Symplectic structures from almost symplectic structures

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Background

Symplectic topology: existence problem

Definition

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- $d\omega = 0$.
- ω non-degenerate.

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Existence problem

Which manifolds are symplectic?

Formal symplectic structures

Obvious *topological* obstruction:

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A **formal** symplectic manifold is (M^{2n}, Ω, θ) where

- Ω non-degenerate 2-form (i.e. *almost* symplectic).
- $\theta \in H^2(M, \mathbb{R})$, with $\theta^n \neq 0$ if M closed.

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Existence problem (McDuff–Salamon, Problem 1)

Given (M^{2n}, Ω, θ) a formal symplectic manifold, does there exist a symplectic form ω on M such that

- ω is homotopic to Ω through almost symplectic forms, and
- $[\omega] = \theta$?

Some known facts

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For **closed** manifolds, the situation is very different:

Theorem (Taubes '94)

There exist formal symplectic closed 4-manifolds which are not symplectic.

For instance, take $M = (k\mathbb{C}P^2)\#(h\overline{\mathbb{C}P^2})$, with $k \geqslant 3$ odd and $h \geqslant 0$.

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Theorem (Bertelson–Meigniez '21)

*If (M, Ω) closed almost symplectic manifold, then Ω can be homotoped to a **conformal** symplectic structure.*

The symplectic existence problem is wide open in dim $\geqslant$ 6. To my knowledge, what follows are the first higher-dimensional results.

Results

Stabilized existence problem

We will consider a "stabilized" existence problem.

Definition

Given a formal symplectic manifold (M, Ω, θ), its **stabilization** is

$$(M \times \mathbb{T}^2, \Omega + \mu, \theta + [\mu]),$$

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Stabilized existence problem

When can the stabilization of a formal symplectic manifold be realized by a symplectic form?

Main theorem

Definition

A (positive) **symplectic divisor** in a formal symplectic manifold (M, Ω, θ) is a codimension-2 submanifold Σ such that:

1. Up to homotopy of Ω among almost symplectic structures, $\Omega|_\Sigma$ is a symplectic structure on Σ.
2. $\theta = \mathrm{PD}(\Sigma)$ is the Poincaré dual of Σ.
3. $c_1(N_\Sigma) = [\Omega|_\Sigma] \in H^2(\Sigma; \mathbb{R})$.

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Our main theorem is the following.

Theorem (Gironella–M.–Presas–Toussaint '24)

Let (M^{2n}, Ω, θ) be a formal symplectic manifold which admits a positive symplectic divisor. Then, its stabilization can be realized by a symplectic form.

Recall: Up to scaling, symplectic manifolds have positive symplectic divisors (Donaldson '96).

Corollary

Let M be a closed 4-manifold.

1. *If M is simply-connected and admits a formal symplectic structure, then $M \times \mathbb{T}^2$ admits a symplectic structure.*
2. *If M is almost symplectic and $\min(b_2^+, b_2^-) \geqslant 2$, then $M \times \mathbb{T}^2$ admits a symplectic structure.*

In particular, if $M = (k\mathbb{C}P^2)\#(h\overline{\mathbb{C}P^2})$, with $k \geqslant 3$ odd and $h \geqslant 0$, then $M \times \mathbb{T}^2$ is symplectic, even though M is *not*.

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Remarks

- Formal symplecticity is *topological* (cf. Wu '52), and in 4D so is having symplectic divisors (Bohr '00).
- (1) builds on Donaldson ('83) and Freedmann's ('82) celebrated work.
- (2) uses main thm, and existence of holomorphic curves (Bohr '00).

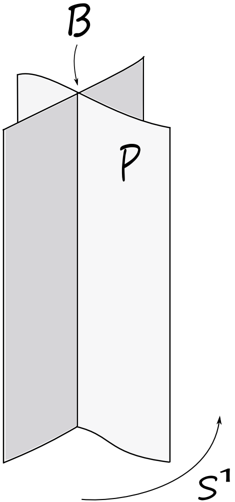
Preliminaries

Open book decompositions

An **OBD** on M is a fibration

$$\pi : M \backslash B \to S^1,$$

with $B \subset M$ codim-2, and $\pi(b, r, \theta) = \theta$ on $B \times \mathbb{D}^2$.



Notation

We write $M = \textbf{OBD}(P, \phi)$, where

- $P = \overline{\pi^{-1}}(pt) = \textit{page}$;
- $B = \partial P = \textit{binding}$;
- $\phi : P \xrightarrow{\cong} P$ *monodromy*, $\phi|_B = \textit{id}$.

Contact manifolds

A **contact manifold** is $(N^{2n-1}, \xi = \ker \alpha)$, α 1-form, $\alpha \wedge d\alpha^{n-1} > 0$.

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Dichotomy: Rigidity vs. Flexibility.

- **tight** *(rigid/geometric)*;
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- **tight** *(rigid/geometric)*;
- **overtwisted** *(flexible/topological)*.

Eliashberg '93, Borman–Eliashberg–Murphy '15:

- Every **almost** contact structure is homotopic to a unique overtwisted contact structure.
- In particular, every contact structure is homotopic to a unique overtwisted contact structure (through **almost** contact structures).

Giroux correspondence

Giroux: Contact structures are *supported* by open books.

Given (N, ξ), there is $N = \mathbf{OBD}(P, \phi)$ and a *Giroux form* α for ξ with

- $d\alpha|_{int(P)} > 0$, and
- $\alpha|_B$ contact.

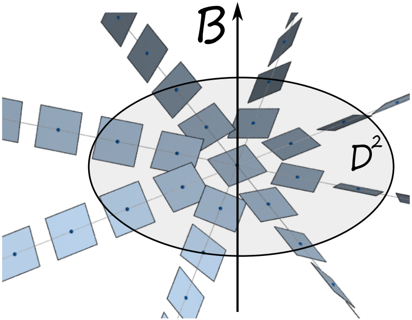

Figure: Supported contact structure.

Bourgeois contact structures

Theorem (Bourgeois '02)

Open book $(M, \xi) = \textbf{\textit{OBD}}(P, \phi) \rightsquigarrow$ *contact structure* $(M \times \mathbb{T}^2, \xi_{BO})$.

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If α Giroux form, $(q_1, q_2) \in \mathbb{T}^2$, and $\Phi = (\Phi_1, \Phi_2) : M \to \mathbb{R}^2$ with $\pi = \frac{\Phi}{|\Phi|}$, then

$\alpha_{BO} = \alpha + \Phi_1 dq_1 - \Phi_2 dq_2$, **Bourgeois form** on $M \times \mathbb{T}^2, \xi_{BO} = \ker \alpha_{BO}$.

Denote $(M \times \mathbb{T}^2, \xi_{BO}) = BO(P, \phi)$.

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Lemma

If $(M,\xi) = \textbf{\textit{OBD}}(P_1,\phi_1) = \textbf{\textit{OBD}}(P_2,\phi_2)$*, there is a symplectic cobordism* $([0,1] \times M \times \mathbb{T}^2, \omega)$ *between* $BO(P_1,\phi_1)$ *and* $BO(P_2,\phi_2)$.

Eliashberg–Murphy's h-principle

Let (W, Ω) be an **almost** symplectic cobordism of dimension $2n \geqslant 4$ between non-empty contact manifolds $(N_\pm, \xi_\pm)$.

Theorem (Eliashberg–Murphy '23)

Suppose that ξ_- is overtwisted, and if $n = 2$, assume ξ_+ is overtwisted.

Then there exists an exact symplectic form $\omega = d\lambda$ on W which is homotopic to Ω as almost symplectic structures, ("conformally") relative boundary.

Main auxiliary result

Stabilized h-principle

Let (W, Ω) be an almost symplectic cobordism of dimension $2n \geqslant 4$ between non-empty contact manifolds $(N_\pm, \xi_\pm)$. Fix an area form μ on $\mathbb{T}^2$.

Theorem (Gironella–M.–Presas–Toussaint '24)

$W \times \mathbb{T}^2$ admits a symplectic form ω such that:

- (i) *ω is homotopic to $\Omega + \mu$, through almost symplectic forms, ("conformally") relative boundary.*
- (ii) *$[\omega] = [\mu] \in H^2(W \times \mathbb{T}^2)$.*

This is the main technical input for the main result.

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Note: no assumptions on overtwistedness.

Proof of main theorem

Let (M, Ω, θ) be a formal symplectic manifold, with Σ a positive symplectic divisor.

Lemma

Up to homotopy of Ω, Σ admits a symplectically concave neighbourhood N_Σ, with contact boundary $(Y = \partial N_\Sigma, \xi_Y)$.

Proof of main theorem.

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$$M = B \cup X \cup N_\Sigma,$$

with X almost symplectic cobordism from $(\mathbb{S}^{2n-1}, \xi_{st})$ to (Y, ξ_Y).

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with X almost symplectic cobordism from $(\mathbb{S}^{2n-1}, \xi_{st})$ to (Y, ξ_Y). The pieces $B \times \mathbb{T}^2$, $N_\Sigma \times \mathbb{T}^2$ have obvious product symplectic structures. And $X \times \mathbb{T}^2$ has a symplectic structure by the stabilized h-principle. $\square$

Proof of the main theorem

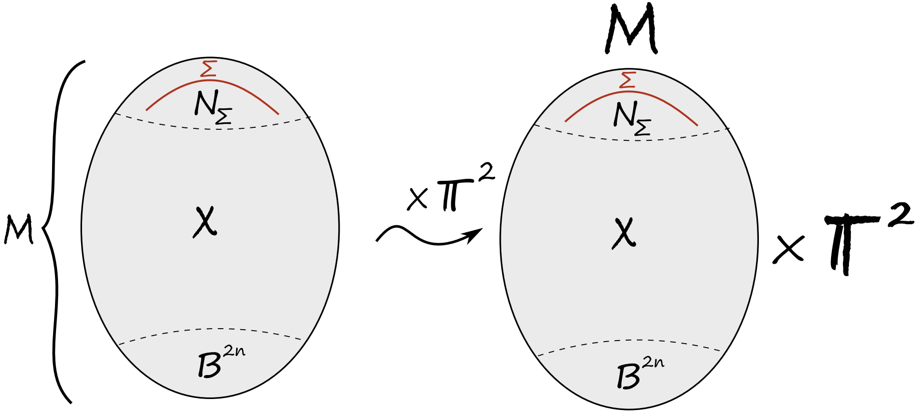

Figure: The splitting $M = B \cup X \cup N_\Sigma$.

Proof of main auxiliary result

Case of a concordance

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Fact

There is always an almost symplectic concordance

$$(X = [0, 1] \times N, \Omega)$$

between a closed contact manifold (N^{2n-1}, ξ) and (N^{2n-1}, ξ_{ot}), where ξ_{ot} is the unique overtwisted structure with $\xi_{ot} \simeq \xi$.

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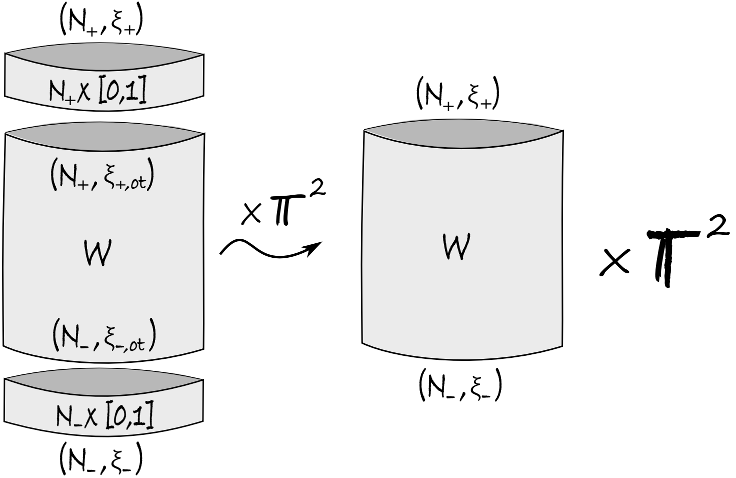
Theorem (GMPT '24, Stabilized h-principle for concordances)

The stabilized h-principle holds on $X \times \mathbb{T}^2$.

From concordance to general case



The general stabilized h-principle follows: add such concordances near the boundary, use the above, and Eliashberg–Murphy in the complement (which now has overtwisted ends).

Monodromy inversion theorem

(N, ξ) contact manifold, ξ_{ot} = unique overtwisted structure with $\xi_{ot} \simeq \xi$.

Theorem (Gironella–M.–Presas–Toussaint '24)

There exist open books $N = OBD(P, \phi) = OBD(P_{ot}, \phi_{ot})$ respectively supporting ξ and ξ_{ot}, such that

$$\overline{N} = OBD(P, \phi^{-1}) = OBD(P_{ot}, \phi_{ot}^{-1})$$

support two contact structures ξ^{inv}, ξ_{ot}^{inv}, which are overtwisted and isomorphic.

Sketch of proof of monodromy inversion theorem

By connected sum, suffices with $(N, \xi) = (\mathbb{S}^{2n-1}, \xi_{st})$.

Almost contact structures on spheres, Harris '63

$$\text{ACont}(\mathbb{S}^{2n-1}) \cong \pi_{2n-1}(SO(2n)/U(n)) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z}_2 & \text{if } n \equiv 0 \mod 4 \\ \mathbb{Z}_{(n-1)!} & \text{if } n \equiv 1 \mod 4 \\ \mathbb{Z} & \text{if } n \equiv 2 \mod 4 \\ \mathbb{Z}_{(n-1)!/2} & \text{if } n \equiv 3 \mod 4 \end{cases}$$

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- **Case 1:** n **odd** (finite group).
- **Case 2:** n **even** (infinite group).

Case 1: finite group.

Take $1 < k = |\text{ACont}(\mathbb{S}^{2n-1})| < \infty$. Let

- $(\mathbb{S}^{2n-1}, \xi_{st}) = OBD(D^*\mathbb{S}^{n-1}, \tau)$, where $\tau =$ Dehn–Seidel twist.
- Choose $(\mathbb{S}^{2n-1}, \xi_{ot}) = OBD(P, \phi)$.

- Since ξ_{st} represents zero:

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Invert the monodromy $\rightsquigarrow$ two negative stabilizations (hence over-twisted by Casals–Murphy–Presas) representing zero, done.

Case 2: infinite group.

Similar argument, but matching elements in AContt$(\mathbb{S}^{2n-1})$ is trickier and needs:

Lemma (Gironella–M.–Presas–Toussaint '24)

Let $(\mathbb{S}^{2n-1}, \xi_{neg}) = OBD(D^*\mathbb{S}^{n-1}, \tau^{-1})$*. If n even,* ξ_{neg} *has infinite order in AContt*$(\mathbb{S}^{2n-1})$*.*

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Remark: If $n = 2$, ξ_{neg} corresponds to $-1 \in \pi_3(SO(4)/U(2)) = \mathbb{Z}$. For $n > 2$, seems unknown (?).

Proof of stabilized h-pple for concordances

$X = N \times [0,1]$ almost symplectic concordance between (N, ξ) and (N, ξ_{ot}). Let $\overline{N} = OBD(P, \phi^{-1}) = OBD(P_{ot}, \phi_{ot}^{-1})$ supporting $\xi^{inv} \cong \xi_{ot}^{inv}$. From properties of Bourgeois manifolds:

Lemma

Since $\xi^{inv} \cong \xi_{ot}^{inv}$, there is a symplectic concordance

$$(\overline{Y} = [0,1] \times \overline{N} \times \overline{\mathbb{T}^2}, \overline{\omega})$$

between the Bourgeois manifolds $BO(P, \phi^{-1})$ and $BO(P_{ot}, \phi_{ot}^{-1})$, with $[\overline{\omega}] = -[\mu]$.

Recall: $BO(P, \phi) \cong BO(P, \phi^{-1})$, $BO(P_{ot}, \phi_{ot}) \cong BO(P_{ot}, \phi_{ot}^{-1})$.

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These diffeomorphisms are both isotopic to

$$N \times \mathbb{T}^2 \to \overline{N} \times \overline{\mathbb{T}^2},\ (p, q_1, q_2) \mapsto (p, q_1, -q_2),$$

and so extend to a concordance:

$$F : Y = [0,1] \times N \times \mathbb{T}^2 \xrightarrow{\cong} \overline{Y} = [0,1] \times \overline{N} \times \overline{\mathbb{T}^2}.$$

We let $\omega = F^*\overline{\omega}$.

End of proof

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- **(homology class)** The symplectic form ω lies in the cohomology class $\theta + [\mu]$ by construction.

Thank you!