# The Lagrangian cobordism group of Weinstein manifolds

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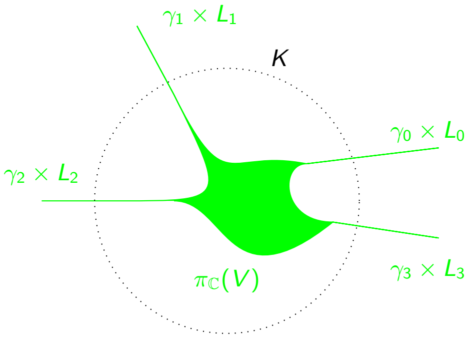
# Lagrangian Cobordisms

Let $X$ be a symplectic manifold.

## Definition (Arnold '80)

A *Lagrangian cobordism* is a Lagrangian submanifold $V$ in $\mathbb{C} \times X$ if there is a compact set $K \subset \mathbb{C}$ such that

$$\pi_{\mathbb{C}}^{-1}(\mathbb{C} \setminus K) \cap V = \bigcup_{j=0}^{m} (\gamma_j \times L_j).$$



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Let $\mathcal{L}^*(X)$ be a class of Lagrangian submanifolds, i.e. $*$ is a subset

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$$\Omega^*(X) = \bigoplus_{L \in \mathcal{L}^*(X)} \mathbb{Z}L \bigg/ \substack{L_0 + \cdots + L_m = 0 \\ \text{if there is a Lag cob } V \in \mathcal{L}^*(\mathbb{C} \times X) \\ \text{with ends } L_0, \dots, L_m}$$

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- *(Lagrangian suspension) If* $L_0$ *and* $L_1$ *are Hamiltonian isotopic then* $L_0 = L_1$ *in* $\Omega^*(X)$.
- *(Lagrangian surgery) If* $L_0 \pitchfork L_1 = \{p\}$ *then* $L_0 \#_p L_1 = L_0 + L_1$ *in* $\Omega^*(X)$.

# The Lagrangian cobordism group of Weinstein manifolds

**Theorem (B. '23)**

*Let $X^{2n}$ be a Weinstein manifold and $* = \{\text{oriented, exact conical at } \infty\}$. Then*

$$\Omega^*(X) \cong H^n(X) \cong H_n(X_0, \partial X_0)$$

*induced by sending a Lagrangian L to its relative homology class.*
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## Example

$X = T^*M$, $M$ *compact. Then*

$$\Omega^*(T^*M) \cong H^n(T^*M) \cong H^*(M) \cong \begin{cases} \mathbb{Z}, & M \text{ orientable}, \\ \mathbb{Z}/2\mathbb{Z}, & M \text{ not orientable}, \end{cases}$$

*generated by a fiber.*

# Previous results

| | $X$ | * | $\Omega^*(X)$ |
|---|---|---|---|
| Arnold '80 | $T^*S^1$ | oriented, immersed circles | $H_1(S^1) \oplus \mathbb{R} \oplus \mathbb{Z}$ |

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| B. '22 | $\Sigma$ with $\partial$ | oriented arcs and oriented exact circles | $H^1(\Sigma) \cong H_1(\Sigma, \partial\Sigma)$ |

# Relation to Floer-theoretic invariants

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\Omega^*(X) & \xrightarrow{\;i\;} & H^n(X) & \xrightarrow{\;\mathcal{A}\;} & SH^n(X) \\
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*$\Theta$ is not always an isomorphism.*

Let $X$ be flexible Weinstein and not subcritical

$$K_0(\mathcal{W}(X)) = 0, \qquad \Omega^*(X) \cong H^n(X) \neq 0.$$

# Viterbo restriction

**Corollary**

*Let $X$ be a Liouville manifold, $Y \subset X$ Weinstein subdomain. Then*

$$\Omega^*(X) \xrightarrow{i} H^n(X) \longrightarrow H^n(Y) \xrightarrow{i^{-1}} \Omega^*(Y)$$

*is well-defined.*

# Proof $i$ well-defined

Let $X$ be a Liouville manifold. Want to show that

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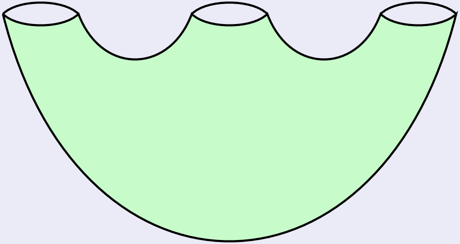
Well-definedness: Look at a cobordism $V \subset \mathbb{C} \times X$ as a relative $(n+1)$-cycle in $X$ that defines singular homology. It has boundary $L_0 + \cdots + L_m$.

# Weinstein manifolds

## Definition

$X^{2n}$ Liouville is *Weinstein* if the Liouville flow is gradient-like for a Morse function $f$ on $X$.

## Example (Pair of pants)



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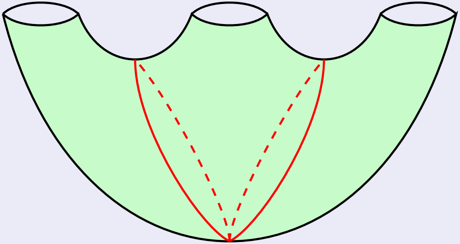

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- Its cocores are the unstable manifolds of the critical points of $f$ of index $n$.

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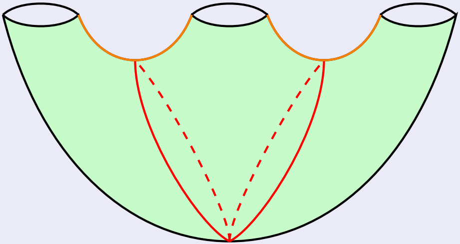

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is surjective: Let $x \in \mathrm{Crit}_n(f)$. Then

$$[i(\mathrm{cocore}(x))] = [x].$$

So $i$ is surjective.

# Proof $i$ injective for Weinstein $X$

Let $X$ be Weinstein.

**Theorem (Hanlon-Hicks '22)**

*If $L \pitchfork \mathrm{core}_X$ then there is a Lagrangian cobordism with ends cocores at $L \pitchfork \mathrm{core}_X$ and $L$.*

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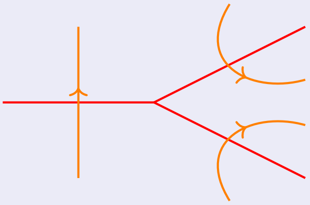

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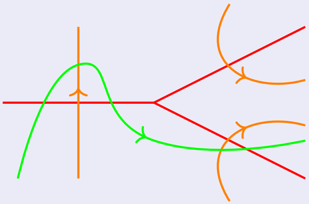

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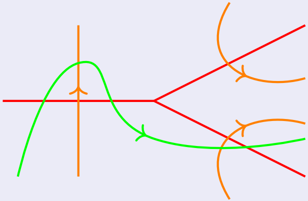

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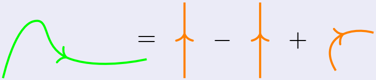

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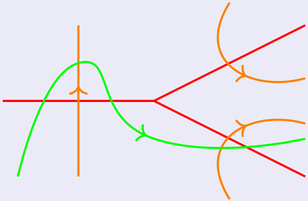

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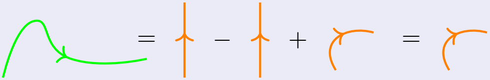

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$$H^n(X) = \bigoplus_{x\in \mathrm{Crit}_n(f)} \mathbb{Z}x \Bigg/ \begin{matrix} x_0+\cdots+x_m=0 \\ \text{if } y \rightsquigarrow x_j \text{ are all Morse} \\ \text{tracectories for } y \in \mathrm{Crit}_{n-1}(f) \end{matrix}$$

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The map

$$\Omega^*(X) \xrightarrow{i} H^n(X)$$

is injective if the relation:

$$\mathrm{cocore}(x_0) + \cdots + \mathrm{cocore}(x_m) = 0$$

whenever

$$y \rightsquigarrow x_j \text{ are all Morse tracectories for } y \in \mathrm{Crit}_{n-1}(f)$$

is induced by a Lagrangian cobordism.

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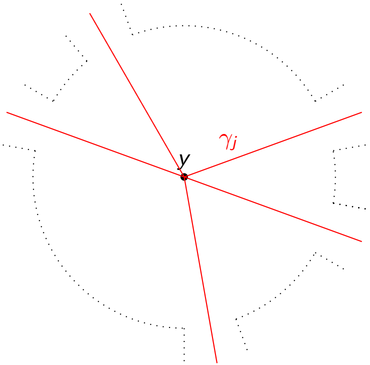
Following Lazarev '22: An $(n-1)$-handle is modelled on $T^*B^{n-1} \times B^2$.



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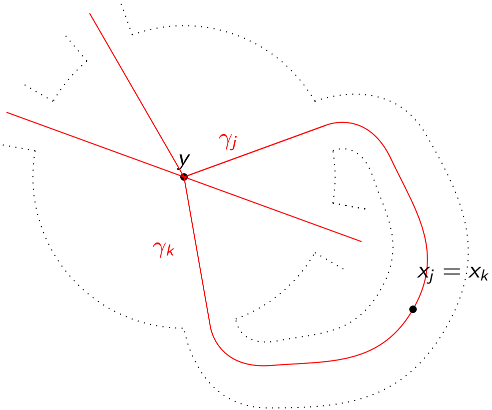

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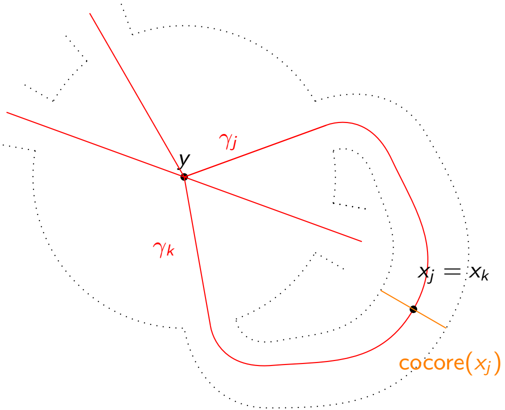

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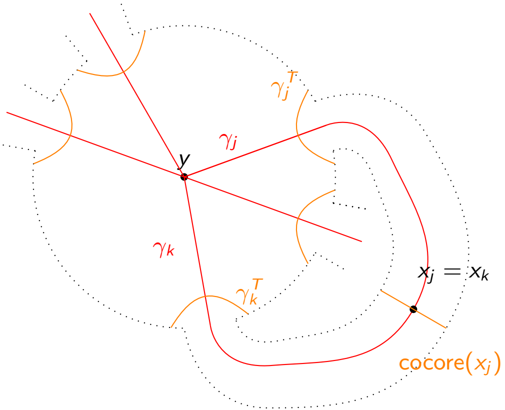

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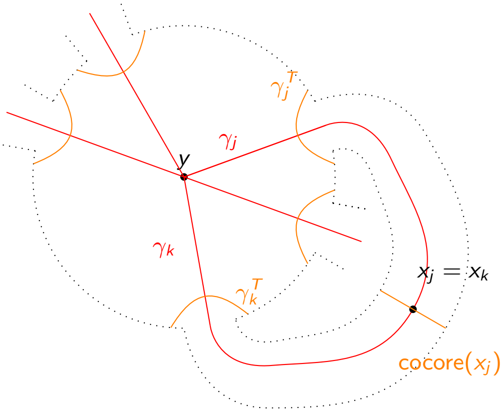


Note that $B^{n-1} \times \gamma_j^T \cong \mathrm{cocore}(x_j)$.

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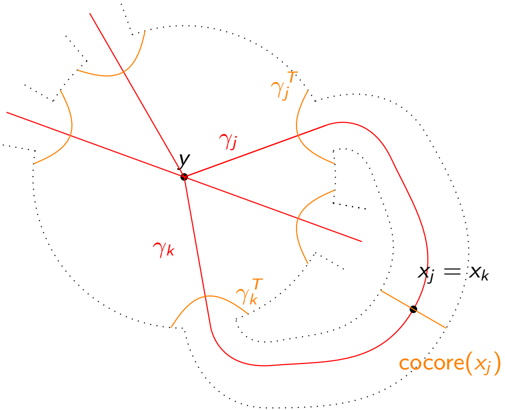


Note that $B^{n-1} \times \gamma_j^T \cong \mathrm{cocore}(x_j)$. By Lagrangian surgery we get that

$$(B^{n-1} \times \gamma_0) \# \cdots \# (B^{n-1} \times \gamma_m)$$

is nullhomotopic, hence in $\Omega^*(X)$:

$$\mathrm{cocore}(x_0) + \cdots + \mathrm{cocore}(x_m) = 0.$$