# Floer-theoretic corrections to the geometry of moduli spaces of Lagrangian tori

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based on arXiv: 2309.13010

Symplectic Zoominar – Feb. 23, 2024

# SYZ mirror symmetry

Strominger-Yau-Zaslow (SYZ) construction of mirror to $(X,\omega)$ (Kähler)
rel. anticanonical divisor $D \subset X$ ($[D] = c_1(TX)$) (log CY pair)

① find a Lagrangian torus fibration $\pi : X - D \longrightarrow B$ (w/ fibers of vanishing Maslov class, generically unobstructed, ..)

$\rightsquigarrow$ uncorrected mirror of $X-D$: $\left\{ (F_b, \nabla) \,\middle/\, \begin{array}{l} F_b = \pi^{-1}(b) \text{ fiber of } \pi \\ \nabla \text{ unitary rk. 1 loc. system}/F \end{array} \right\}$

analytic space, loc. $\simeq (\mathbb{K}^*)^n$ (moduli space of Lagr. tori + $\nabla$ in $X-D$).

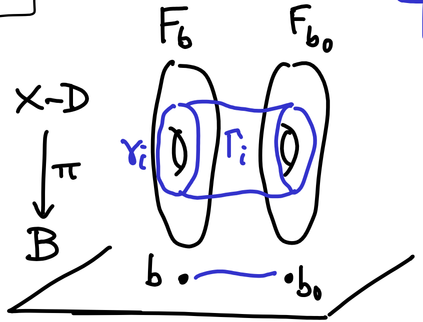
② Floer-theoretic corrections to geometry of $X^\vee$ from holomorphic discs in $(X, F_b)$

- Maslov index $\mu(u) = 2[D]\cdot[u] = 0$ discs deform analytic structure of $X^\vee$ (& fix issues w/ singular fibers of $\pi$)
- $\mu = 2$ discs ($[D]\cdot[u] = 1$) $\rightsquigarrow$ superpotential $W \in \mathcal{O}(X^\vee)$
  $(X^\vee, W)$ Landau-Ginzburg model.

NEW PHENOMENON

- $\mu < 0$ discs ($[D]\cdot[u] < 0$) $\rightsquigarrow$ extended deformations - $X^\vee$ no longer analytic space!!

# The uncorrected moduli space $X_0^\vee$



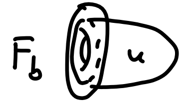

$\pi : X \setminus D \longrightarrow B$ Lagr. torus fibration

$$\rightsquigarrow \quad X_0^\vee := \left\{ (F_b, \nabla) \;\middle|\; \begin{array}{l} F_b = \pi^{-1}(b) \text{ smooth fiber of } \pi \\ \nabla \in \hom(\pi_1(F_b), U(1)_{\mathbb{K}}) = H^1(F_b, U(1)_{\mathbb{K}}) \end{array} \right\}$$

analytic space $/\mathbb{K}$ (away from sing. fibers):

$\{\gamma_i\}$ basis of $H_1(F_b)$ + base point $b_0$ $\rightsquigarrow$ local coords. $z_i(F_b, \nabla) = T^{\omega(\Gamma_i)} \cdot \nabla(\gamma_i) \in \mathbb{K}^*$.

Here $\mathbb{K} = \left\{ \sum a_i T^{\lambda_i} \;\middle|\; a_i \in k,\ \begin{array}{l} \lambda_i \in \mathbb{R} \\ \lambda_i \to +\infty \end{array} \right\}$ Novikov field in formal variable $T$

- Floer-theoretic weights of holom. discs in $(X, F_b)$: 

$[u] = \beta \in \pi_2(X, F_b)$.

  $z_\beta = T^{\omega(\beta)} \nabla(\partial\beta) \in \mathbb{K}^*$ are monomials in $z_1, \dots, z_n$

  $\Rightarrow$ Lagr. Floer theory has analytic dependence on $(F, \nabla) \in X_0^\vee$.

  $\rightsquigarrow$ Family Floer approach to mirror symmetry (Fukaya, Abouzaid, Tu, Yuan, ...)

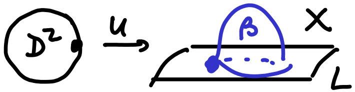
# Holomorphic discs and the Floer-theoretic obstruction

If there are no holom. discs, $X_0^\vee := \{(F_b,\nabla) \mid F_b = \pi^{-1}(b),\ \nabla \in \hom(\pi_1(F_b), U(1)_{\mathbb{K}})\}$
is a moduli space of objects of $\mathcal{F}(X\setminus D)$, with its natural analytic structure.

Holomorphic discs deform the Fukaya category. Floer obstruction (FOOO) $m_0^L \in CF(L,L)$

$$\mathcal{M}_1(X,L;J,\beta) = \left\{ u:(D^2,\partial)\to(X,L) \;\middle|\; \begin{matrix}\bar\partial_J u = 0\\ [u]=\beta\end{matrix} \right\} \Big/ \mathrm{Aut}(D^2,1) \xrightarrow{ev_\beta} L$$

expected $\dim_{\mathbb{R}} = n-2+\mu(\beta)$.




$ev_\beta : u \mapsto u(1) \in L$

$$\boxed{m_0^{(F_b,\nabla)} = \sum_{\beta\in\pi_2(X,F_b)} ev_{\beta *}[\overline{\mathcal{M}}_1(X,F_b;J,\beta)]\ \underbrace{T^{\omega(\beta)}\nabla(\beta)}_{z_\beta} \in C^*(F_b;\mathbb{K}).}$$

↳ stable map compactif$^n$. $(\deg = 2-\mu(\beta))$.

Expect: → $\mu=0$ discs (eg. discs in $X\setminus D$) occur along union of (thickened) $\mathrm{codim}_{\mathbb{R}}\, 1$ walls in $B$

→ outside of walls, $F_b$ is weakly unobstructed: min $\mu=2$, so $m_0 = W_{(F,\nabla)}\cdot 1$

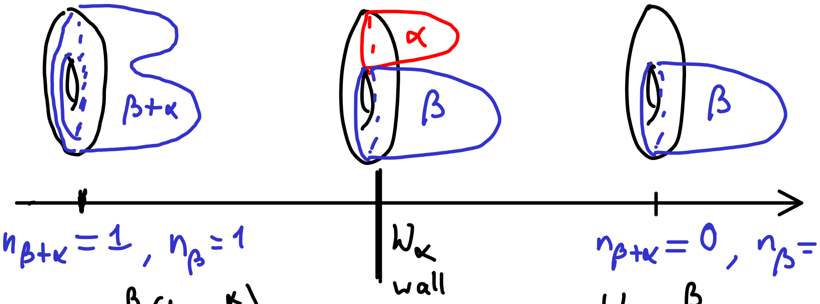
$\Downarrow$ Floer cohomology still well-defined.

superpotential $W(F_b,\nabla) = \sum_{\mu(\beta)=2} n_\beta z_\beta \in \mathcal{O}(X_0^\vee)$, $n_\beta = \deg(ev_\beta) \in \mathbb{Z}$.

4)

# Wall-crossing

The superpotential $W(F_b,\nabla) = \sum_{\mu(\beta)=2} n_\beta z_\beta$ is analytic over domains delimited by the walls in $B$, but counts of $\mu=2$ discs have wall-crossing discontinuities due to bubbling of $\mu=0$ discs.




$n_{\beta+\alpha}=1$, $n_\beta=1$

$W = z^\beta(1+z^\alpha)+\dots$

$n_{\beta+\alpha}=0$, $n_\beta=1$

$W = z^\beta + \dots$

But: formulas for $W$ match under wall-crossing coord. changes $\varphi: z^\beta \longmapsto z^\beta(1+z^\alpha+\dots)^{[\partial\beta]\cdot[W_\alpha]}$

→ Corrected mirror $X^\vee$ = reglue local pieces of $X^{\vee 0}$ via these coord. changes; $W \in \mathcal{O}(X^\vee)$ globally.

Consistency of wall-crossing transformations (cocycle condition) ⇒ $X^\vee$ well-defined analytic space

Kontsevich-Soibelman algorithm to construct consistent scattering diagram of walls (Gross-Siebert)

... but consistency fails when there are $\mu<0$ discs !!!

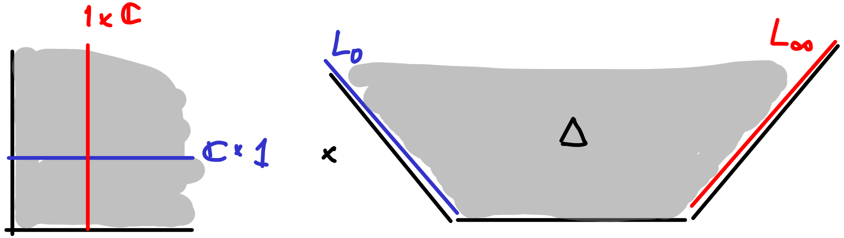
# Main example

$X$ = blowup of toric 4-fold $\mathbb{C}^2 \times K_{\mathbb{P}^1}$ along $\mathbb{C}\times\{1\}\times L_0 \,\cup\, \{1\}\times\mathbb{C}\times L_\infty$

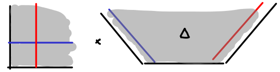
$D$ = proper transform of $\cup$ toric strata

$\mathcal{O}_{\mathbb{P}^1}(-2)$

fibers of $K_{\mathbb{P}^1} \to \mathbb{P}^1$ over $0$ and $\infty$

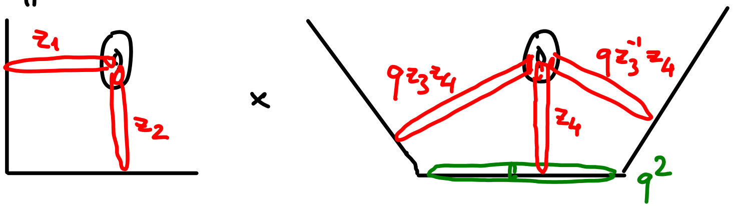



$\pi : X \setminus D \longrightarrow B \simeq \mathbb{R}^2_+ \times \mathrm{int}(\Delta)$

w/ singular fibers over 


fibers away from exc. divisors $E_0 \cup E_\infty$ = lifts of product tori in $\mathbb{C}^2 \times K_{\mathbb{P}^1}$

For $\mathbb{C}^2 \times K_{\mathbb{P}^1}$, the mirror is $(K^*)^4$, $W = z_1 + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4$

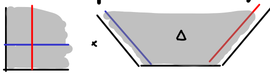



Toric $\mu = 2$ discs along coord. axes, intersecting one toric divisor

$+\, q^2 z_4$ term: disc $\cup$ $S$ ← zero section in $K_{\mathbb{P}^1}$, $N = \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(-2)$.

6

$X = Bl(\mathbb{C}^2 \times K_{\mathbb{P}^1}, \mathbb{C}\times 1 \times L_0 \cup 1 \times \mathbb{C} \times L_\infty)$ $\quad \pi : X \setminus D \longrightarrow B \simeq \mathbb{R}^2_+ \times \mathrm{int}(\Delta)$

w/ singular fibers over 



fibers away from exc. divisors $E_0 \cup E_\infty$ = lifts of product tori in $\mathbb{C}^2 \times K_{\mathbb{P}^1}$

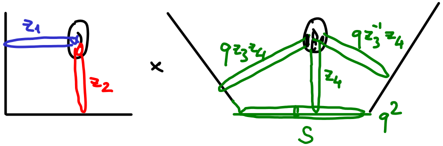
Wall-crossing occurs as radii $r_1, r_2$ in $\mathbb{C}^2$ $\nearrow$ through 1: new $\mu=2$ discs in $X$: proper transforms of $\mu \geq 4$ discs in $\mathbb{C}^2 \times K_{\mathbb{P}^1}$ through $\mathbb{C}\times 1 \times L_0$ $(r_2 > 1)$ or $1 \times \mathbb{C} \times L_\infty$ $(r_1 > 1)$.

| | $r_1 < 1$ | $r_1 > 1$ |
|---|---|---|
| $r_2 > 1$ | $W_{-+} = z_1 + z_2(1 + qq' z_4 + q' z_3 z_4) + (1 + q^2 + q z_3 + q z_3^{-1}) z_4$ | $W_{++} = z_1(1 + qq'' z_4 + q'' z_3^{-1} z_4) + z_2(1 + qq' z_4 + q' z_3 z_4) + q' q'' z_1 z_2 z_4 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4$ |
| $r_2 < 1$ | $W_{--} = z_1 + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4$ | $W_{+-} = z_1(1 + qq'' z_4 + q'' z_3^{-1} z_4) + z_2 + (1 + q^2 + q z_3 + q z_3^{-1}) z_4$ |

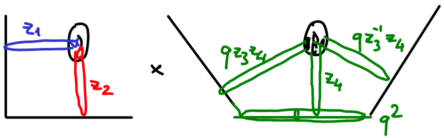
Notes on $W_{-+}$: $qq'z_4$: $(x_2, x_4)$ disc st. $x_2 = 1$ @ $x_4 = 0$ union $\hat{S}_{x_2=1}$; $q'z_3z_4$: $(x_2, x_3x_4)$ disc st. $x_2 = 1$ @ $L_0$.

Note on $W_{++}$: $q'q''z_1z_2z_4$: $(x_1, x_2, x_4)$ disc st. $(x_1, x_2) = (1,1)$ @ $x_4 = 0$ union $\hat{S}_{(x_1,x_2)=(1,1)}$.

same as for $\mathbb{C}^2 \times K_{\mathbb{P}^1}$ (no discs through $E_0 \cup E_\infty$)



# 7) The mirror of $X = Bl(\mathbb{C}^2 \times K_{\mathbb{P}^1},\ \mathbb{C}\times 1 \times L_0 \cup 1 \times \mathbb{C} \times L_\infty)$




Wall crossing at $r_1 = 1$ preserves $z_2, z_3, z_4$
$r_2 = 1$ preserves $z_1, z_3, z_4$

$$\varphi_{0+}: z_1 \mapsto z_1\left(1 + qq''z_4 + q''z_3^{-1}z_4 + q'q''z_2z_4\right)$$

$r_2 > 1$

$$W_{-+} = z_1 + z_2\left(1 + qq'z_4 + q'z_3z_4\right) + \left(1 + q^2 + qz_3 + qz_3^{-1}\right)z_4$$

$$W_{++} = z_1\left(1 + qq''z_4 + q''z_3^{-1}z_4\right) + z_2\left(1 + qq'z_4 + q'z_3z_4\right) + q'q''z_1z_2z_4 + \left(1 + q^2 + qz_3 + qz_3^{-1}\right)z_4$$

$$\varphi_{-0}: z_2 \mapsto z_2\left(1 + qq'z_4 + q'z_3z_4\right)$$

$$\varphi_{+0}: z_2 \mapsto z_2\left(1 + qq'z_4 + q'z_3z_4 + q'q''z_1z_4\right)$$

$r_2 < 1$

$$W_{--} = z_1 + z_2 + \left(1 + q^2 + qz_3 + qz_3^{-1}\right)z_4$$

$$W_{+-} = z_1\left(1 + qq''z_4 + q''z_3^{-1}z_4\right) + z_2 + \left(1 + q^2 + qz_3 + qz_3^{-1}\right)z_4$$

$r_1 < 1$

$$\varphi_{0-}: z_1 \mapsto z_1\left(1 + qq''z_4 + q''z_3^{-1}z_4\right)$$

$r_1 > 1$

$\varphi_{+0} \circ \varphi_{0-} \neq \varphi_{0+} \circ \varphi_{-0}$ : the wall-crossing diagram is inconsistent!!

This is caused by $\mu = -2$ stable disc at $(r_1, r_2) = (1,1)$: $z_4$-disc $\cup\ \hat{S}_{(x_1,x_2)=(1,1)}$ (weight $q'q''z_4$).

# $X^\vee$ as a deformed Landau-Ginzburg model.

The mirror constructed above is consistent mod $q'q''$. The extra terms can be viewed as a deformation of (alg. geom of) $(X^\vee, W)$ by a class in

$$HH^*(MF(X^\vee, W)) = \mathbb{H}^*(X^\vee, (\Lambda^* T_{X^\vee}, \iota_{dW}))$$

determined by $\mu = -2$ discs: leading term $w^{(2)} \in \check{C}^2(X^\vee, \Lambda^2 T_{X^\vee})$ takes value $w^{(2)}_{00} = q'q'' z_4 \, \partial_{\log z_1} \wedge \partial_{\log z_2}$ on overlap of coord. charts around $(r_1, r_2) = (1,1)$.

(complete to a cocycle by adding $q'q''$ terms in $\varphi_{+0}, \varphi_{0+}, W_{++}$:

$$\left.\begin{aligned} w^{(1)}_{+0} &= q'q'' z_1 z_4 \, \partial_{\log z_2} \\ w^{(1)}_{0+} &= q'q'' z_2 z_4 \, \partial_{\log z_1} \\ w^{(0)}_{++} &= q'q'' z_1 z_2 z_4 \end{aligned}\right\} \Rightarrow \begin{aligned} \iota_{dW}(w^{(2)}) &= \delta w^{(1)} \\ \iota_{dW}(w^{(1)}) &= \delta w^{(0)} \end{aligned}$$

$\to X^\vee$ deformed Landau-Ginzburg model: $W$ loc. analytic, but coord. charts fail to satisfy cocycle condition by $\iota_{dW}(w^{(2)})$.

9)

# A family Floer approach to correcting the mirror

$\pi : X \setminus D \to B$ Lagr. torus fibration, $\quad X^{\vee 0} = \left\{ (F_b, \nabla) \;\middle|\; \begin{array}{l} F_b = \pi^{-1}(b) \text{ smooth fiber} \\ \nabla \in \hom(\pi_1 F_b, U(1)_{\mathbb{K}}) \end{array} \right\}$

$$\begin{array}{c} X^{\vee 0} \\ \downarrow \pi^\vee \\ B^0 \end{array} \text{ (smooth part)} \qquad \text{uncorrected mirror}$$

$\pi^\vee_* \mathcal{O} =: \mathcal{O}_{an}$ sheaf on $B^0$

$=$ completion of $\mathbb{K}[H_1(F_b)] = \mathbb{K}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ ($\ni$ monomials $z^\beta$ = disc weights)

Lagr. Floer theory for discs in $(X, F_b)$ $\forall b \in B^0$, with universal weights $\in \mathcal{O}_{an}$

$\Rightarrow$ curved $A_\infty$-operations on $C^*(B^0; C^*(F_b) \hat{\otimes} \mathcal{O}_{an})$.

$$m_0 = \sum_{\beta \in \pi_2(X, F_b)} ev_{\beta *} \Big[ \bigsqcup_b \overline{\mathcal{M}}_1(X, F_b; J, \beta) \Big] z^\beta \in C^*(B^0; C^*(F_b) \hat{\otimes} \mathcal{O}_{an}).$$

$\pi_2(X, F_b)$: local system of abelian groups on $B^0$ $\qquad$ ($\deg = 2 - \mu(\beta)$).

**Expect:**

1) $\exists\, \alpha^{(i)} \in C^i(B^0; H^i(F_b) \hat{\otimes} \mathcal{O}_{an})$ st. $m_0 = \underset{\mu=2}{\alpha^{(0)}} + \underset{\mu=0}{\alpha^{(1)}} + \underset{\mu=-2}{\alpha^{(2)}} + \dots$

codim. $i$ walls of $\mu = 2-2i$ discs

Fukaya, K. Irie + ... $\to$ 2) $m_0$ satisfies <u>master equation</u> $\delta m_0 + \frac{1}{2}\{m_0, m_0\} = 0$

deg. $-1$ bracket

# The master equation & mirror geometry

- $m_0 = \alpha^{(0)} + \alpha^{(1)} + \ldots \in \bigoplus_{i\geq 0} C^i(B^0; H^i(F_b)\hat{\otimes}\mathcal{O}_{an})$  $\alpha^{(i)}$ records codim. $i$ walls of $\mu = 2-2i$ discs

on $H^*(F_b)\otimes \mathbb{K}[H_1(F_b)]$ : $\{z^\gamma\alpha, z^{\gamma'}\alpha'\} := z^{\gamma+\gamma'}(\alpha\wedge \iota_\gamma \alpha' + (-1)^{|\alpha|}\iota_{\gamma'}\alpha\wedge\alpha')$ (deg. $-1$)

$(\simeq H_*(\mathcal{L}F_b)$, Chas-Sullivan string bracket$)$  $\forall \alpha,\alpha' \in H^*(F_b)$ $\gamma,\gamma'\in H_1(F_b)$.

$\rightsquigarrow$ extend (w/ cup-product) to $\{\cdot,\cdot\}$ on $C^*(B^0; H^*(F_b)\hat{\otimes}\mathcal{O}_{an})$.

- Using $H^1(F_b,\mathbb{R}) \simeq T_bB$, get $H^*(F_b)\hat{\otimes}\mathcal{O}_{an} \xrightarrow{\sim} \pi^\vee_*(\Lambda^* T_{X^{\vee 0}})$

$$z^\gamma \gamma^*_{j_1}\wedge\ldots\wedge\gamma^*_{j_k} \longmapsto z^\gamma \partial_{\log z_{j_1}}\wedge\ldots\wedge\partial_{\log z_{j_k}}$$

under which $m_0 = \alpha^{(0)} + \alpha^{(1)} + \ldots \longmapsto \mathbb{W} = W^{(0)} + W^{(1)} + \ldots \in \bigoplus_{i\geq 0} C^i(X^{\vee 0}, \Lambda^i T_{X^{\vee 0}})$

$\{\cdot,\cdot\} \longmapsto$ Schouten-Nijenhuis bracket

$$\delta m_0 + \frac{1}{2}\{m_0, m_0\} = 0 \longmapsto \delta\mathbb{W} + \frac{1}{2}[\mathbb{W},\mathbb{W}] = 0$$

# The geometry of the master equation

$\mathbb{W} = W^{(0)} + W^{(1)} + \dots \in \bigoplus_{i\geq 0} C^i(X^{\nu 0}, \Lambda^i T_{X^{\nu 0}})$ Schouten-Nijenhuis bracket

$$\delta\mathbb{W} + \frac{1}{2}[\mathbb{W},\mathbb{W}] = 0 \iff (\delta + [\mathbb{W},\cdot])^2 = 0 \iff \begin{cases} (\delta+[W^{(1)},\cdot])\,W^{(0)}=0 & (1)\\ (\delta+[W^{(1)},\cdot])^2 = [\iota_{dW^{(0)}}(W^{(2)}),\cdot\,] & (2)\\ (\delta+[W^{(1)},\cdot])\,W^{(2)} = \iota_{dW^{(0)}}(W^{(3)}) & (3)\\ \dots \end{cases}$$

View $\delta+[W^{(1)},\cdot]$ as deform$^{\underline{n}}$ of analytic structure

$\begin{cases} \text{Čech: deform gluing of } U_i \longleftrightarrow U_j \text{ by vector field } W^{(1)}_{ij} \\ \text{Dolbeault: deform } \bar\partial \text{ operator by } W^{(1)} \in \Omega^{0,1}\otimes T^{1,0} \end{cases}$

(1) $\Rightarrow$ $W^{(0)}$ is analytic wrt deformed structure

(2) $\Rightarrow$ deformation actually fails to be an analytic space by $\iota_{dW^0}(W^{(2)}) \in C^2(X^\nu, T_{X^\nu})$

$\begin{cases} \text{Čech: failure of coord. changes to satisfy cocycle condition} \\ \text{Dolbeault: failure of integrability of the deformed cx. structure = Nijenhuis tensor} \end{cases}$

<u>Note</u>: crit$(W^{(0)})$ is still honestly analytic, since $\iota_{dW^{(0)}}(W^{(2)}) = 0$ along critical locus.

# A Morse model for Floer operations on $C^*(B^0; H^*(F_b)\hat{\otimes}\mathcal{O}_{an})$

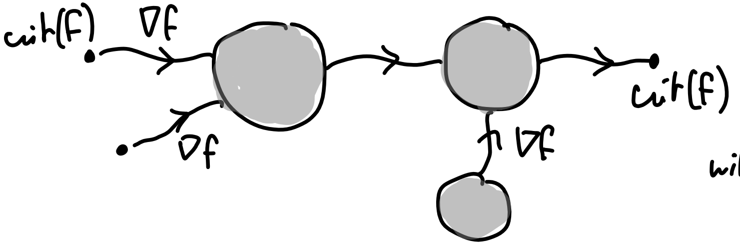
Fix a Morse function $f: X^0 = \pi^{-1}(B^0) \to \mathbb{R}$ (+ Morse-Smale metric)

triangulation $\mathcal{P}$ of $B^0$ $\rightsquigarrow$ Morse $f^n$ on $B^0$ with 1 crit pt $b_\sigma$ index $k$ $\forall \sigma \in \mathcal{P}^{[k]}$, $\overline{W^-(b_\sigma)} = \sigma$.

+ add perfect Morse function on $F_{b_\sigma}$.

$\rightarrow$ $CM^*(f) \simeq \check{C}^*(B^0; H^*(F_b))$ for open cover of $B^0$ by stars of $\mathrm{vert}(\mathcal{P})$

Define Floer operations $(m_k)_{k\geq 0}$ on $CM^*(f;\mathcal{O}_{an})$ by counting (perturbed) J-holom. treed discs




$\left\{ \begin{array}{l} \text{gradient flow lines of } f \text{ in } X^0 \\ J\text{-hol. discs in } (X, F_b)\ \forall b \in B^0 \end{array} \right.$

with weight $z_\beta = \sum [u(D_i)] \in H_2(X, F_{b_{out}})$.

- achieve transversality using domain stabilization (cf. Charest-Woodward; divisors vary over $B^0$)
  + domain-dependent perturbations
- convergence under assumption that $\mathcal{P}$ fine enough ($\Rightarrow \mathrm{val}(z_{\beta_i}) > 0$ at $b_{out}$)