Quantitative Floer theory and coefficients

Yusuke Kawamoto
(joint work with Egor Shelukhin)

ETH Zürich

Symplectic Zoominar
March 15 (Fri), 2024

Introduction

- Homology theory depends on the choice of a coefficient ring.
- HF over different rings:
  1. Some Lagrangians have non-zero HF only over specific fields, e.g. $\mathbb{R}P^2$ in $\mathbb{C}P^2$.
  2. Arnold conjecture: obtain better bounds of the fixed points of Hamiltonian diffeomorphisms;
     - over $\mathbb{F}_p$ (a field of characteristic p) by Abouzaid–Blumberg,
     - over $\mathbb{Z}$ by Bao–Xu.

Theme

How much does the choice of a coefficient ring to set-up Floer theory impact the quantitative information (i.e. spectral invariants)?

Quick review of spectral invariants

- Spectral invariants are important quantitative information of Floer theory along with boundary depths (or barcodes).
- Pick a ring R; we get $HF(H;R)$ and $QH(M;R)$, which are related by the PSS-map $PSS_{H;R} : QH(M;R) \xrightarrow{\sim} HF(H;R)$.
- For a pair of a Hamiltonian H and a quantum homology class $a \in QH(M;R) \backslash \{0\}$, we define

$$c_R(H,a) := \inf\{\tau \in \mathbb{R} : PSS_{H;R}(a) \in Im(i_*^\tau)\} \tag{1}$$

where $i_*^\tau : HF^{<\tau}(H;R) \to HF(H;R)$ is the map coming from inclusion.

- Spectral invariants give rise to a metric on $\mathrm{Ham}(M)$:

$$\gamma_R(\phi) := \inf_{\phi_H=\phi} \gamma_R(H), \ \gamma_R(H) := c(H,[M]) + c(\overline{H},[M]),$$

$$d_{\gamma_R}(\phi,\phi') := \gamma_R(\phi^{-1}\phi').$$

Main result

- It is widely known that for $\mathbb{C}P^n$, the spectral norm over a field $\mathbb{K}$ is uniformly bounded (Entov-Polterovich 04):

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{K}}(\phi) \leqslant 1.$$

- This property was crucial in some important work on $\mathbb{C}P^n$, e.g. Ginzburg-Gürel on pseudo-rotations, Shelukhin on Viterbo's conjecture.

Theorem A (K-Shelukhin 23)

For $\mathbb{C}P^n$ with $n > 1$, we have

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{Z}}(\phi) = +\infty. \tag{2}$$

- Remark: for $\mathbb{C}P^2$, we have $\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} \gamma_{\mathbb{Z}/14}(\phi) = +\infty$.

Plan of the talk

I will discuss

- Proof of Thm A.
- Applications of Thm A.
- What is behind the contrast between field coefficients and $\mathbb{Z}$-coefficients (i.e. boundedness vs. divergence)?

Application: Hingston's question

- To study closed geodesics, Hingston uses "spectral invariants": for $\alpha \in H_*(\Lambda M; R)$ (homology of the loop space over ring R), you get $c_R(\alpha) \in \mathbb{R}$ via a variational procedure and posed the following question.

Hingston's question

Does there exist a manifold M and a homogeneous non-torsion class $\alpha \in H_*(\Lambda M; R)$ (R is a ring) such that

$$c_R(k \cdot \alpha) < c_R(\alpha)$$

for some $k \in \mathbb{N}$?

- This question remains widely open; Chambers–Liokumovich showed that for S^2, k odd and $|\alpha| = 1$, the answer is actually negative.

Application: Hingston's question

- We consider the following symplectic counterpart:

Symplectic version of Hingston's question

Does there exist a symplectic manifold (M, ω) and a Hamiltonian H on it such that

$$\inf_{k \in \mathbb{Z}} c_{\mathbb{Z}}(H, k \cdot [M]) < c_{\mathbb{Z}}(H, [M])?$$

Theorem B (K-Shelukhin 23)

Consider $\mathbb{C}P^n$ with $n > 1$. For every non-zero class $a \in QH(\mathbb{C}P^n; \mathbb{Z})$, there is a Hamiltonian H such that

$$\inf_{k \in \mathbb{N}} c_{\mathbb{Z}}(H, k \cdot a) < c_{\mathbb{Z}}(H, a). \tag{3}$$

Application: Pseudo-rotations

- Pseudo-rotations are Hamiltonian diffeomorphisms that have the 'minimal' expected periodic points from the viewpoint of the Arnold conjecture (that is, $n+1$ for $\mathbb{C}P^n$).
- Their dynamical behavior has been studied extensively, but the geometry of the entire set of pseudo-rotations was not studied.

Application: Pseudo-rotations

- Pseudo-rotations are Hamiltonian diffeomorphisms that have the 'minimal' expected periodic points from the viewpoint of the Arnold conjecture (that is, $n+1$ for $\mathbb{C}P^n$).
- Their dynamical behavior has been studied extensively, but the geometry of the entire set of pseudo-rotations was not studied.

Question

What does the set $PR(M,\omega) := \{\phi \in \mathrm{Ham}(M,\omega) \colon \phi \text{ is a pseudo-rotation}\}$ look like in $\mathrm{Ham}(M,\omega)$ wrt the Hofer metric?

- We prove the first result in this direction, which states that the set $PR(M,\omega)$ is "small" in $\mathrm{Ham}(M,\omega)$.

Application: Pseudo-rotations

- Pseudo-rotations are Hamiltonian diffeomorphisms that have the 'minimal' expected periodic points from the viewpoint of the Arnold conjecture (that is, $n+1$ for $\mathbb{C}P^n$).
- Their dynamical behavior has been studied extensively, but the geometry of the entire set of pseudo-rotations was not studied.

Question

What does the set $PR(M,\omega) := \{\phi \in \mathrm{Ham}(M,\omega) : \phi \text{ is a pseudo-rotation}\}$ look like in $\mathrm{Ham}(M,\omega)$ wrt the Hofer metric?

- We prove the first result in this direction, which states that the set $PR(M,\omega)$ is "small" in $\mathrm{Ham}(M,\omega)$.

Theorem C (K-Shelukhin 23)

Consider $\mathbb{C}P^n$ with $n > 1$. Then

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} d_{\mathrm{Hof}}(\phi, PR(\mathbb{C}P^n)) = +\infty. \tag{4}$$

Proof of Thm A

- We look at the case of $\mathbb{C}P^2$.
- Key point: we have two distinguished Lagrangians, namely the Chekanov torus T^2_{Chek} and $\mathbb{R}P^2$ that satisfy the following remarkable properties:
  1. They are disjoint, $T^2_{\mathrm{Chek}} \cap \mathbb{R}P^2 = \emptyset$.
  2. They are both superheavy; the Chekanov torus wrt $1_{\mathbb{C}}$ and $\mathbb{R}P^2$ wrt $1_{\mathbb{Z}/2}$.

Definition: Superheaviness for $\mathbb{C}P^n$

On $\mathbb{C}P^n$, we define the asymp. spectral invariant of $1_R \in QH_*(\mathbb{C}P^n; R)$;

$$\zeta_R : C^\infty(\mathbb{C}P^n) \to \mathbb{R},\ \zeta_R(H) := \lim_{k \to +\infty} \frac{c_R(k \cdot H, 1_R)}{k}.$$

A subset $S \subset \mathbb{C}P^n$ is superheavy wrt. the unit $1_R = [\mathbb{C}P^n] \in QH(\mathbb{C}P^n; R)$ iff for any H, we have $\inf_{x \in S} H(x) \leqslant \zeta_R(H) \leqslant \sup_{x \in S} H(x)$.

- Obvious corollary: if S is 1_R-superheavy, then for a Hamiltonian H s.t. $H|_S = \tau$, we have $\zeta_R(H) = \tau$.
- We now study

$$\mu(H) := \zeta_{\mathbb{C}}(H) + \zeta_{\mathbb{Z}/2}(\overline{H})$$

for a Hamiltonian H ($\overline{H}$ is the inverse Hamiltonian of H).

- Pick any $a \in \mathbb{R}$. Take a Hamiltonian G_a such that $G|_{T^2_{\mathrm{Chek}}} = a$ and $G|_{\mathbb{R}P^2} = 0$ (remember that $T^2_{\mathrm{Chek}} \cap \mathbb{R}P^2 = \emptyset$). The superheaviness implies

$$\zeta_{\mathbb{C}}(G_a) = a, \ \zeta_{\mathbb{Z}/2}(\overline{G_a}) = 0.$$

- Thus,

$$\mu(G_a) = a + 0 = a.$$

- It is easy to see that

$$\zeta_{\mathbb{C}}(H) \leqslant c_{\mathbb{C}}(H, [\mathbb{C}P^2]), \ \zeta_{\mathbb{Z}/2}(H) \leqslant c_{\mathbb{Z}/2}(H, [\mathbb{C}P^2]),$$

so we have

$$\mu(H) \leqslant c_{\mathbb{C}}(H, [\mathbb{C}P^2]) + c_{\mathbb{Z}/2}(\overline{H}, [\mathbb{C}P^2]).$$

Key Lemma

Let R and R' be rings and suppose you have a homomorphism $j : R \to R'$. Let $j\colon QH(M;R) \to QH(M;R')$ be the map induced by it. Then, we have

$$c_{R'}(H, j(a)) \leqslant c_R(H, a)$$

for every Hamiltonian H and $a \in QH(M;R)$.

Proof:

$$\begin{array}{ccccc} HF^{\tau}(H;R) & \xrightarrow{i_*^{\tau}} & HF_*(H;R) & \xleftarrow{PSS_{H;R}} & QH(M;R) \\ \downarrow j & & \downarrow j & & \downarrow j \\ HF^{\tau}(H;R') & \xrightarrow{i_*^{\tau}} & HF(H;R') & \xleftarrow{PSS_{H;R'}} & QH(M;R'). \end{array}$$

- By considering $\mathbb{Z} \to \mathbb{Z}/2$ and $\mathbb{Z} \to \mathbb{C}$, we get, for any H,

$$c_{\mathbb{C}}(H, [\mathbb{C}P^2]) \leqslant c_{\mathbb{Z}}(H, [\mathbb{C}P^2]),$$
$$c_{\mathbb{Z}/2}(H, [\mathbb{C}P^2]) \leqslant c_{\mathbb{Z}}(H, [\mathbb{C}P^2]).$$

Finishing the proof

- Recall that we had
$$\mu(H) \leqslant c_{\mathbb{C}}(H, [\mathbb{C}P^2]) + c_{\mathbb{Z}/2}(\overline{H}, [\mathbb{C}P^2]).$$
- Key lemma implies
$$\mu(H) \leqslant c_{\mathbb{Z}}(H, [\mathbb{C}P^2]) + c_{\mathbb{Z}}(\overline{H}, [\mathbb{C}P^2]) = \gamma_{\mathbb{Z}}(H).$$
- Take $H = G_a$; we get $a = \mu(G_a) \leqslant \gamma_{\mathbb{Z}}(G_a)$ for every $a \in \mathbb{R}$.
- Thus,
$$\sup_H \gamma_{\mathbb{Z}}(H) = +\infty.$$
- This implies
$$\sup_\phi \gamma_{\mathbb{Z}}(\phi) = +\infty.$$
- For $\mathbb{C}P^n$ with $n > 2$, we need to find a pair of Lagrangians that have nice properties (disjointness and superheaviness). We use $\mathbb{R}P^n$ and a Chekanov-type torus by Chanda–Hirschi–Wang ('lifted Vianna tori').

Proof of Thm B

- Recall that we want to prove the following:

Theorem B (K-Shelukhin 23)

For every non-zero class $a \in QH(\mathbb{C}P^n; \mathbb{Z})$ with $n > 1$, there is a Hamiltonian H such that

$$\inf_{k \in \mathbb{N}} c_{\mathbb{Z}}(H, k \cdot a) < c_{\mathbb{Z}}(H, a). \tag{5}$$

- We focus on the case $a = [\mathbb{C}P^n]$;

$$\beta_{\mathrm{spec}}(H) := c_{\mathbb{Z}}(H, [\mathbb{C}P^n]) - \inf_{k \in \mathbb{N}} c_{\mathbb{Z}}(H, k \cdot [\mathbb{C}P^n]) > 0. \tag{6}$$

- Theorem B (or (6)) follows from the following:

Thm (K-Shelukhin 23)

For $\mathbb{C}P^n$ with $n > 1$, we have

$$\gamma_{\mathbb{Z}}(H) \leqslant 1 + \beta_{\mathrm{spec}}(H) + \beta_{\mathrm{spec}}(\overline{H}).$$

- Notice that, as we know from Thm A that there is H s.t. $\gamma_{\mathbb{Z}}(H) > 1$, for such H, Thm implies $\beta_{\text{spec}}(H) > 0$ or $\beta_{\text{spec}}(\overline{H}) > 0$ and we obtain Hingston's question.
- To prove Thm, we need the following lemma:

$\mathbb{Z}$ vs $\mathbb{Q}$ Lemma (K-Shelukhin 23)

On (M, ω), for every Hamiltonian H, we have

$$\inf_{k \in \mathbb{N}} c_{\mathbb{Z}}(H, k \cdot [M]) = c_{\mathbb{Q}}(H, [M]) \tag{7}$$

- By $\mathbb{Z} vs \mathbb{Q}$ Lemma, we have

$$\begin{aligned}
\gamma_{\mathbb{Z}}(H) &= c(H, 1_{\mathbb{Z}}) + c(\overline{H}, 1_{\mathbb{Z}}) \\
&= \left(c(H, 1_{\mathbb{Z}}) - c(H, 1_{\mathbb{Q}})\right) + \left(c(\overline{H}, 1_{\mathbb{Z}}) - c(\overline{H}, 1_{\mathbb{Q}})\right) \\
&\qquad + c(H, 1_{\mathbb{Q}}) + c(\overline{H}, 1_{\mathbb{Q}}) \\
&= \beta_{\text{spec}}(H) + \beta_{\text{spec}}(\overline{H}) + \gamma_{\mathbb{Q}}(H) \\
&\leqslant \beta_{\text{spec}}(H) + \beta_{\text{spec}}(\overline{H}) + 1.
\end{aligned}$$

Proof of Thm C

- Recall that what we want to prove is

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} d_{\mathrm{Hof}}(\phi, PR(\mathbb{C}P^n)) = +\infty. \tag{8}$$

- The main difficulty to study the (Hofer) geometry of the set $PR(M, \omega)$ $\rightarrow$ there was no measurement that distinguishes pseudo-rotations and other Hamiltonian diffeomorphisms
(Boundary depth? For the cases where we know that the boundary depth can diverge, e.g. the 2-torus, there are no pseudo-rotations).

- Our proposal is to use $\gamma_{\mathbb{Z}}$ as a measurement that distinguishes pseudo-rotations and other Hamiltonian diffeomorphisms; in fact, for a pseudo-rotation $\phi \in \mathrm{Ham}(\mathbb{C}P^n)$, we have

$$\gamma_{\mathbb{Z}}(\phi) \leqslant 1.$$

- We have
  1. On $\mathbb{C}P^n$, $\gamma_{\mathbb{Z}}$ can diverge, but for PR's it stays small.
  2. $\gamma_{\mathbb{Z}} \leqslant d_{\mathrm{Hof}}$.
- Thus,

$$\sup_{\phi \in \mathrm{Ham}(\mathbb{C}P^n)} d_{\mathrm{Hof}}(\phi, PR(\mathbb{C}P^n)) = +\infty.$$

Behind $\mathbb{K}$ vs. $\mathbb{Z}$: Poincaré duality

- Boundedness $\gamma_{\mathbb{K}} \leqslant 1$ comes from Poincaré duality formula for spectral invariants:

$$-c_{\mathbb{K}}(\overline{H}, a) = \inf\{c_{\mathbb{K}}(H, b) : \Pi(a, b) \neq 0\} \tag{9}$$

  where $\Pi : QH(M; \mathbb{K}) \otimes QH(M; \mathbb{K}) \to \mathbb{K}$ is some pairing.

- From (9), for $\mathbb{C}P^n$ we obtain

$$c_{\mathbb{K}}(\overline{H}, [\mathbb{C}P^n]) = -c_{\mathbb{K}}(H, [pt]),$$

  and thus, by using the quantum relation $[pt] * [\mathbb{C}P^{n-1}] = [\mathbb{C}P^n]t^{-1}$ and the triangle inequality, we get $\gamma_{\mathbb{K}} \leqslant 1$.

- However, over $\mathbb{Z}$, we do NOT have $\gamma_{\mathbb{Z}} \leqslant 1$ (Thm A).
- This means that Poincaré duality formula fails over $\mathbb{Z}$.

- Why?
- Over $\mathbb{K}$, as there are no torsion classes, i.e. $\forall \tau$,

$$Ext(HF_*^{<\tau}(H;\mathbb{K}),\mathbb{K}) = 0,$$

so we have the identification between $HF^*_{<\tau}(H;\mathbb{K})$ and $Hom(HF_*^{<\tau}(H;\mathbb{K}),\mathbb{K})$ (universal coefficient Theorem).
- Over $\mathbb{Z}$, it can be $Ext(HF_*^{<\tau}(H;\mathbb{Z}),\mathbb{Z}) \neq 0$ for some τ, i.e. there are torsion classes (which cannot be seen over $\mathbb{K}$-coefficients).
- So how can be describe $c_{\mathbb{Z}}(\overline{H}, a)$ in terms of $HF^{<\tau}(H;\mathbb{Z})$?

Poincaré duality formula over $\mathbb{Z}$ (KS23)

We have

$$\inf_{\Pi(a,b)\neq 0} c_{\mathbb{Z}}(H,b) - \beta_{\text{tor}}(H) \leqslant -c_{\mathbb{Z}}(\overline{H},a) \leqslant \inf_{\Pi(a,b)\neq 0} c_{\mathbb{Z}}(H,b) \qquad (10)$$

where $\beta_{\text{tor}}(H)$ measures the "persistence" of the torsion classes in $Ext(HF_*^{<\tau}(H;\mathbb{Z}),\mathbb{Z})$.

Summary

- For $\mathbb{C}P^n$, $\gamma_{\mathbb{K}} \leqslant 1$ ($\mathbb{K}$: field), but $\gamma_{\mathbb{Z}} \to +\infty$.
- The "persistence" of the torsion classes (in $Ext(HF_*^{\leqslant \tau}(H;\mathbb{Z}),\mathbb{Z})$) is responsible for this contrast.
- This solves symplectic ver of Hingston's question.
- This has application to geometry of pseudo-rotations.
- Thank you for your attention!