расслоения;

Условие выпуклости “единичных сфер” $L(x,v) = 1$ во всех касательных пространствах $T_x M$, усиленное ещё дополнительным требованием, чтобы кривизна “единичной сферы” (вычисленная по отношению к произвольной эвклидовой метрике в $T_x M$) нигде не обращалась в нуль.

Some thoughts of Anosov

Anosov [from "Geodesics in Finsler geometry"]:

``My interest in Finslerian geometry is partly...caused by the desire to highlight in problems of Riemannian geometry that which is associated with their variational nature alone, and does not depend on other specific Riemannian features…

In the problem of closed geodesics on a sphere, the most complete results relate to the two-dimensional sphere. They were obtained by Lyusternik and Shnirelman in 1929...their presentation is limited to Riemannian metrics...The corresponding reasoning can be modified...as to extend to Finslerian metrics. For invertible Finsler metrics, the final result is the same.

For irreversible metrics, only the existence of two closed geodesics can be guaranteed...This last result is of interest because it is sharp. The corresponding example is given by Katok...The example suggests the following: although the number of closed geodesics that can be expected on the basis of calculus of variations is usually very small, perhaps it is unimprovable?"

A concrete question

Question (Alvarez Paiva, Burns and Matveev, Long)

Does every Finsler metric on S^2 have either two or infinitely many prime closed geodesics?

What is now known about these "two or infinity" questions?

(Setup: Y a closed connected three-manifold, λ a contact form.)
Current state:

- There are always at least two simple Reeb orbits. (CG-Hutchings, '13)
- When there are exactly two simple Reeb orbits, Y is a lens space and ξ is universally tight. (CG-Hryniewicz-Hutchings-Liu, '21)
- When λ is nondegenerate, there are either two or infinitely many simple Reeb orbits. (Colin-Dehornoy-Rechtman, '20)
- (For non-specialists: think of nondegenerate as like "Morse"; holds generically.)

(Many other important contributions/partial results...see discussion in arxiv:2310.07636)

Today's main theorem

Theorem (CG-Hryniewicz-Hutchings-Liu)

Let Y be a closed connected three-manifold, λ a contact form, and assume that $c_1(\xi)$ is torsion. Then there are always either two or infinitely many simple Reeb orbits.

Corollaries:

- HWZ's conjecture.
- Two or infinity conjecture for Finsler geodesics
- Affirmative answer to Hutchings' question, in the special case where $c_1(\xi)$ is torsion (e.g. Y a rational homology sphere.)

Current state of these questions

Upshot/summary: we now understand the answers to all of the "two or infinity" questions I mentioned, except for the case of degenerate contact forms with $c_1(\xi)$ not torsion (which one would guess have infinitely many simple Reeb orbits.)

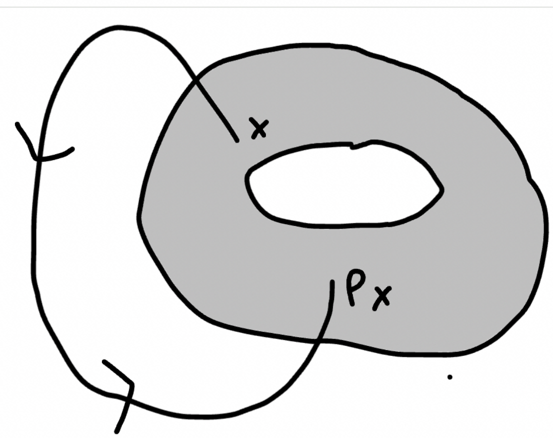
Section 2

Key ideas in the proof

Global surfaces of section

Goal: Under assumptions of theorem (i.e. $c_1(\xi)$ torsion) + finitely many simple Reeb orbits, find an *annular global surface of section*, i.e. a compact annulus S such that:

- S is immersed in Y, the interior $int(S)$ is embedded and transverse to the Reeb vector field, the boundary of S is on Reeb orbits.
- every flow line hits S both forwards and backwards in time.



Proof, given a global surface of section

If such a GSS S exists, can define an area-preserving *first return map*

$$P : int(S) \longrightarrow int(S).$$

by taking a point to the next place at which the flow line through the point hits $int(S)$. Then, the periodic points of P are in bijection with the Reeb orbits.

Now apply:

Theorem (Franks)

Any area-preserving homeomorphism of an open annulus with at least one periodic point has infinitely many.

Global surfaces of section and pseudoholomorphic curves

How to find the desired global surface of section S?

Basic idea: Find S via the projection of an index 2 J-holomorphic cylinder C, in $X = \mathbb{R} \times Y$.

Why might one try this?: To first approximation, if deformations of C foliate X, its projection will be a GSS. Thus, the problem of finding a GSS is turned into a PDE problem.

This approach has a long history, though there are various new aspects here. e.g. λ is degenerate, but a large part of J-holomorphic curve theory is in the nondegenerate case

The problem, in more detail

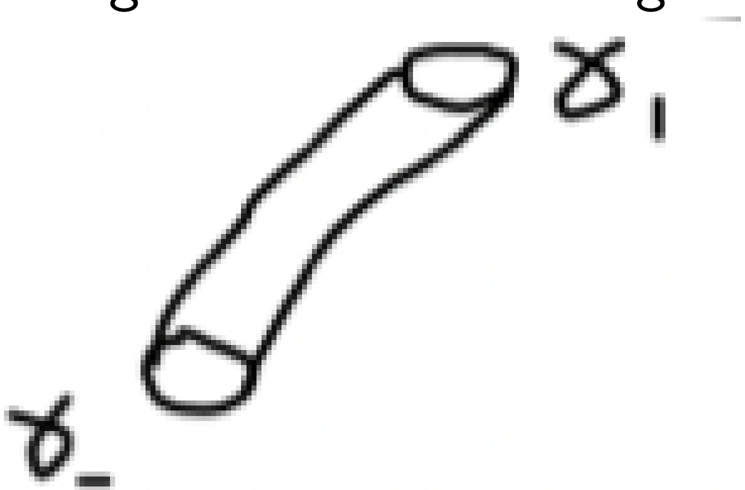
Thus, there are two (linked) aspects that must be dealt with for this approach to work:

- (P1) Find criteria guaranteeing that a J-holomorphic cylinder C in $X = \mathbb{R} \times Y$ projects to a GSS for λ.
- (P2) Prove that a J-holomorphic cylinder C satisfying these criteria actually exists.

$P1$: the basic point

(P1: Find criteria guaranteeing that a J-holomorphic cylinder C in $X = \mathbb{R} \times Y$ projects to a GSS for λ, without assuming nondegeneracy)

Key idea: If C *converges exponentially fast* to the Reeb orbits at its ends, much of the (well-developed) theory from the nondegenerate case has a good analogue.



The nondegenerate case

In the nondegenerate case, for a J-holomorphic cylinder C from γ_+ to γ_- to project to a GSS, it is known that the following suffice. (J assumed admissible.)

1. $ind(C) = 2$
2. C is embedded in $\mathbb{R} \times Y$
3. Let $\mathcal{M}_C$ denote the component of the moduli space containing C. Then $\mathcal{M}_C$ is compact (modulo translation)
4. (for specialists: $\gamma_\pm$ have odd Conley-Zehnder index)
5. (for specialists: $gcd(m, \lfloor m\theta \rfloor) = 1$, when γ_+ is an m-fold cover of a simple orbit γ with rotation number θ. Similarly for γ_-.)

Rough idea: $(1+3) \implies$ images of curves in $\mathcal{M}_C$ cover all of X; $(2+4) \implies$ curves in $\mathcal{M}_C$ disjoint. (5) controls return time near boundary

$P1$: Putting it together

Thus, to prove $P1$, it remains to find analogues of the 5 criteria on the previous slide, under the condition of exponential convergence.

Basic idea (for specialists): Use "exponentially weighted Sobolev spaces": there is a corresponding Fredholm index, Conley-Zehnder index, automatic transversality, etc. A long history.

We won't say much more about this here, to leave time for $P2$.

$P2$: the scheme

(P2: Prove that a J-holomorphic cylinder C satisfying the criteria from $P1$ actually exists.)

Let us at least give a sense for how this is proved, though it is quite involved. Outline of the argument:

1. Approximate λ by (particular) nondegenerate contact forms λ_n.
2. For each λ_n, find J_n-holomorphic cylinders C_n, using "embedded contact homology".
3. Find the desired cylinder C by taking a limit of the C_n.

A crucial point: That the C_n converge to something with exponential convergence requires luck! e.g. what if the decay rate of the C_n is tending to 0??

Background: embedded contact homology

A crash course:

- $ECH(Y,\lambda_n)$ homology of a chain complex $ECC(Y,\lambda_n)$; requires λ_n nondegenerate.
- $ECC(Y,\lambda_n)$ generated by (certain) sets $\{(\alpha_i, m_i)\}$, where the α_i are distinct embedded Reeb orbits and the m_i are positive integers.
- Chain complex differential ∂ counts $I = 1$ J_n-holomorphic curves in $\mathbb{R} \times Y$, where I is the "ECH index". $I = 1$ (mostly) forces the curves to be embedded.
- $ECH(Y,\lambda_n) \simeq \widehat{HM}(Y)$ (Taubes), where $\widehat{HM}(Y)$ is the Seiberg-Witten Floer cohomology. Implies that ECH is far from vanishing.

More ingredients from ECH theory

We will need the following two facts:

F1: U-map. There is an endomorphism map $U : ECH \longrightarrow ECH$, counting $I = 2$ curves in X. By Taubes' isomorphism, it is non-zero. In fact, for any N, U^N is nontrivial.

F2: Partition conditions. The ends of U-map curves are determined combinatorially by the "rotation number" of the local flow around the corresponding Reeb orbit. See figures from [CGHS], [Hutchings] on next page:

A U-map curve and the partition conditions

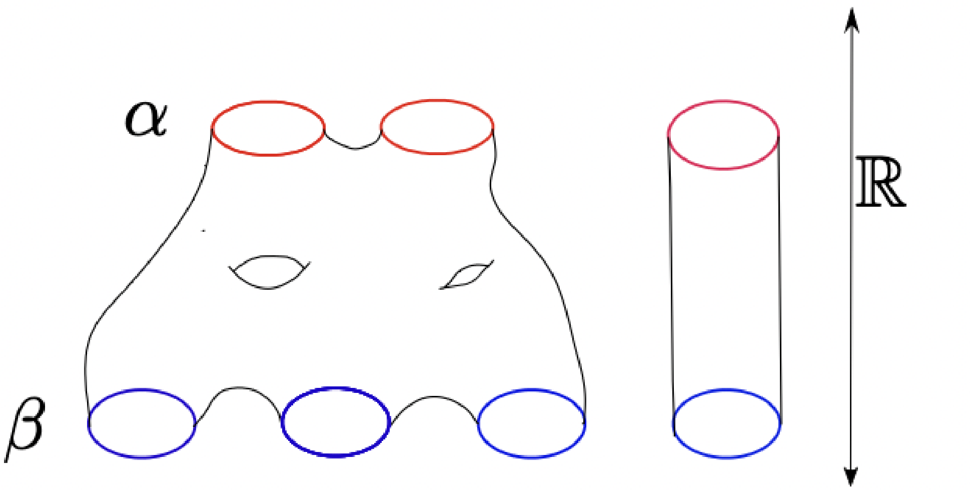


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Figure 1: The positive partitions $p^+_\theta(m)$ for $2 \leq m \leq 8$ and all θ. The left column shows the interval in which θ mod 1 lies, and the top row indicates m. (Borrowed from [21])

P2: key ideas

Let λ_n be close to λ.

The approach: Fix N large. The non-triviality of U^N gives N index 2 curves $C(i)$. Want: at least one of these curves satisfy criteria for a GSS.

Key new idea: For N large, "most" of the $C(i)$ are cylinders. **[This is the only place where the assumption $c_1(\xi)$ torsion is used.]**

Context: No a priori bound on topology of ECH curves! Requires a new invariant "the score".

Another input: It was previously known (for specialists: as a consequence of the ECH Weyl law) that most of the $C(i)$ have $\int_{C(i)} d\lambda_n$ small.

Why are cylinders helpful?

Upshot from previous slide: for each λ_n close to λ, we have a U-map cylinder C_n, with $\int_{C_n} d\lambda$ as small as we'd like.

Key calculation: Let μ_n denote the rate of convergence of C_n to the Reeb orbit at its positive end. Then μ_n is bounded away from 0. (The same holds for the negative end.)

Outline of the calculation:

- Let θ_n denote the rotation number for the Reeb orbit at the positive end and assume $\theta_n \longrightarrow 0$.
- By known theory ("asymptotic analysis"), this is closely related to μ_n; working through this, to bound μ_n, it suffices to show that θ_n is slightly negative.
- Given that C_n is a cylinder, this follows from the partition conditions when $\int_{C_n} d\lambda_n$ is small enough (!!)

P2: Putting it all together – sketch

Let's now conclude by giving a sense for how the argument goes:

- Approximate λ by λ_n.
- Find J-holomorphic cylinders C_n for λ_n from the U-map. As in the previous slide, their exponential convergence is **not** tending to 0 with n.
- Show that each C_n satisfies the criteria to project to a GSS. (We did not explain this, but it uses that $\int_{C_n} d\lambda_n$ is small + the partition conditions.)
- Extract the desired curve C as a limit, and check that it satisfies the criteria in $P1$. *Basic idea of the proof:* we've chosen the $P1$ conditions to be analogues of the nondegenerate case, and the C_n satisfy those.

Throughout, there are certainly other points too: e.g. we also have to be careful to guarantee that the length of the orbits for the C_n stays bounded.

Section 3

Open questions

Some questions

- When there are infinitely many simple Reeb orbits, what can we say about the *growth rate*? i.e., what lower bound is there on the number of simple Reeb orbits of action $\leq T$?
- Are there always infinitely many simple Reeb orbits when $c_1(\xi)$ is not torsion?
- What kinds of Reeb orbits must exist? e.g, does S^3 always have an elliptic orbit?
- What about other kinds of vector fields, e.g: stable Hamiltonian (see Cardona-Rechtman)? hypersurfaces in symplectic manifolds that are not contact type (see Prasad)?

Section 4

Bonus: The topology of ECH curves

How do we control the topology of ECH curves?

Basic idea:

- *Step 1.* U^N is non-trivial, so can find N curves $C(1), \ldots, C(N)$ in a row.
- *Step 2.* We know: $I(\sum_{i=1}^{N} C(i)) = 2N$
- *Step 3.* There is another index J_0. By the ECH Weyl Law + **assumption that $c_1(\xi)$ is torsion**, $J_0 \approx I = 2N$
- *Step 4.* Hence, most curves $C(i)$ have $J_0 \approx 2$.

By general theory, $J_0(C) = -2 + 2g(C_1) + e(C)$, where $C = C_0 \cup C_1$ and e is a certain measurement of the ends of C. (For each orbit γ_+ at which C_1 has positive ends, define $e(\gamma_+)$ to be twice the number of ends at γ_+, minus 1 if C_0 does not have any ends at γ_+. We define $e(\gamma_-)$ analogously. Now sum up.)

About the score

However, a $J_0 = 2$ curve need not be a cylinder. It is quite a delicate matter, and “the score” is built to deal with this. The partition conditions are key.