Augmentation Varieties and Disk-Potentials

Soham Chanda
Rutgers University

joint work with K.Blakey, Y.Sun and C.Woodward

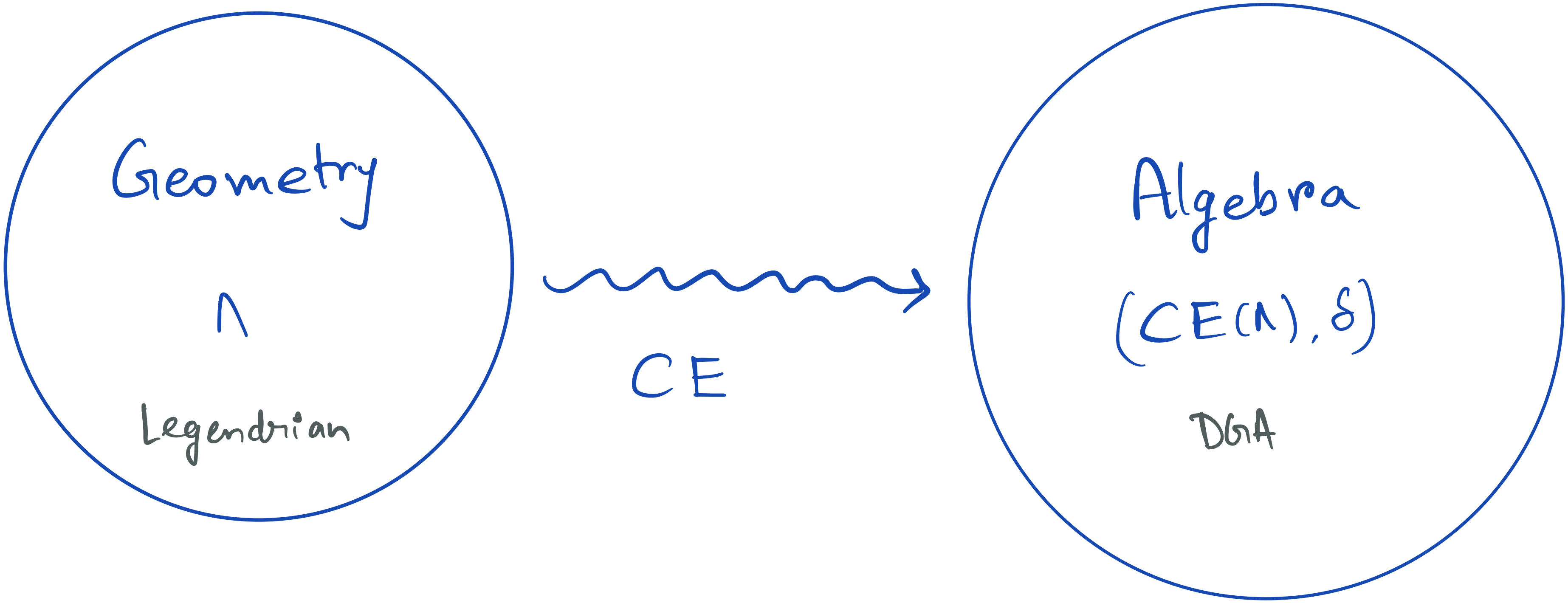
• Goals

- Chekanov-Eliashberg algebra for lifted Legendrians in circle-fibered contact manifolds.
- Augmentation ideal and augmentation variety from CE (Λ).
- Recovering augmentation variety from disk-potential.

$$\boxed{\mathrm{Aug}(\Lambda) \text{ "="} \; W_{p(\Lambda)}^{-1}(0)}$$

— Rizell-Golovko conjecture.

Big-Picture


Geometry
Λ
Legendrian
CE
Algebra
$(CE(\Lambda), \partial)$
DGA

Augmentations!

- Aganagic-Ng-Ekholm-Vafa, Diogo-Ekholm, Gao-Shen-Weng, Sabloff. Ng-Rutherford-Shende-Sivek-Zaslow, Rutherford-Sullivan.

Geometry

Λ

Legendrian

$\xrightarrow{\ CE\ }$

Algebra

$(CE(\Lambda), \delta)$

DGA

Augmentations of $CE(\Lambda)$

Augmentation Variety

$Aug(\Lambda) \subseteq (\mathbb{C}^*)^n$

Augmentations!

- Aganagic-Ng-Ekholm-Vafa, Diogo-Ekholm, Gao-Shen-Weng, Sabloff.
  Ng-Rutherford-Shende-Sivek-Zaslow

Geometry

Λ

Legendrian

$\xrightarrow{\text{LCH.}}$

Algebra

$(CE(\Lambda), \delta)$

DGA

Augmentations of $CE(\Lambda)$

Augmentation Variety

$Aug(\Lambda) \subseteq (\mathbb{C}^*)^n$

L is a filling of Λ

Ideal induced from L.

$Aug_L(\Lambda)$

$Aug(\Lambda) \supseteq Aug_L(\Lambda)$.

I. Circle-fibered contact manifold (Prequantum Bundle).

$$Z \xrightarrow[/S^1]{p} Y \qquad S^1\text{-bundle}$$

$$\left.\begin{array}{l} \alpha - \text{connection 1-form on } Z. \\ d\alpha = p^*\omega_Y, \quad \omega_Y \text{ symplectic on } Y. \end{array}\right\} \Rightarrow (Z, \alpha) \text{ contact}$$

Reeb-field $\propto$ infinitesimal action of S^1

i.e. $\varphi^t_R(-) = e^{2\pi i t} \cdot (-)$

Length of Reeb chord $\propto$ "angle change" from the S^1 action.

Example.

$$\left(S^{2n-1}, \xi_{std}\right) \xrightarrow[\text{Hopf action}]{} \left(\mathbb{P}^{n-1}, \omega_{FS}\right)$$

$$e^{i\theta} \cdot (z_1 \dots z_n) = (e^{i\theta} z_1, \dots, e^{i\theta} z_n)$$

Lifting Lagrangians from Y to Legendrians in Z.

– Dimitroglou-Rizell–Golovko 19.

Bohr-Sommerfeld Lagrangian

$$\begin{tikzcd}
(i^*Z, i^*\alpha) \arrow[r] \arrow[d] & (Z,\alpha) \arrow[d, "p"] \\
L \arrow[r, "i"] & (Y,\omega_Y)
\end{tikzcd}$$

$d\alpha = p^*\omega_Y$.

Lagrangian immersion

$i^*\alpha$ is trivial connection on i^*Z.

Take horizontal lift $\longrightarrow$ Get immersed Leg. in Z.

When Y - simply connected, can get canonical Bohr-Sommerfeld immersions

$$\begin{tikzcd}
\Lambda \arrow[d, "\text{covering}"'] \arrow[dr] & \\
\Pi \arrow[r, hook, "\text{embeded}"'] & Y
\end{tikzcd}$$

Lagrangian.

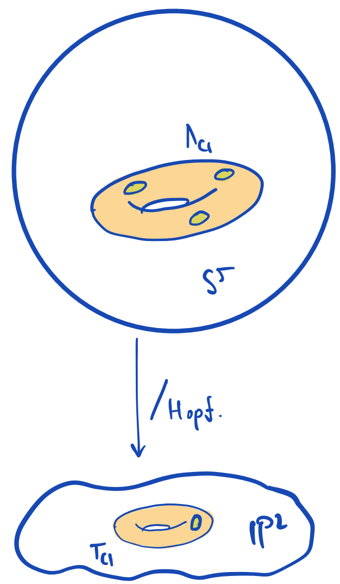
> If $\Pi \hookrightarrow Y$ is monotone, and ω_Y is integral, Π admits a Bohr-Sommerfeld cover.

Example

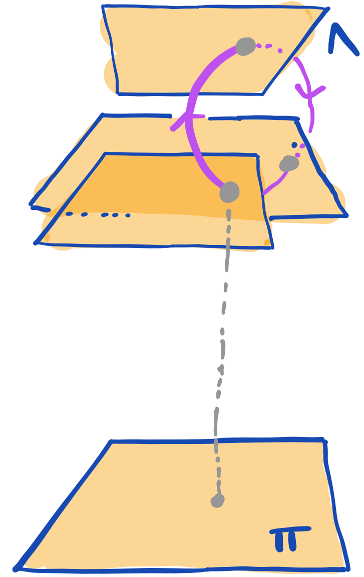
$$T_{cl} \hookrightarrow \mathbb{P}^{n-1} \quad \text{monotone.}$$

$$\Lambda_{cl} = \left\{ (z_1 \dots z_n) \in S^{2n-1} \,\middle|\, |z_1| = |z_2| \dots = |z_n|,\ \mathrm{Im}(z_1 \dots z_n) = 0 \right\}$$

$$\Lambda^{n-1}_{cl} \xrightarrow{n:1} T^{n-1}_{cl}$$



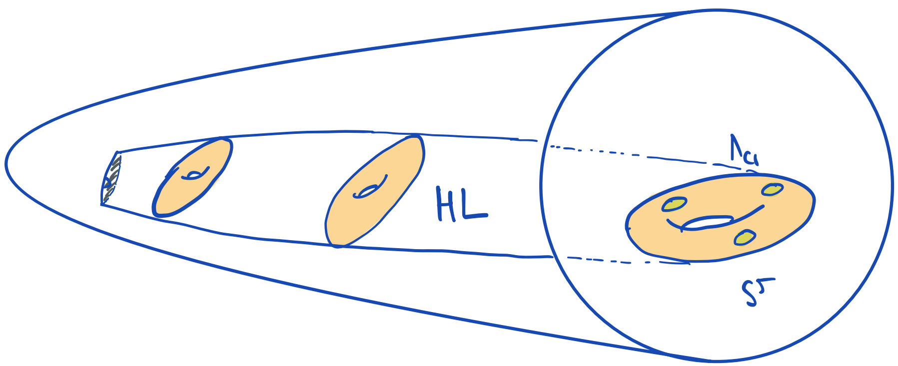
II. Chekanov-Eliashberg algebra by treed-disc count



- Morse-Bott degeneracy for $\mathcal{R}(\Lambda)$
  Reeb chords on Λ.

- Allow non-exact, but tame Lagrangian cobordisms
  "tame" := satisfies some topological constraints.

e.g. we want non-exact fillings like Harvey-Lawson filling of Λ^2_{cl} to produce augmentations.



$$HL = \left\{ (z_1 \ldots z_3) \;\middle|\; \begin{array}{l} |z_1|^2 = |z_2|^2 = |z_3|^2 - 1, \\ z_1 z_2 z_3 \in [0,\infty) \end{array} \right\} \subseteq \mathbb{C}^3$$

$$u : \mathbb{D}^2 \longrightarrow \mathbb{C}^3 \qquad z \mapsto (0, 0, z)$$

has boundary on HL and positive symplectic area.

Theorem: [Blakey-C-Sun-Woodward]

If Y is integral symplectic with minimum chern number atleast 2, Π is compact-oriented-spin, -monotone lag

and $\Lambda \longrightarrow \Pi$ is Bohr-Somm. cover.

- $(CE(\Lambda), \delta)$ is dga whose homology is invariant.
- tame Lagrangian cobordisms induce dga-maps, i.e we have TFT-axioms.
- $Aug(\Lambda)$, $I(\Lambda)$ are Legendrian isotopy invariant.
- When Λ is connected, and min-maslov of Π is 2,

$$I(\Lambda) = P_*\left(x^{-\nu}\underbrace{W_{\Pi}}_{\text{disc-potential of } \Pi}\right)$$

Application:

Lemma

Let L_1, L_2 be monotone, compact, spin, oriented in Y,
$\Lambda_i \longrightarrow L_i$ are Bohr-Sommerfeld cover
if $W_{L_1} \neq W_{L_2}$ (up to change of variables $x_1 \dots x_k$)
then Λ_1 is not Leg. isotopic to Λ_2

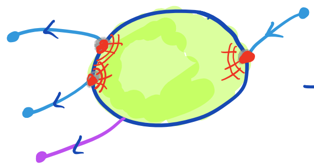
Vianna 14: infinitely many $T^2 \overset{lag}{\hookrightarrow} \mathbb{P}^2$ with distinct disk-potentials.

C-Hirschi-Wang 23, Diogo-Tonkonog-Vianna-Wu: infinitely many $T^k \overset{lag}{\hookrightarrow} \mathbb{P}^k$ with distinct disk-potentials.

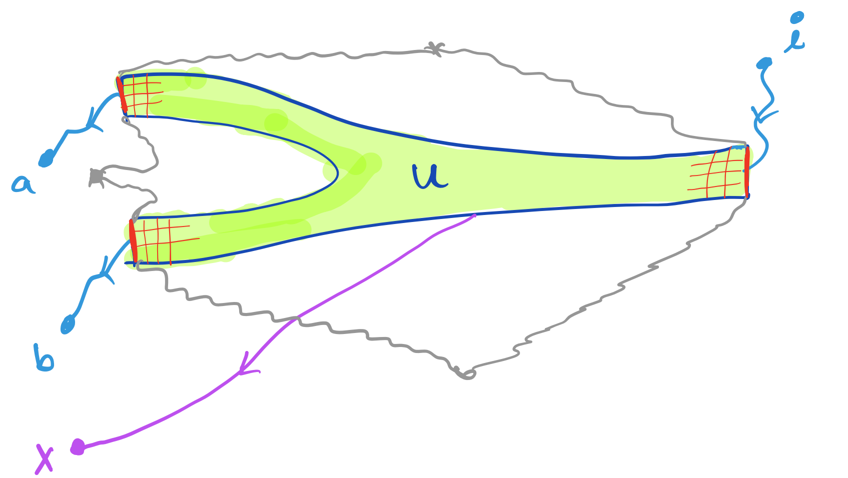
Cor:

$\exists$ infinitely many Legendrian, $T^n \hookrightarrow S^{2n+1}$, in the standard contact sphere

Contribution to the Differential



$$\xrightarrow{\;u\;} (\mathbb{R}\times Z,\ \mathbb{R}\times\Lambda)$$



$$\delta(i) = ab{x} + \dots$$

Generators of CE

$f_0 : \mathcal{R}(\Lambda) \longrightarrow \mathbb{R}$

$f_\bullet : \Lambda \longrightarrow \mathbb{R}$.

$a, b, i \in \mathrm{crit}(f_0)$

$x \in \mathrm{crit}(f_\bullet)$

Ingredients for $CE(\Lambda)$

Geometric

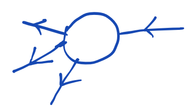
- $f_0, f_\bullet$ morse
- J - cylindrical on $\mathbb{R} \times Z$
  $J|_{\ker \alpha}$ = compatible
- capping path.

Algebraic

- $C = \mathrm{crit}(f_0) \cup \mathrm{crit}(f_\bullet)$.
- W = finite words which letters from C } Generators
- $G(\Lambda) = \mathbb{C}[H_1(\Lambda)]$ } coefficient

$$CE(\Lambda) := \left\{ \sum_{i=1}^{\infty} c_i w_i \;\middle|\; \begin{array}{l} w_i \in W \\ c_i \in G(\Lambda) \\ \ell(w_i) \longrightarrow \infty \end{array} \right\}, \quad \delta =$$

weighted



count

Gradings: $\mathbb{R}$ - grading from Reeb chord length and morse index.

I. Augmentations

$$\begin{tikzcd}
 & {} \\
 & 0 \arrow[u] \\
 & 0 \arrow[u] \\
\varphi : CE(\Lambda) \arrow[r, "\text{chain map}"'] & \mathcal{R} \arrow[u] \\
 & 0 \arrow[u] \\
 & 0 \arrow[u] \\
 & {} \arrow[u]
\end{tikzcd}$$

$$\hat{\varphi} : CE^{ab}(\Lambda) \longrightarrow \mathcal{R}$$

$$\alpha\beta = (-1)^{|\alpha||\beta|}\beta\alpha$$

Aug ideal

Set basis $(\mu_1, \dots, \mu_k)$ for $H_1(\Lambda)^{free}$ and $c_1 \dots c_k \in CE_\bullet(\Lambda)$, dual.

$$\underbrace{\mathbb{C}[y_1, y_1^{-1} \dots, y_k, y_k^{-1}]}_{\mathbb{C}[Rep(\Lambda)]} \xrightarrow{i} CE^{ab}(\Lambda)$$

$$y_i \longmapsto [\mu_i]\, e^{c_i} \qquad \left(e^{c_i} = 1 + c_i + \frac{c_i^2}{2} + \dots \right)$$

$$\boxed{I(\Lambda) := \{ p \in \mathbb{C}[\dots] \mid \varphi(i(p)) = 0 \ \ \forall \text{ augmat}^n\ \varphi \text{ of } CE \}}$$

e.g. for Λ^2_{cl}, $\quad i(1 + y_1 + y_2) = \delta^{ab}(a)$

this is true for any Λ conn.

thus $1 + y_1 + y_2 \in I(\Lambda^2_{cl})$. $\qquad \delta^{ab}(a) = (x^{-\nu} W_\pi)(i(y_1), \dots, i(y_k))$.

$$\boxed{Aug(\Lambda) := Var(\, I(\Lambda))}$$

Thanks!