Invariant Sets and Hyperbolic Periodic Orbits of Reeb Flows

Based on joint work with

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References: arXiv:2309.04576, arXiv:2401.01421

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Motivation: Hyperbolic Periodic Orbits $\Longrightarrow$ Interesting Dynamics

Phenomenon: In some instances, the presense of one or several *hyperbolic* or even *locally maximal* periodic orbits forces a Hamiltonian system to have interesting dynamics.

Some examples (for Hamiltonian diffeomorphisms):

- **Homoclinic intersections:** A hyperbolic periodic orbit with transverse homoclinic intersections $\Rightarrow$ a horseshoe, positive entropy, etc. Note: This is a C^1-generic condition (Hayashi '97, Xia '96).

- **Spectral norm:** Sufficiently many hyperbolic periodic orbits of $\varphi \Rightarrow$ a lower bound on the spectral norm $\gamma(\varphi^k) > \epsilon > 0$, $\forall k \in \mathbb{N}$; Çineli–G.–Gürel, arXiv:2207.03613 and arXiv:2310.00470. Note: This is a C^∞-generic condition.

Motivation

- **Multiplicity:** A hyperbolic fixed point of $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n \Rightarrow |\operatorname{Per}(\varphi)| = \infty$, G.–Gürel '14.

  Closely related: Franks Theorem (Franks '92, '96): $|\operatorname{Per}(\varphi)| = 2$ or ∞ for $\varphi\colon S^2 \to S^2$. Generalizations to $\mathbb{CP}^n$ – the Hofer–Zehnder conjecture: "$|\operatorname{Per}(\varphi)| > n + 1 \Rightarrow |\operatorname{Per}(\varphi)| = \infty$" (Shelukhin 22').

- **Invariant sets:** Moreover, a *locally maximal* fixed point of $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n \Rightarrow |\operatorname{Per}(\varphi)| = \infty$, G.–Gürel '18.
  Def: Locally maximal = isolated as an invariant set; e.g., hyperbolic fixed point is locally maximal.

  Corollary: for a *Hamiltonian pseudo-rotation (PR)* of $\mathbb{CP}^n$ no fixed point is locally maximal.
  Def: $\varphi\colon \mathbb{CP}^n \to \mathbb{CP}^n$ is a PR if $|\operatorname{Per}(\varphi)| = n + 1$.

  Closely related: For S^2: Le Calvez–Yoccoz '97, Franks '99.

Goal: Analogs of the last two results to Reeb flows on S^{2n-1}.

Main results: Setting

Mainly interested in: The contact sphere (S^{2n-1}, α); $\ker \alpha =$ the standard contact structure; $\varphi^t =$ the Reeb flow of α. Think of (S^{2n-1}, α) as the boundary of a star-shaped domain $W \subset \mathbb{R}^{2n}$.

Closed Reeb orbits: $\mathcal{P} = \mathcal{P}(\alpha)$ is the collection of closed Reeb orbits; $\mathring{\mathcal{P}}$ is the set of simple closed Reeb orbits.

Dynamical Convexity (DC): $\mu(x) \geq n + 1$ for all $x \in \mathcal{P}$, where μ is the lower semi-continuous extension of the Conley–Zehnder index (Hofer–Wysocki-Zehnder '98). Often weaker requirements of this type suffice. Ubiquitous in proofs in higher dimensions.

Remark: Convexity $\Rightarrow$ DC; but a DC hypersurface in $\mathbb{R}^{2n}$ need not be symplectomorphic to a convex hypersurface (Chaidez–Edtmair '22; Cristofaro-Gardiner–Hind '23; Dardennes–Gutt–Ramos–Zhang '23).

Many counterparts of the proof work in a more general setting: $M = \partial W^{2n}$ where (W, α) is a Liouville domain, etc.

Main results: Multiplicity

Notation: $\hat{\mu}(x) := \lim_{k\to\infty} \mu_-(x^k)/k$ is the mean index of x; $2\nu(x)$ is the algebraic multiplicity of the eigenvalue 1 of the Poincaré return map of x.

Theorem A (ÇGGM, arXiv:2309.04576)

Assume that (S^{2n-1}, α) has a hyperbolic (simple) closed Reeb orbit z with $\hat{\mu}(z) > 0$ and

$$\mu(x) \geq \max\{3,\, 2+\nu(x)\} \quad \textit{(DC type condition)} \tag{1}$$

for all $x \in \mathcal{P}(\alpha)$ with $\hat{\mu}(x) > 0$. Then the Reeb flow of α has infinitely many simple periodic orbits: $|\mathring{\mathcal{P}}(\alpha)| = \infty$.

Remark: DC $\Rightarrow$ (1). As a consequence: DC + a hyperbolic orbit $\Rightarrow$ $|\mathring{\mathcal{P}}(\alpha)| = \infty$. **Note:** No non-degeneracy conditions.

Main results: Invariant sets

Theorem B (ÇGGM, arXiv:2401.01421)

Assume that $(S^{2n-1\geq 3}, \alpha)$ is DC, non-degenerate and its Reeb flow has only finitely many simple closed orbits (aka Reeb PR). Then no closed orbit is locally maximal, i.e., isolated as an invariant set.

Remark: Hyperbolic closed orbits are locally maximal. Hence,

$$\text{Theorem B} \overset{\text{almost}}{\rightsquigarrow} \text{Theorem A}$$

up to non-degeneracy and a stronger DC type condition.

Remark: Reeb PR's can have interesting dynamics: $\exists$ C^∞-small ergodic PR perturbations of irrational ellipsoids (Katok '73; Albers–Geiges–Zehmisch '22).

Main results: Bonus – Reeb barcode entropy

More general setting:

- A Liouville domain (W, α); Reeb flow φ^t on ∂W.
- The filtered symplectic homology (non-equivariant, ungraded) persistence module $\mathrm{SH}(W) := \{\mathrm{SH}^s(W) \mid s \in \mathbb{R}\}$.
- $\mathfrak{b}_\epsilon(s) = |\{ \text{ bars } > \epsilon \text{ beginning } < s\}|$.
- The ϵ-barcode entropy and barcode entropy of (W, α)

$$\hbar_\epsilon(W) := \limsup_{s\to\infty} \frac{\log^+ \mathfrak{b}_\epsilon(s)}{s} \text{ and } \hbar(W) := \lim_{\epsilon\to 0+} \hbar_\epsilon(W) \in [0, \infty],$$

where $\log^+ = \max\{\log, 0\}$.

Main results: Bonus – Reeb barcode entropy

Theorem: $\hbar(\alpha) \le \mathrm{h_{top}}(\varphi)$ (Fender–Lee–Sohn '23). In particular, $\hbar(\alpha) < \infty$.

Theorem C (ÇGGM, arXiv:2401.01421)

Let $K \subset \partial W$ be a compact hyperbolic invariant set of φ^t. Then

$$\mathrm{h_{top}}(\varphi|_K) \le \hbar(W).$$

Combining these two theorems with the results of Lian–Young '12 or Lima–Sarig '19 extending Katok '80 to flows, we have

Corollary (ÇGGM, arXiv:2401.01421)

Assume that $\dim \partial W = 3$. *Then* $\hbar(W) = \mathrm{h_{top}}(\varphi)$.

Discussion and context: Reeb flows in 3D

Disclaimer: Theorems A and B are mainly of interest when $\dim > 3$**.**

Multiplicity in 3D has been extensively studied and well understood. The 2-or-∞ conjecture has been proved for most of Reeb flows in 3D: Hofer–Wysocki–Zehnder '98, Cristofaro-Gardiner, Hutchings, Ramos, Pomerleano, Hryniewicz, Liu '16–'23, Colin–Dehornoy–Rechtman '23. Nothing as precise as that is true when $\dim > 3$. The (expected) orbit bounds depend very much on the underlying contact manifold and much less is known even for $S^{2n-1\geq 5}$.

Invariants sets in 3D: Theorem B in 3D $\Leftarrow$ the Franks–Le Calvez–Yoccoz theorem (2D); for the latter theorem is in fact local.

Related result in a similar spirit: In 3D, the union of proper closed invariant sets is dense (Cristofaro-Gardiner–Prasad 24'). This does not follow from the Franks–Le Calvez–Yoccoz theorem and the proof also implies Theorem B in 3D.

Discussion and context: Multiplicity for $S^{2n-1\geq 5}$

The question originates in classical mechanics and calculus of variations (Lyapunov, Moser, Rabinowitz, Weinstein, Ekeland, ...).

Conjecture : For a Reeb flow on the standard contact S^{2n-1} either $|\mathring{\mathcal{P}}| = n$ and all orbits are elliptic or $|\mathring{\mathcal{P}}| = \infty$ and at least one of the orbits is degenerate or not elliptic. (Along the lines of the Reeb HZ Conjecture aka the Reeb Franks "Theorem".)

Comment: A long shot given how little is known! Theorem A is one of the first steps in the "or" direction.

Unknown: If the Reeb flow on the standard contact $S^{2n-1\geq 5}$ must have > 1 simple closed Reeb orbits or > 2 in the non-degenerate case, without a DC type index condition! (Nondegeneracy $\Rightarrow |\mathring{\mathcal{P}}| \geq 2$; Gürel '15; Abreu–Gutt–Kong–Macarini '19, ...)

Discussion and context: Multiplicity for $S^{2n-1\geq 5}$

Lower bounds on $|\mathring{\mathcal{P}}|$ with index requirements – Extensively studied:

- DC type conditions + non-degeneracy $\Rightarrow |\mathring{\mathcal{P}}| \geq n$.
- DC type conditions without non-degeneracy $\Rightarrow |\mathring{\mathcal{P}}| \geq\sim n/2$; improvements in lower dimensions... .

Credits: Breakthrough: Long–Zhu '02. Then in various combinations: Long, Liu, Wang, Hu '02–'24; Gutt–Kang '16; Abreu, Macarini, Gürel, G. '16–'19; ... .

Related work: Some upper bounds for "perfect" flows on the sphere and other manifolds; multiplicity results for other manifolds, the contact Conley conjecture, ...

Discussion and context: Invariant sets in dim > 3

Theorem B is the first result of this type. Nothing else seems to be known. No general conceptual picture.

Somewhat related work: No hypersurfaces in $\mathbb{R}^4$ with minimal characteristic flow (Fish–Hofer '23) + refinements (Prasad '24); Invariant probability measures (Prasad '21); No hypersurfaces in $\mathbb{R}^{2n}$ with uniquely ergodic characteristic flow (G.–Niche '15).

Discussion and context: Barcode entropy

Some related results and constructions:

Barcode entropy:

- Barcode entropy for Hamiltonian diffeomorphisms: ÇGG '21–'23
- Barcode entropy for geodesic flows: GGM '23
- Barcode entropy for Reeb flows: Fender–Lee–Sohn '23, Fernandes '24
- Relation of categorical entropy to h_{top}: Bae–Lee '22
- Lower semicontinuity of Lagrangian volume: ÇGG '22
- Triangulated persistence categories: Biran–Cornea–Zhang '22, '23

$\mathfrak{b}_\epsilon$: In some other settings, $\mathfrak{b}_\epsilon$ carries useful geometrical info: Cohen-Steiner–Edelsbrunner–Mileyko '10, I.+L. Polterovich–Stojisavljević '17, Buhovsky–Payette–I.+L. Polterovich–Shelukhin–Stojisavljević '21.

About proofs: Background

Three main ingredients:

- Boundary depth upper bound
- Crossing energy lower bound – The key new ingredient (Çineli)
- Index recurrence (IR)

Need to work with specific Hamiltonians rather than symplectic homology and things get a bit technical.

About proofs: Background

Convenient choice: *Semi-admissible* Hamiltonians.

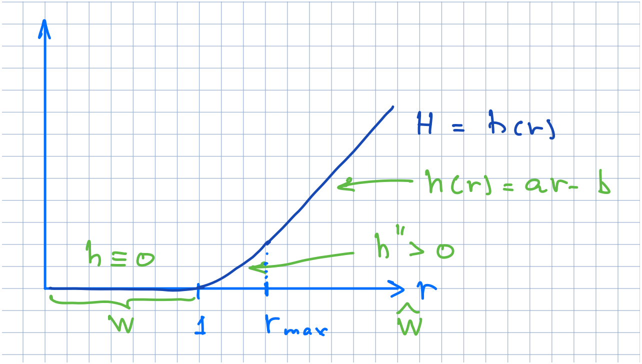

Fig 1: A semi-admissible Hamiltonian

Fact: $\mathrm{SH}^{\tau}(W) \cong \mathrm{HF}^{f(\tau)}(H)$ where $f(\tau) \approx \tau$ when $\tau \ll slope(H)$.

About proofs: Boundary depth upper bound

Notation: $\mathrm{SH}^\infty(W)$ is the total symplectic homology, i.e., the action range is $[0, \infty)$; e.g., $\mathrm{SH}^\infty(W) = 0$ when W is displaceable (Viterbo '99, Cieliebak–Frauenfelder–Oancea '10, Sugimoto '16, ...); $\beta_{\mathrm{max}}(W)$ is Usher's boundary depth, i.e., the maximal bar in $\mathrm{SH}(W)$.

Theorem (Irie, Shon–G. '18): $\mathrm{SH}^\infty(W) = 0 \Longrightarrow \beta_{\mathrm{max}} < \infty$.

Remark: Upper bound = non-equivariant *SH*-capacity. In fact, we need a more precise result:

Theorem (ÇGGM '23): Assume that $\mathrm{SH}(W) = 0$. Fix $a > 0$ and let H be a semi-admissible Hamiltonian with $slope(H) > a$. Then there exists a constant $C > 0$ depending only on H such that for every sufficiently large $k \in \mathbb{N}$ and any $\tau < ka$ the inclusion/quotient map

$$\mathrm{HF}^\tau(kH) \to \mathrm{HF}^{\tau+C}(kH) \text{ is zero.}$$

Hence, every bar I ending $< ka$ has $|I| < C$. (Note: $\mathrm{HF}^\infty(kH) \neq 0$.)

About proofs: Crossing energy

Ingredients:

- z is a locally maximal (e.g., hyperbolic) closed Reeb orbit of period T.
- H is semi-admissible with $slope(H) > T$.
- $\tilde{z}$ is the corresponding orbit (never locally maximal) of H.
- Iterated orbits – z^k and $\tilde{z}^k$. Note: $\tilde{z}^k$ is a one-periodic orbit of kH.
- An admissible almost complex structure.

Theorem (Crossing Energy, ÇGGM 2309.04576): Under a minor additional requirement on H, there exists $\sigma > 0$ such that $E(u) \geq \sigma$ for any $k \in \mathbb{N}$ and any Floer cylinder $u\colon \mathbb{R} \times S^1 \to \widehat{W}$ of kH asymptotic, at either end, to $\tilde{z}^k$.

Remark: A similar result for periodic orbits z in a locally maximal hyperbolic set of the Reeb flow (ÇGGM, arXiv:2401.01421) $\Rightarrow$ applications to barcode entropy (Theorem C).

About proofs: Crossing energy

Key point of the proof (Çineli): u cannot get too close to W in $\widehat{W}$!

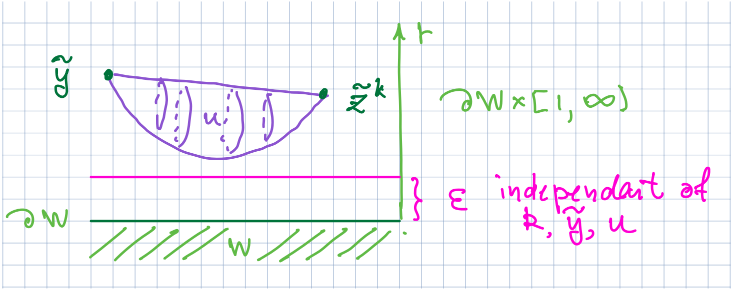

Fig 2: Key point: u stays away from W.

Remark: This is a new result and it does not follow from any previously known fact about the behavior of Floer cylinders in $\widehat{W}$.

About proofs: Index recurrence

Setting and notation: r non-degenerate elements $\Phi_1, \ldots, \Phi_r$ in $\widetilde{Sp}(2n)$ with positive mean indices $\hat{\mu}(\Phi_i) > 0$. Set $\mu_i(k) := \mu\big(\Phi_i^k\big)$ for $k \neq 0$ ... .

Index Recurrence Theorem – Non-degenerate Version; GG '20): For every $N > 0$ (large) and every $\epsilon > 0$ (small), there exist r integer sequences $k_{ij} \to \infty$ as $j \to \infty$ and $i = 1, \ldots, r$, and an integer sequence $d_j \to \infty$ such that for every $1 \leq |\ell| \leq N$

(i) $\big|\hat{\mu}_i\big(k_{ij}\big) - d_j\big| < \epsilon$ and

(ii) $\mu_i\big(k_{ij}\big) = d_j + \mu_i(\ell)$.

Explanation: Arbitrary long segments $[\mu_i(-N), \ldots, \mu_i(N)]$ (with $\mu_i\big(0\big)$ deleted) repeat themselves infinitely many times in the sequences $\mu_i\big(k\big)$ up to a common index shift d; in the derivative sequence $\mu_i(k) - \mu_i(k-1)$ every interval repeats itself infinitely many times. Hence, *recurrence*! An *IR event:* $\{d_j, k_{1j}, \ldots, k_{rj}\}$.

Closely related: The common jump theorem; Long–Zhu '02, ...

About proofs: Index recurrence

We need a very particular case of the IRT.

Corollary: Assume that all Φ_i are dynamically convex: $\mu(\Phi_i) \geq n+1$. Then there exist r integer sequences $k_{ij} \to \infty$ as $j \to \infty$ and $i = 1, \ldots, r$, and an integer sequence $d_j \to \infty$ such that

(i) $\left|\mu_i(k_{ij}) - d_j\right| \leq n-1$ and $\left|\hat{\mu}_i(k_{ij}) - d_j\right| \leq \epsilon$, and

(ii) $\left|\mu_i(k_{ij}) - \mu_i(k)\right| \geq n+1$ when $k \neq k_{ij}$.

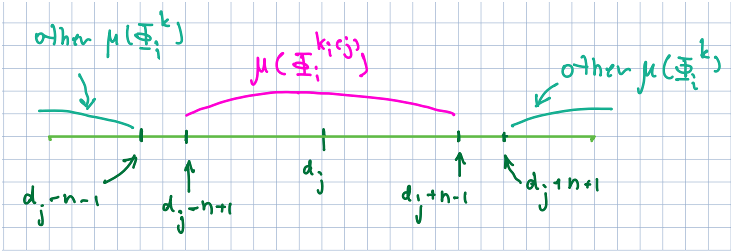

Fig 3: Indices: an IR event in a DC setting.

About proofs: Outline

Simplifying assumptions:

- Non-degeneracy and DC.
- Working with SH(W) including crossing energy rather than HF(H).
- Focus on Theorem A.

Theorem (a weaker version of Theorem A): A non-degenerate dynamically convex Reeb flow on S^{2n-1} with a hyperbolic closed Reeb orbit has infinitely many closed Reeb orbits.

About proofs: Outline

Generators of the complex CSH(W) where W is a star shaped domain filling of (S^{2n-1}, α): Two generators $\check{y}$ and $\hat{y}$ with $|\check{y}| = \mu(y)$ and $|\hat{y}| = \mu(y) + 1$ for every $y \in \mathcal{P}$ and one generator of degree n for the interior of W.

By contradiction, assume that $\mathring{\mathcal{P}}$ is finite: $\mathring{\mathcal{P}} = \{x_0 = z, x_1, \ldots, x_r\}$ with actions $a_0, \ldots, a_r$; z is hyperbolic. Can assume $a_0 / \hat{\mu}(z) = 1$.

Consider an *IR event*: $\{d, k_0, \ldots, k_r\}$ suppressing j (large!) in the notation. Note: $d = \hat{\mu}(z) = \mu(z)$.

Key observation: $\check{z}^{k_0}$ is a non-exact cycle in CSH(W) $\Rightarrow$ SH(W) $\neq 0 \Rightarrow$ contradiction.

About proofs: Outline

Two groups of orbits:

- Group I: $a_i/\hat{\mu}(x_i) = a_0/\hat{\mu}(z) = 1$; action close to d.
- Group II: $a_i/\hat{\mu}(x_i) \neq a_0/\hat{\mu}(z)$; action far from d.

No differential connecting to $\check{z}$:

- Iterates of Group I within an IR event: action difference is too small (Energy Crossing).
- Iterates of Group II within an IR event: action difference is too large (Upper bound on the boundary depth).
- Other iterates: index difference > 1.

About proofs: Outline

Visualizing an IR event on the action/index plane:

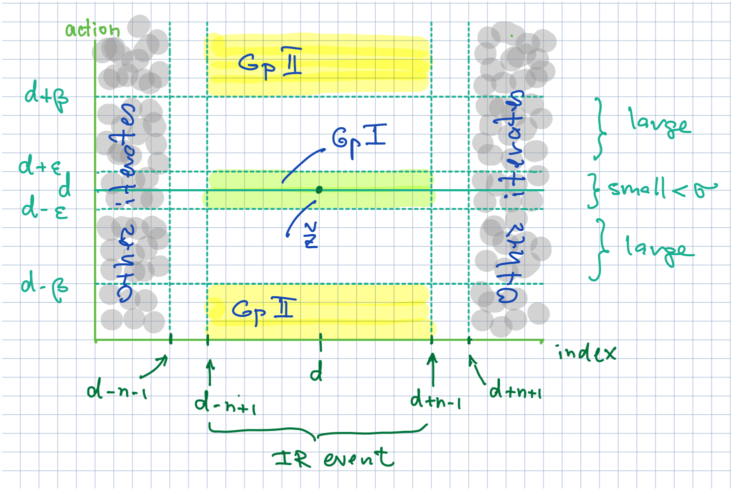

Fig 4: IR event on the index/action plane.

Thanks!