Cabling families of Legendrian embeddings

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The problem

Consider the standard contact 3-sphere (S^3, ξ_{std}), where $\xi_{\text{std}} = TS^3 \cap i(TS^3)$. Associated to this contact 3-manifold there are two natural spaces of interest:

- The space of Legendrian embeddings $\mathcal{L}(S^3, \xi_{\text{std}}) = \mathcal{L}$
- The space of formal Legendrian embeddings $\mathcal{FL}(S^3, \xi_{\text{std}}) = \mathcal{FL}$.

There is an obvious inclusion

$$i : \mathcal{L} \hookrightarrow \mathcal{FL}$$

Problem

Determine the behaviour of the induced maps $\pi_k(i)$.

The problem

For $k = 0$ this is classical Legendrian knot/link theory, i.e. classification of Legendrians up to Legendrian isotopy. We know a lot about $\pi_0(i)$:

- (Bennequin) $\pi_0(i)$ is not surjective.
- (Chekanov) $\pi_0(i)$ is not injective.
- (Eliashberg-Fraser, Etnyre-Honda, many others...) For some knot (or link) types K the natural restriction

  $$\pi_0(i) : \pi_0(\mathcal{L}(K)) \to \pi_0(\mathcal{FL}(K))$$

  is injective and its image can be determined,e.g. $K = U$, $K = T_{p,q}$.

Instrumental technique in every classification type result: Giroux Convex Surface Theory.

The problem

For $k > 0$ we know very little.

- (Fuchs-Tabachnikov,Casals-del Pino, Murphy) If the base point is "sufficiently stabilized" $\pi_k(i)$ is surjective.
- (F-Martínez Aguinaga-Presas) $\pi_k(i)$ is not surjective in general.
- (F-Martínez Aguinaga-Presas) $\pi_k(i)$ is injective and its image can be determined for $K = U$ and $tb + |rot| = -1$ and $K = T_{n,n}$ with tb maximal.

The following remains open

Problem

Find a base point for which $\ker(\pi_k(i)) \neq 0$.

Today: Places where not to look for such a base point.

Cables

- Let $K \subseteq S^3$ be a knot, μ a meridian of K and λ a Seifert longitude. Take N to be a neighborhood of K. For relatively prime integers $p > 0$ and q, we define a (p, q)-cable $K_{p,q}$ of K to be a knot on ∂N with homology class $p[\lambda] + q[\mu] \in H_1(\partial N)$ and define the slope of $K_{p,q}$ to be q/p.
- If $q/p > \overline{tb}(K) + 1$, we say $K_{p,q}$ is a sufficiently positive cable of K.
- The (np, nq)-cable link of K for $n > 1$, denoted by $K_{np,nq}$ to be n parallel copies of $K_{p,q}$ on ∂N. We also say that $K_{np,nq}$ is sufficiently positive if $K_{p,q}$ it is sufficiently positive.

Main results

Sufficiently positive Legendrian cables have been classified by Chakraborty–Etnyre–Min. In particular, if L is a max-tb Legendrian then its possible to find a max-tb representative $L_{p,q}$ in an standard neighborhood N of L and vice-versa. Moreover, $tb(L_{p,q}) = pq - |q - ptb(L)|$. Importantly, there is a model in which ∂N is foliated by parallel Legendrians isotopic to $L_{p,q}$. This gives a loop

$$S^1 \hookrightarrow \mathcal{L}(L_{p,q})$$

"generalized Kàlmàn loop".

Theorem (F-Min)

Let L be a max-tb Legendrian and $L_{p,q}$ be max-tb sufficiently positive cable.

$$\mathcal{L}(L_{p,q}) \cong S^1 \times \mathcal{L}(L)$$

Main results

- The previous result is the Legendrian version of a known result of Hatcher for long smooth embeddings. In particular, the map $\pi_k(\mathcal{L}(L_{p,q})) \to \pi_k(\mathcal{FL}(L_{p,q}))$ is (non)injective iff $\pi_k(\mathcal{L}(L)) \to \pi_k(\mathcal{FL}(L))$ is (non)injective.
- If L is the $tb = -1$ unknot, $\mathcal{L}(L) \cong U(2)$ by F-Martínez Aguinaga-Presas. Our formula then implies that if $\hat{L}$ is an n-iterated sufficiently positive cable of L with the maximal tb number then

$$\mathcal{L}(\hat{L}) \cong U(2) \times (S^1)^n$$

Generalized cables

Given a knot K we define $1 \cdot K = K$ and $0 \cdot K = \emptyset$ and $K_{0,0} = \emptyset$. Consider a link $L = (L^1, \ldots, L^n)$, $B = \{i_1, \ldots, i_n\} \in \{0,1\}^n$ and $C = \{(p_1, q_1), \ldots, (p_n, q_n)\} \in (\mathbb{Z}^2)^n$. We define the generalized (B, C)-cable of L to be

$$L_C^B = (i_1 \cdot L^1, \ldots, i_n \cdot L^n, L^1_{p_1,q_1}, \ldots, L^n_{p_n,q_n}).$$

We say that L_C^B is sufficiently positive if L_{p_j,q_j} is sufficiently positive or $L_{p_j,q_j} = L_{0,0} = \emptyset$ for every j.

Main results

The complement of a Legendrian L is denoted by $(C(L), \xi_{\text{std}}) = (S^3 \backslash \mathcal{O}p(L), \xi_{\text{std}})$. We denote by $\mathcal{C}(C(L), \xi_{\text{std}})$ the space of contact structures on $C(L)$ that coincide with ξ_{std} near $\partial C(L)$ and isotopic to ξ_{std}, rel boundary.

Definition

L satisfies the C-property if $\mathcal{C}(C(L), \xi_{\text{std}})$ is contractible.

Lemma

If L satisfies the C-property then $\pi_k(\mathcal{L}(L)) \to \pi_k(\mathcal{FL}(L))$ is injective for all k.

Theorem (F-Min)

If L is max-tb and L^B_C is a sufficiently positive max-tb Legendrian cable then

$$\mathcal{C}(C(L), \xi_{\text{std}}) \cong \mathcal{C}(C(L^B_C), \xi_{\text{std}})$$

Main results

- The previous result gives ∞-many new components of the space of Legendrians for which $\pi_k(i)$ is injective. The Legendrian unknot with $tb = -1$ satisfies the C-property so you can apply the Theorem in a iterative way. For instance, every max-tb Legendrian algebraic link.
- A special case are max-tb (positive) Seifert fibered Legendrian links. In this case, Etnyre-LaFountain-Tosun observed that the complement of these Legendrians are Legendrian Seifert spaces. This allow us to fully describe the homotopy type of these components of the space of Legendrian embeddings in terms of pure braid groups.

About the proofs

To study Legendrian embedding spaces we follow Budney-Hatcher scheme in the smooth case.

1. Let $\mathcal{K}(K)$ be the space of smooth embeddings realizing the knot type K. Define the space of long smooth embeddings to be $\mathcal{K}_{(p,v)}(K) = \{\gamma \in \mathcal{K}(K) : (\gamma(0), \gamma'(0)) = (p, v)\}$.
2. Both spaces are related by a fibration
$$\mathcal{K}_{(p,v)}(K) \to \mathcal{K}(K) \to V_{4,2} = S^3 \times S^2$$
3. There is a fibration
$$\mathrm{Diff}(C(K)) \to \mathrm{Diff}(D^3) \to \mathcal{K}_{(p,v)}(K)$$
4. The group $\mathrm{Diff}(D^3)$ is contractible (Hatcher) so $\Omega\mathcal{K}_{(p,v)}(K) \cong \mathrm{Diff}(C(K))$

About the proofs

Assume that K is Legendrian and define $\mathcal{L}_{(p,v)}(K) = \mathcal{L}(K) \cap \mathcal{K}_{(p,v)}(K)$

1. There is a homotopy equivalence $\mathcal{L}(K) \cong U(2) \times \mathcal{L}_{(p,v)}(K)$.
2. There is a fibration

$$\mathrm{Cont}(C(L), \xi_{\mathrm{std}}) \to \mathrm{Cont}(D^3, \xi_{\mathrm{std}}) \to \mathcal{L}_{(p,v)}(K)$$

3. Finally, the group $\mathrm{Cont}(D^3, \xi_{\mathrm{std}})$ is also contractible (Eliashberg-Mishachev). Therefore, $\Omega\mathcal{L}_{(p,v)}(K) \cong \mathrm{Cont}(C(L), \xi_{\mathrm{std}})$. This also works for Legendrian link embeddings.

Take away 1: Instead of Legendrian embedding spaces we study the contactomorphism group of the complement.
Take away 2: Instead of the contactomorphism group of the complement we study the space of contact structures on the complement (Gray Stability).

Quick review of convexity in $3D$.

A surface $\Sigma \subseteq (M^3, \xi = \ker(\alpha))$ is convex iff there exists a contact vector field X transverse to Σ. The dividing set of Σ is $\Gamma = \{p \in \Sigma : \alpha(X_p) = 0\}$.

- Giroux: every surface can be perturbed to become convex (Giroux Genericity), the dividing set encodes all the contact topological information (Giroux flexibility) and the tightness prevents some configurations of dividing sets (Giroux Tightness Criterion).
- Honda and Colin: changes on the dividing set produced under smooth isotopies are encoded by sequences of bypasses.

Relation between $C(L)$ and $C(L_{p,q})$

Recall that L is a max-tb Legendrian knot and $L_{p,q}$ is a sufficiently positive max-tb Legendrian cable of L. Fix an standard neighborhood of N of L. So ∂N is convex and has a dividing set Γ given by a pair of parallel $(1, tb(L))$-cables of L. We may assume (Giroux flexibility) that the characteristic foliation of ∂N is given by (p, q)-curves that are (Legendrian rulings) and two Legendrians parallel to Γ (Legendrian divides). The Legendrian $L_{p,q}$ is one of those Legendrian rullings.

Relation between $C(L)$ and $C(L_{p,q})$

- The complement of $L_{p,q}$ in ∂N is a convex annulus A with Legendrian boundary, foliated by the rulings and with dividing set given by $2n = 2|q - tb(L)p|$ parallel curves. Importantly, for $q/p > tb(L) + 1$ we have that $n > 1$.
- We have that $C(L_{p,q}) \setminus A = \mathcal{O}p(L) \sqcup C(L)$
- For generalized cables, the description is analogous but with many annuli.

n-standard annulus

Fix an inclusion $j : A \hookrightarrow (M, \xi)$ of an n-standard annulus into a tight contact manifold. Consider the space $E(A, M)$ of embeddings of A that coincide with j near the boundary and are smothly isotopic to j. Define $E(A, (M, \xi)) \subseteq E(A, M)$ to be the subspace of n-standard embeddings.

Theorem (F-Min)

Assume that ∂A maximizes the twisting number with respect to any given framing and $n > 1$. If $n = 1$ assume that A unwraps in some covering and the boundary still maximizes the twisting number. Then, the inclusion $E(A, (M, \xi)) \hookrightarrow E(A, M)$ is a weak equivalence.

Corollary (Gluing)

Assume that (M_A, ξ_A) is obtained by gluing two n-standard annuli $A_1, A_2 \subseteq \partial(M, \xi)$ that satisfy the previous conditions. Then, (M_A, ξ_A) is tight. Conversely, if (M_A, η) is tight, $\eta_{|\mathcal{O}p(\partial M_A)} = \xi_{|\mathcal{O}p(\partial A)}$ and $\partial(A = A_1 = A_2)$ satisfies the previous conditions then $\eta = \xi_A$ for some ξ on M. Moreover, there is a weak equivalence $\mathcal{C}(M, \xi) \cong \mathcal{C}(M_A, \xi_A)$.

The proof of the result about generalized cables follows easily from the gluing result. The "sufficiently positive" condition ensures that $n > 1$ for every annuli and the *tb* condition implies the maximality of the twisting number. Notice that $\mathcal{C}(\mathcal{O}p(L), \xi_{\text{std}}) = \mathcal{C}(J^1 S^1, \xi_{\text{std}})$ is contractible because the Legendrian unknot satisfies the C-property.

The inclusion $E(A,(M,\xi)) \hookrightarrow E(A,M)$

To prove that $E(A,(M,\xi)) \hookrightarrow E(A,M)$ is a weak equivalence we use the microfibration trick. This is a recipe to prove weak equivalences between embedding spaces. The two conditions that one must check are

1. **Density:** Given $e \in E(A,M)$ and $\mathcal{O}p(e(A))$ there is some $\hat{e} \in E(A,(M,\xi))$ such that $\hat{e}(A) \subseteq \mathcal{O}p(e(A))$.
2. **Local Equivalence:** If $(M,\xi) = (A \times I, \xi)$ is an I-invariant neighborhood of A the statement is true.

The density property follows from the fact that the annuli $j(A)$ does not admit non-trivial bypasses in (M,ξ). The local equivalence follows from a fibration argument since $\mathrm{Cont}(A \times I, \xi) \cong \mathrm{Diff}(A \times I) \cong *$.

Thanks

Thanks for your attention!