Kähler-type embeddings of balls into symplectic manifolds

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Starting question

(M^{2n}, ω) – a closed symplectic manifold.

$B^{2n}(r) \subset \mathbb{R}^{2n}$ – the closed ball of radius r centered at 0.

Q.: When can two symplectic embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M^{2n}, \omega)$ be mapped into each other by $\mathrm{Symp}_0(M, \omega)$ ($\Leftrightarrow$ by $\mathrm{Ham}(M, \omega)$)? By $\mathrm{Symp}(M, \omega)$? By intermediate subgroups of $\mathrm{Symp}(M, \omega)$?

Thm (McDuff, 1997; previous partial results by McDuff, Lalonde, Biran): *If* $\dim_{\mathbb{R}} M = 4$ *and* (M, ω) *has "enough non-trivial Gromov-Witten invariants" (e.g. if* (M, ω) *is a rational or ruled surface), then for any* $\bigsqcup_{i=1}^k B^{2n}(r_i)$ *any two symplectic embeddings* $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (M^{2n}, \omega)$ *(if they exist!) can be mapped into each other by* $\mathrm{Symp}_0(M, \omega)$ *(*$\Leftrightarrow$ *the space of such sympl. emb. is path-connected).*

Q.: What can be said about other (in part., higher-dim.) (M, ω)?

Will consider a more restrictive class of symplectic embeddings that can be studied using the tools of *complex geometry*.

Kähler structures

M – closed connected oriented manifold, $\dim_{\mathbb{R}} M = 2n$.

A **Kähler structure** on a manifold M is a pair (ω, J), where

1. ω is a symplectic form on M,
2. J is an (*integrable*) complex structure on M compatible with ω.

A symplectic form, or a complex structure, on M is said to be of **Kähler type**, if it appears in *some* Kähler structure.

Further on: ω is a fixed Kähler-type symplectic form on M; all symplectic forms and complex structures on M are assumed to be compatible with the orientation.

$\mathrm{Symp}_H(M, \omega)$ – the symplectom-s of (M, ω) acting trivially on $H_*(M)$.

$\mathcal{C}(M)$ – the space of (Kähler-type) complex structures on M.

$\mathcal{C}(M, \omega) \subset \mathcal{C}(M)$ – the space of (Kähler-type) complex structures on M compatible with ω.

Teichmüller space

$\texttt{Teich}(M) := \mathcal{C}(M)/\mathrm{Diff}_0(M)$ – **Teichmüller space** (of Kähler-type complex structures on M),

$pr : \mathcal{C}(M) \to \texttt{Teich}(M)$ – the natural projection.

$[I] := pr(I)$ – the $\mathrm{Diff}_0(M)$-orbit of I.

Kähler-type embeddings (I)

Definition

Let $f : \coprod_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ be a symplectic embedding.

First, fix a complex structure $J \in \mathcal{C}(M, \omega)$.

Then f is called **Kähler** (w.r.t. the Kähler structure (ω, J)) if it is both symplectic (w.r.t. ω) and holomorphic (w.r.t. J). For such an f, the Kähler-metric $\omega(\cdot, J\cdot) + \sqrt{-1}\omega(\cdot, \cdot)$ is flat on the image of f.

Now assume that no $J \in \mathcal{C}(M, \omega)$ is fixed in advance.

f is called **Kähler-type** if, in addition to being symplectic w.r.t. ω, it is also holomorphic with respect to some *(not a priori fixed)* complex structure $J \in \mathcal{C}(M, \omega)$.

If this J can be chosen to lie in the $\mathrm{Diff}_0(M)$-orbit $[I]$ of a complex structure $I \in \mathcal{C}(M, \omega)$, we say that f is of $[I]$**-Kähler type**.

More generally, if such a J can be picked from a certain subset (e.g. a connected component) of $\mathcal{C}(M)$, we say that f **favors** that subset.

Kähler-type embeddings (II)

Remarks:

1. f is of **Kähler type** $\Leftrightarrow$ f is of $[I]$-Kähler type for some $I \in \mathcal{C}(M, \omega)$.

2. $\exists$ $[I]$-Kähler-type embedding into (M, ω) $\Longleftrightarrow$ $\exists$ *Kähler* embedding into (M, ω') for some Kähler form ω' on (M, I), s.t. $[\omega] = [\omega']$.

For a fixed I, the existence of $[I]$-Kähler-type embeddings was previously studied from this angle by Eckl, Witt Nystrom, Fleming, Luef-Wang, Trusiani.

3. In principle, a Kähler-type embedding may be of $[I]$-Kähler type for different non-isotopic $I \in \mathcal{C}(M, \omega)$ and may favor several different connected components of $\mathcal{C}(M, \omega)$ or of $\mathcal{C}(M)$.

4. There do exist symplectic embeddings that are *not* Kähler-type (will see below).

Kähler-type embeddings $f, f' : \bigsqcup_{i=1}^k B^{2n}(r_i) \to (M, \omega)$ can be connected by a smooth path in the space of the Kähler-type embeddings

$$\Updownarrow \text{ (isotopy extension)}$$

f, f' lie in the same $\mathrm{Symp}_0(M, \omega)$-orbit

$$\Downarrow$$

f, f' are holomorphic with respect to complex structures compatible with ω and lying in the same $\mathrm{Symp}_0(M, \omega)$-orbit, hence in the same path-connected component of $\mathcal{C}(M, \omega)$, hence in the same connected component of $\mathcal{C}(M)$

$$\Downarrow$$

f, f' favor the same path-connected component of $\mathcal{C}(M, \omega)$ (and the same connected component of $\mathcal{C}(M)$)

In general – very little info on $\mathrm{Symp}_0(M, \omega)$-orbits in $\mathcal{C}(M, \omega)$...

However, in some cases – better info on $\mathrm{Diff}_0(M)$-orbits of c.s. in $\mathcal{C}(M, \omega)$, leading to results on $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$-action on Kähler-type emb. (Note that $\mathcal{C}(M, \omega)$ is <u>not</u> $\mathrm{Diff}_0(M)$-invariant!)

Blow-ups

$\hat{M}^k$:= the space of k-tuples of pairwise distinct points of M,
$\mathbb{x} := (x_1, \ldots, x_k) \in \hat{M}^k$.

I – a complex structure on M.

$\widetilde{M}_{I,\mathbb{x}}$ – the complex blow-up of (M, I) at $x_1, \ldots, x_k$.

$\widetilde{I}$ – the lift of I to $\widetilde{M}_{I,\mathbb{x}}$.

$\Pi : \widetilde{M}_{I,\mathbb{x}} \to M$ – the natural projection.

$e_1, \ldots, e_k \in H^2(\widetilde{M}_{I,\mathbb{x}}; \mathbb{R})$ – the cohomology classes Poincaré-dual to the homology classes of the exceptional divisors.

Definition

For $r_1, \ldots, r_k > 0$, $\mathbb{r} = (r_1, \ldots, r_k)$, define $K(\mathbb{r}) \subset \mathcal{C}(M, \omega)$ as
$K(\mathbb{r}) := \Big\{ I \in \mathcal{C}(M, \omega) \;\Big|\; \exists \mathbb{x} \in \hat{M}^k \; \Pi^*[\omega] - \pi \sum_{i=1}^k r_i^2 e_i \in H^2(\widetilde{M}_{I,\mathbb{x}}; \mathbb{R})$ *is Kähler w.r.t.* $\widetilde{I} \Big\}$.

Existence – main result

Theorem

Let $r_1, \ldots, r_k > 0$, $\mathbb{r} = (r_1, \ldots, r_k)$. *The following are equivalent:*

- $\exists$ *a Kähler-type embedding* $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$.
- $K(\mathbb{r}) \neq \emptyset$ *(i.e.,* $\exists\ I \in \mathcal{C}(M, \omega)$ *and* $\mathbb{x} \in \hat{M}^k$ *s.t.* $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,\mathbb{x}}; \mathbb{R})$ *is Kähler with respect to* $\widetilde{I}$*).*

More precisely, if $I \in \mathcal{C}(M, \omega)$*, then the following are equivalent:*

- $\exists \mathbb{x} \in \hat{M}^k$ *s.t.* $\Pi^*[\omega] - \pi \sum_{i=1}^{k} r_i^2 e_i \in H^2(\widetilde{M}_{I,\mathbb{x}}; \mathbb{R})$ *is Kähler w.r.t.* $\widetilde{I}$.
- $\exists$ $[I]$*-Kähler-type embedding* $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$.

Remarks:

1. The proof is a modification of the proof of a similar result for *symplectic* embeddings of balls (McDuff-Polterovich, 1994).

2. A similar existence result holds for $[I]$-Kähler-type embeddings into $(M \setminus \Sigma, \omega)$ for a proper complex submanifold $\Sigma \subset (M, I)$.

Connectedness – main result

Theorem

Let $\mathcal{C}_0$ *be a connected component of* $\mathcal{C}(M)$*.*
Assume that pr $(K(\mathfrak{r}) \cap \mathcal{C}_0) \subset \texttt{Teich}(M)$ *is connected.*

Then any two K.-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ *favoring* $\mathcal{C}_0$ *lie in the same* $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$*-orbit. In particular, both embeddings are of* $[I]$*-Kähler type for the same* I*.*

If, in addition, $\mathrm{Symp}_H(M)$ *acts transitively on the set of connected components of* $\mathcal{C}(M)$ *intersecting* $\mathcal{C}(M, \omega)$*, then any two Kähler-type embeddings* $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ *lie in the same* $\mathrm{Symp}_H(M, \omega)$*-orbit.*

Remark:

Assume $\dim_{\mathbb{R}} M = 4$ and (M, ω) has "enough non-trivial Gromov-Witten invariants" (e.g. if (M, ω) is a rational or ruled surface).

Then, for any $\bigsqcup_{i=1}^{k} B^4(r_i)$, any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (M^4, \omega)$ (if they exist!) can be mapped into each other by $\mathrm{Symp}_0(M, \omega)$ (since, by McDuff's thm., the same is true for any two *symplectic* embeddings).

Consequently, as long as there exists *a* Kähler-type embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (M^4, \omega)$, *all symplectic* embeddings $\bigsqcup_{i=1}^{k} B^4(r_i) \to (M^4, \omega)$ are of Kähler-type – because the set of Kähler-type embeddings is $\mathrm{Symp}_0(M, \omega)$-invariant.

Sample applications: $\mathbb{C}P^n$ (I)

Theorem

In the following cases we have necessary and sufficient conditions for the existence of Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS})$*:*

A. $k = l^n$, $r_1 = \ldots = r_k =: r$*:* $\mathrm{Vol}\,(\bigsqcup_{i=1}^k B^{2n}(r)) < \mathrm{Vol}\,(\mathbb{C}P^n, \omega_{FS})$.

B. $n = 2$, $1 \leq k \leq 8$*:* $\mathrm{Vol}\,(\bigsqcup_{i=1}^k B^4(r_i)) < \mathrm{Vol}\,(\mathbb{C}P^2, \omega_{FS})$ *& additional explicit quadratic inequalities on* $r_1, \ldots, r_k > 0$ *(coming from the description of the Kähler cone of the blow-up of* $\mathbb{C}P^2$ *at* k *generic points).*

In both cases for any complex structure I *on* $\mathbb{C}P^n$ *compatible with* ω_{FS} *(and, in particular, for the standard complex structure* I_{st}*), there exists an* $[I]$*-Kähler-type embedding* $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS})$.

Sample applications: $\mathbb{C}P^n$ (II)

Corollary

Let I_{st} be the standard complex structure on $\mathbb{C}P^n$. Assume that $\mathrm{Vol}\,(B^{2n}(r)) < \mathrm{Vol}\,(\mathbb{C}P^n, \omega_{FS})$.
Then there exists a Kähler form ω on $(\mathbb{C}P^n, I_{st})$ isotopic to ω_{FS} and such that the Kähler manifold $(\mathbb{C}P^n, I_{st}, \omega)$ admits a Kähler (that is, both holomorphic and symplectic) embedding of $B^{2n}(r)$ with the standard flat Kähler metric on it.

For $n = 2$ this was previously proved by Eckl (2017).

Remark:
For any $\bigsqcup_{i=1}^{k} B^4(r_i)$, any Kähler-type embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ (if it exists!) is, in fact, of $[I_{st}]$-Kähler-type.
If $k = l^2$ and $r_1 = \ldots = r_k$ or if $1 \leq k \leq 8$, then any symplectic embedding $\bigsqcup_{i=1}^{k} B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ is of Kähler-type – in fact, of $[I_{st}]$-Kähler-type.

Sample applications: $\mathbb{C}P^n$ (III)

Theorem

For any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$, any two Kähler-type embeddings $\bigsqcup_{i=1}^k B^{2n}(r_i) \to (\mathbb{C}P^n, \omega_{FS})$ can be mapped into each other by $\mathrm{Symp}(\mathbb{C}P^n, \omega_{FS}) = \mathrm{Symp}_H(\mathbb{C}P^n, \omega_{FS})$. They can be mapped into each other by $\mathrm{Symp}(\mathbb{C}P^n, \omega_{FS}) \cap \mathrm{Diff}_0(\mathbb{C}P^n)$ if and only if they favor the same connected component of $\mathcal{C}(\mathbb{C}P^n)$.

Remarks:

1. For $n = 2$ the group $\mathrm{Symp}(\mathbb{C}P^2, \omega_{FS})$ is connected (Gromov) and thus for any $k \in \mathbb{Z}_{>0}$ and $r_1, \ldots, r_k > 0$ the space of Kähler-type embeddings $\bigsqcup_{i=1}^k B^4(r_i) \to (\mathbb{C}P^2, \omega_{FS})$ is path-connected (as also follows from McDuff's thm. about *symplectic* embeddings in dim. 4).

2. For $n = 3$ (Kreck-Su) and $n = 4$ (Brumfiel) the space $\mathcal{C}(\mathbb{C}P^n)$ has more than one connected component. Unknown for other n.

3. For $n > 2$ it is unknown if $\mathrm{Symp}(\mathbb{C}P^n, \omega_{FS})$ is connected or lies in $\mathrm{Diff}_0(\mathbb{C}P^n)$.

Sample applications: tori and K3 surfaces (I)

Assume M is either $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ or a smooth manifold underlying a K3 surface; ω is a Kähler-type symplectic form on M.

Remark: The Kähler-type symplectic/complex structures on $\mathbb{T}^{2n}$ (compatible with the standard orientation) are exactly the ones that can be mapped by a diffeomorphism of $\mathbb{T}^{2n}$ to a linear symplectic/complex structure. It is unknown if there exist non-Kähler-type symplectic forms on $\mathbb{T}^{2n}$, $n > 1$, or on K3 surfaces.

Definition

The form ω is called **rational** if $[\omega] \in H^2(M;\mathbb{R})$ is proportional to a rational homology class, and **irrational** otherwise.

Sample applications: tori and K3 surfaces (II)

Theorem

Assume M is either $\mathbb{T}^{2n} = \mathbb{R}^{2n}/\mathbb{Z}^{2n}$ or a smooth manifold underlying a K3 surface.
Let ω be a Kähler-type symplectic form on M. Assume that either ω is irrational or $M = \mathbb{T}^2$.
Then:

$$\mathrm{Vol}\left(\bigsqcup_{i=1}^{k} B^{2n}(r_i)\right) < \mathrm{Vol}\,(M,\omega)$$
$$\Updownarrow$$
$$\exists \text{ a Kähler-type embedding } \bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M,\omega)$$

Remark:
For $M = \mathbb{T}^{2n}$ and one *ellipsoid* – and, in particular, for one ball – this was previously proved by Luef and Wang (2021) using a similar method. Their work relates the problem for $M = \mathbb{T}^{2n}$ to Gabor frames (an important notion in signal processing).

Sample applications: tori and K3 surfaces (III)

Theorem

Let ω be a Kähler-type symplectic form on M. Assume that either ω is irrational or $M = \mathbb{T}^2$.
Then for any $k \in \mathbb{Z}_{>0}$ and any $r_1, \ldots, r_k > 0$ any two Kähler-type embeddings $\bigsqcup_{i=1}^{k} B^{2n}(r_i) \to (M, \omega)$ can be mapped into each other by $\mathrm{Symp}_H(M, \omega)$. *They can be mapped into each other by* $\mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$ *if and only if they favor the same connected component of* $\mathcal{C}(M)$.

Remarks:

1. It is unknown whether $\mathrm{Symp}_H(\mathbb{T}^{2n}, \omega) = \mathrm{Symp}_0(\mathbb{T}^{2n}, \omega)$ for any Kähler-type symplectic form ω on $\mathbb{T}^{2n}$, $n > 1$.

2. In the K3 case, for at least some irrational ω:
$\mathrm{Symp}_0(M, \omega) \subsetneqq \mathrm{Symp}_H(M, \omega)$ (Sheridan-Smith, 2020),
$\mathrm{Symp}_0(M, \omega) \subsetneqq \mathrm{Symp}(M, \omega) \cap \mathrm{Diff}_0(M)$ (Seidel, 2000; Smirnov, 2022).

Sample applications: tori and K3 surfaces (IV)

Remark: If the Kähler-type form ω on $\mathbb{T}^{2n}$ is *rational*, then there may be obstructions to the existence of Kähler-type embeddings of balls into (M, ω) that are independent of the symplectic volume – for instance, obstructions coming from Seshadri constants.

Example: Let $M = \mathbb{T}^4$, $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, $\mathrm{Vol}\,(\mathbb{T}^4, \omega) = 2$.

For any complex structure I on $\mathbb{T}^4$ compatible with ω one can biholomorphically identify $(\mathbb{T}^4, I)$ with a *principally polarized* abelian variety. A universal upper bound on the Seshadri constants for all such varieties (Steffens, 1998) yields that if $(4/3)^2 < \mathrm{Vol}\,(B^4(r)) < 2$, then there are <u>no</u> Kähler-type embeddings $B^4(r) \to (\mathbb{T}^4, \omega)$.

However, for such r there do exist symplectic embeddings $B^4(r) \to (\mathbb{T}^4, \omega)$ (Latschev-McDuff-Schlenk, 2013; E.-Verbitsky, 2016).

Tame embeddings (I)

Definition

Assume:
M^{2n} *is a closed manifold,* ω *is a Kähler-type symplectic form on* M*;*
$\mathbb{W}$ *is a disjoint union of compact domains with boundary in* $\mathbb{R}^{2n}$*.*

For $\varepsilon > 0$*, a symplectic embedding* $\mathbb{W} \to (M, \omega)$ *is called* ε**-tame** *if it is holomorphic w.r.t some (not a priori fixed!) complex structure* I *on* M *which is "*ε*-almost compatible with* ω*" – i.e., such that*

- I *is tamed by* ω*,*
- *the cohomology class* $[\omega]_I^{1,1}$ *is Kähler,*
- $\left| \left\langle \left([\omega]_I^{2,0} + [\omega]_I^{0,2} \right)^n, [M] \right\rangle \right| < \varepsilon.$

Remark: For symplectic embeddings $\mathbb{W} \to (M, \omega)$:
Kähler-type $\Longrightarrow$ ε-tame for every $\varepsilon > 0$.

Tame embeddings (II)

Theorem

Assume that M is either $\mathbb{T}^{2n}$ or a smooth manifold underlying a K3 surface and $\mathbb{W} := \bigsqcup_{i=1}^{k} W_i$ is a disjoint union of compact domains with boundary specified in the next slide.

Then for any $\varepsilon > 0$ there exists a $\mathrm{Diff}^+(M)$-invariant open dense set $\Theta(\mathbb{W}, \varepsilon)$ of Kähler-type symplectic forms on M (depending on $\mathbb{W}$ and ε and containing, in particular, all irrational Kähler-type symplectic forms on M), so that for each $\omega \in \Theta(\mathbb{W}, \varepsilon)$, the only obstruction to the existence of ε-tame symplectic embeddings $\mathbb{W} \to (M, \omega)$ is the symplectic volume.

This holds (at least) if $\mathbb{W}$ is either of the following...

Tame embeddings (III)

Theorem (continued)

... This holds (at least) if $\mathbb{W}$ is either of the following:

- *a disjoint union of k (possibly different) $2n$-dimensional balls,*
- *a disjoint union of k identical copies of a parallelepiped $P(e_1, \ldots, e_{2n}) := \left\{\sum_{j=1}^{2n} s_j e_j, 0 \leq s_j \leq 1, j = 1, \ldots, 2n\right\}$, spanned by a basis $e_1, \ldots, e_{2n}$ of $\mathbb{R}^{2n}$.*

If $M = \mathbb{T}^{2n}$, we also allow $\mathbb{W}$ to be a disjoint union of k identical copies of a $2n$-dim. polydisk $B^{2n_1}(r_1) \times \ldots \times B^{2n_l}(r_l)$, $n_1 + \ldots + n_l = n$.

Remark: If $M = \mathbb{T}^{2n}$, then for all $\mathbb{W}$ above and all $\varepsilon > 0$, the open dense set $\Theta(\mathbb{W}, \varepsilon)$ appearing in the theorem contains a $\mathrm{Diff}^+(\mathbb{T}^{2n})$-orbit of an irrational K.-type form, so that for each form ω in the orbit, the only obstruction to the existence of Kähler-type symplectic embeddings $\mathbb{W} \to (M, \omega)$ is the symplectic volume. (The orbit is dense in the set of all Kähler-type symplectic forms of a fixed volume on $\mathbb{T}^{2n}$).

THANK YOU!