Equivariant Lagrangian Floer theory on compact toric manifolds

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Outline

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2. The setup
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   - 2. $HF_G((L(u), b)) \neq 0 \Leftrightarrow (u, b) \in \mathrm{Crit}_G^{\Delta}(\mathfrak{P}\mathfrak{O})$
   - 3. $\mathrm{Crit}_G^{\Delta}(\mathfrak{P}\mathfrak{O})$ is a rigid analytic space
   - 4. Non-trivial G-equivariant Lagrangian Floer cohomology implies G-nondisplaceability
   - Examples

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3. Equivariant homological mirror symmetry

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$$\Delta = \mu(X) = \bigcap_{i=1}^{m} \{u \in \mathfrak{t}^* \cong \mathbb{R}^n \mid \langle u, v_i \rangle - \lambda_i \geq 0\} \tag{1}$$

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- Let $0 \leq r \leq n$ and $G \cong T^r$ be a subtorus of T^n with induced action.
- For each $u \in \operatorname{int} \Delta$, $L(u) := \mu^{-1}(u)$ is a T^n-invariant Lagrangian torus with a T^n-invariant relatively spin structure. Choose a basis $\{e_1, \ldots, e_n\}$ for $H^1(\mu^{-1}(u), \mathbb{R})$.

- Our A_∞-algebra takes coefficients either in the universal Novikov ring

$$\Lambda_{0,nov} = \left\{ \sum_{i \in \mathbb{N}} a_i T^{\lambda_i} e^{n_i} \;\middle|\; \begin{array}{ll} a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, n_i \in \mathbb{Z}, & \forall i \in \mathbb{N} \\ 0 \leq \lambda_0 < \lambda_1 < \cdots, & \lim_{i \to \infty} \lambda_i = \infty \end{array} \right\} \tag{2}$$

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- We will also use the Novikov ring

$$\Lambda_0 = \left\{ x = \sum_{i\in\mathbb{N}} a_i T^{\lambda_i} \,\middle|\, x \in \Lambda_{0,nov} \right\} \tag{3}$$

  and $\Lambda = \mathrm{Frac}(\Lambda_0)$.

1. Definition of $HF_G((L(u),b),(L(u),b),\Lambda_{0,nov})$

Theorem 1

We define a G-equivariant Lagrangian Floer cohomology

$$HF_G((L(u),b),(L(u),b),\Lambda_{0,\mathrm{nov}})$$

for each pair $(L(u),b)$ consisting of

- a Lagrangian torus fiber $L(u)=\mu^{-1}(u)$ of the toric moment map and
- an element $b\in H^1(L(u),\Lambda_0/(2\pi i\mathbb{Z}))$.

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Sketch Proof. Recall that the Cartan complex for the G-manifold L is given by

$$(\Omega(L) \otimes S(\mathfrak{g}^*))^G .$$

We define a G-equivariant A_∞ algebra

$$\left(\Omega_G(L, \mathbb{R}) \widehat{\otimes} \Lambda_{0,\mathrm{nov}}, \{(\mathfrak{m}_k^G)^b\}_{k \in \mathbb{N}}\right) .$$

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The G-equivariant A_∞ algebra

$$\left(\Omega_G(L, \mathbb{R})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, \{(\mathfrak{m}_k^G)^b\}_{k\in\mathbb{N}}\right)$$

satisfies $(\mathfrak{m}_1^G)^b \circ (\mathfrak{m}_1^G)^b = 0$. Define the G-equivariant Lagrangian Floer cohomology by

$$HF_G\left((L(u), b), (L(u), b), \Lambda_{0,\mathrm{nov}}\right) = H^*\left(\Omega_G(L, \mathbb{R})\widehat{\otimes}\Lambda_{0,\mathrm{nov}}, (\mathfrak{m}_1^G)^b\right).$$

$(\mathfrak{m}_0)^b_G(1) = \mathfrak{PO}^X(L(u), b)e$

Lemma

$(\mathfrak{m}_0)^b_G(1) = \mathfrak{PO}^X(L(u), b)e$, where the potential function $\mathfrak{PO} : \bigcup_{u \in \mathrm{int}\,\Delta} \{u\} \times H^1(L(u), \Lambda_0/2\pi i\mathbb{Z}) \to \Lambda$ can be extended

$$\begin{array}{ccc} \bigcup_{u \in \mathrm{int}\,\Delta} \{u\} \times H^1(L(u), \Lambda_0/(2\pi i\mathbb{Z})) & \longrightarrow & \Lambda\,. \\ \downarrow \iota & \nearrow_{\mathfrak{PO}} & \\ (\Lambda^*)^n & & \end{array}$$

to a formal Laurent series $\mathfrak{PO} : (\Lambda^*)^n \to \Lambda$ via an embedding

$$\iota\left(u_1, \ldots, u_n, \sum_{i=1}^n x_i e_i\right) = (y_1, \ldots, y_n),$$

where $y_i = \exp^{x_i} T^{u_i}$ for $1 \leq i \leq n$.

Note that Λ^* comes with a non-archimedean valuation function

$$\mathrm{val} : \Lambda^* \to \mathbb{R}, \qquad \mathrm{val}\left(\sum_{i\in\mathbb{N}} a_i T^{\lambda_i}\right) = \min\{\lambda_i \mid a_i \neq 0\}.$$

We call the coordinate-wise valuation map $\mathtt{trop} : (\Lambda^*)^n \to \mathbb{R}$ the tropicalization map.

2. $HF_G((L(u), b)) \neq 0 \Leftrightarrow (u, b) \in \mathrm{Crit}_G^{\Delta}(\mathfrak{PO})$

Theorem 2

The pair $(L(u), b)$ has non-zero G-equivariant cohomology if and only if $\iota(u, b) \in \mathrm{Crit}_G^{\Delta}(\mathfrak{PO})$, where

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = \left\{ y = (y_1, \ldots, y_n) \,\middle|\, (\nabla \mathfrak{PO})|_{H_G^1(L(u))}(y) = 0 \right\} \cap \mathrm{trop}^{-1}(\mathrm{int}\,\Delta).$$

2. $HF_G((L(u), b)) \neq 0 \Leftrightarrow (u, b) \in \mathrm{Crit}^{\Delta}_G(\mathfrak{PO})$

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This follows from

Theorem 3

There is a spectral sequence such that

$$E_2 \cong H_G(L(u), \Lambda_{0,\mathrm{nov}}) \Rightarrow E_\infty \cong HF_G((L(u), b), (L(u), b), \Lambda_{0,\mathrm{nov}}),$$

which collapses at E_2 if and only if $(\mathfrak{m}_1^G)^b|_{H_G(L(u))} = 0$.

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In particular, this allows us to locate the Lagrangian torus fibers with non-trivial G-equivariant Lagrangian Floer cohomology by tropicalizing this rigid analytic space $\mathrm{Crit}_G^{\Delta}(\mathfrak{PO})$.

4. G-nondisplaceability

Theorem 5

A Lagrangian toric fiber $L(u)$ of the moment map with a non-zero G-equivariant Floer cohomology $HF_G(L(u), b)$ is not displaceable by any G-equivariant Hamiltonian diffeomorphism.

Example: $\mathbb{CP}^2$

Consider $(\mathbb{CP}^2, \omega, T^2, \mu)$ associated with the moment polytope

$$\Delta = \left\{ (u_1, u_2) \in \mathbb{R}^2 \,\middle|\, \begin{array}{l} u_i \geq 0 \quad \forall 1 \leq i \leq 2, \\ 1 - u_1 - u_2 \geq 0 \end{array} \right\}.$$

Its potential function is $\mathfrak{PO} = y_1 + y_2 + \frac{T}{y_1 y_2}$.

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Its potential function is $\mathfrak{PO} = y_1 + y_2 + \frac{T}{y_1 y_2}$.
(I) Consider the action by the subtorus $S^1 \hookrightarrow T^2$, $t \mapsto (k_1 t, k_2 t)$, where $k_1, k_2 \in \mathbb{Z}$ are not both zero.

$$\implies 0 = f := \frac{\partial \mathfrak{PO}}{\partial c_2} = -k_2 y_1 + k_1 y_2 + (k_2 - k_1)\frac{T}{y_1 y_2}.$$

Case 1: Suppose $k_1, k_2, k_2 - k_1$ are all non-zero. Then

$$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}, \quad (u_1, u_2) \mapsto \min\{u_1, u_2, 1 - u_1 - u_2\}.$$

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We have

$$W := V(f) \cap \mathrm{trop}^{-1}(\Delta) = \mathrm{Sp}\, \frac{\Lambda\left\langle y_1, y_2, \frac{T}{y_1 y_2}\right\rangle}{\left(-k_2 y_1 + k_1 y_2 + (k_2 - k_1)\frac{T}{y_1 y_2}\right)}$$

$$\cong \mathrm{Sp}\, \frac{\Lambda\langle y_1, y_2, z\rangle}{(-k_2 y_1 + k_1 y_2 + (k_2 - k_1)z, y_1 y_2 z - T)},$$

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and

$$\mathrm{Crit}^{\Delta}_{G}(\mathfrak{PO}) = W \setminus \mathrm{trop}^{-1}\left(\{(0,0), (0,1), (1,0)\}\right).$$

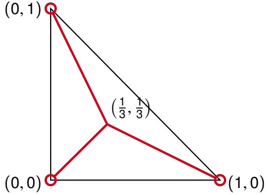

Figure: Case when $k_1, k_2, k_1 - k_2 \neq 0$

Case 2 & 3: $k_1 = 0$ and $k_2 \neq 0$.

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&= V(f) \cap \mathrm{trop}^{-1}(\mathrm{int}\,\Delta) \\
&= \left\{ (y_1, y_2) \in (\Lambda^*)^2 \;\middle|\; \begin{array}{l} -k_2 y_1 + k_2 \dfrac{T}{y_1 y_2} = 0, \mathrm{val}(y_1) > 0, \\ \mathrm{val}\left(\dfrac{T}{y_1^2}\right) > 0, \mathrm{val}\left(\dfrac{T}{y_1 T / y_1^2}\right) > 0 \end{array} \right\} \\
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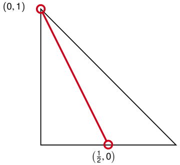

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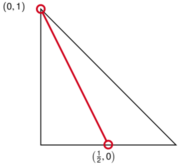

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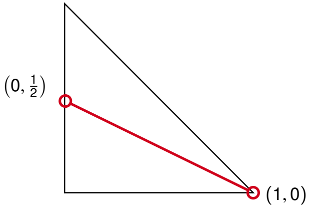

Figure: Case when $k_2 = 0$

$\mathbb{CP}^2$ with trivial subtorus action

(II) Take the subtorus $G = \{e\} \hookrightarrow T^2$.

$$\Longrightarrow \begin{cases} 0 & = f_1 := \frac{\partial \mathfrak{PO}}{\partial x_1} = y_1 - \frac{T}{y_1 y_2} \\ 0 & = f_2 := \frac{\partial \mathfrak{PO}}{\partial x_2} = y_2 - \frac{T}{y_1 y_2}. \end{cases}$$

Then

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) \cong \mathrm{Sp}\, \frac{\Lambda\langle y_1, y_2, z\rangle}{(y_1 - z, y_2 - z, y_1 y_2 z - T)} \cap \mathrm{trop}^{-1}(\mathrm{int}\,\Delta).$$

$$\mathrm{trop}\,\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = \left\{\left(\frac{1}{3}, \frac{1}{3}\right)\right\}$$

is the barycenter of the polytope.

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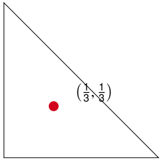

Example: $S^2(c/2) \times S^2(d/2)$

Denote by $S^2(r)$ the 2-sphere with radius r. Let $0 < c < d, c, d \in \mathbb{N}$. Consider $\left(S^2(c/2) \times S^2(d/2), \omega, T^2, \mu\right)$ whose moment polytope is given by

$$\Delta = \left\{(u_1, u_2) \in \mathbb{R}^2 \,\middle|\, 0 \leq u_1 \leq c, 0 \leq u_2 \leq d\right\}.$$

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Case 1: Suppose $k_1, k_2 \neq 0$. Then

$$\mathrm{trop}(f) : \mathbb{R}^2 \to \mathbb{R}, \qquad (u_1, u_2) \mapsto \min\{u_1, c - u_1, u_2, d - u_2\}.$$

We have

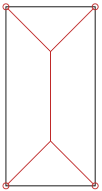
$$W = V(f) \cap \mathrm{trop}^{-1}(\Delta) = \mathrm{Sp}\, \frac{\Lambda \left\langle y_1, y_2, y_1^{-1} T^c, y_2^{-1} T^d \right\rangle}{\left(-k_2 \left(y_1 - y_1^{-1} T^c\right) + k_1 \left(y_2 - y_2^{-1} T^d\right)\right)}$$

$$\cong \mathrm{Sp}\, \frac{\Lambda \langle y_1, y_2, x_1, x_2 \rangle}{\left(-k_2 \left(y_1 - x_1\right) + k_1 \left(y_2 - x_2\right), x_1 y_1 - T^c, x_2 y_2 - T^d\right)}$$

and

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = W \setminus \mathrm{trop}^{-1} \left\{(c, 0), (0, 0), (0, d), (c, d)\right\}$$

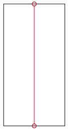
The Lagrangian torus fibers with nontrivial equivariant Floer cohomology can be visualized in the moment polytope via the tropicalization map.



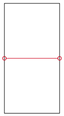
Case 2 & 3: Suppose $k_1 = 0$ and $k_2 \neq 0$. Then $f = -k_2\left(y_1 - \frac{T^c}{y_1}\right)$. Then

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = \left\{\left(T^{\frac{c}{2}}, y_2\right) \mid 0 < \mathrm{val}(y_2) < d\right\} \cup \left\{\left(-T^{\frac{c}{2}}, y_2\right) \mid 0 < \mathrm{val}(y_2) < d\right\},$$

which is a union of analytic annuli. The $k_2 = 0, k_1 \neq 0$ case is similar. The Lagrangians with nontrivial S^1-equivariant Floer cohomology can be visualized in the moment polytope as below.



(a) Case when $k_1 = 0$



(b) Case when $k_2 = 0$

$S^2(c/2) \times S^2(d/2)$ with trivial subtorus action

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$$\Longrightarrow \begin{cases} 0 & = f_1 := \frac{\partial \mathfrak{PO}}{\partial x_1} = y_1 + -y_1^{-1} T^c \\ 0 & = f_2 := \frac{\partial \mathfrak{PO}}{\partial x_2} = y_2 - y_2^{-1} T^d. \end{cases}$$

Then

$$\mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) \cong \mathrm{Sp}\, \frac{\Lambda \langle y_1, y_2, x_1, x_2 \rangle}{(y_1 - x_1, y_2 - x_2, x_1 y_1 - T^c, x_2 y_2 - T^d)} \cap \mathrm{trop}^{-1}(\mathrm{int}\, \Delta).$$

$$\mathrm{trop}\, \mathrm{Crit}_G^{\Delta}(\mathfrak{PO}) = \left\{ \left(\frac{c}{2}, \frac{d}{2} \right) \right\}$$

is the barycenter of the polytope.

(II) Take the subtorus $G = \{e\} \hookrightarrow T^2$.

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$S^2(c/2) \times S^2(d/2)$ with trivial subtorus action

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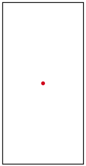

Thank you very much for your attention!