Global Kuranishi charts for Gromov–Witten moduli spaces and a product formula

Symplectic Zoominar

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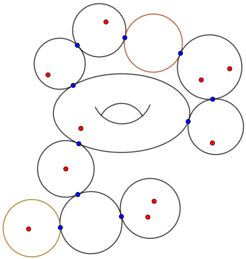
Set-up

Let (X, ω) be a closed symplectic manifold and $J \in \mathcal{J}_\tau(X, \omega)$.

We want to define invariants using the *moduli space of stable maps*

$$\overline{\mathcal{M}}^J_{g,n}(X, \beta).$$

It consists of J-holomorphic maps $u \colon C \to X$ where C is of the form



and

1. g the arithmetic genus,
2. $n = \#$ marked points,
3. $[u] = \beta \in H_2(X, \mathbb{Z})$,
4. stable iff $|\mathrm{Aut}(u, C, x_1, \ldots, x_n)| < \infty$.

Gromov–Witten invariants

$\overline{\mathcal{M}}^J_{g,n}(X,\beta)$ is compact and metrisable. Using

$$\begin{array}{ccc} & \overline{\mathcal{M}}^J_{g,n}(X,\beta) & \\ {}^{\mathrm{ev}}\swarrow & & \searrow^{\mathrm{st}} \\ X^n & & \underbrace{\overline{\mathcal{M}}_{g,n}}_{\text{if } 2g-2+n>0} \end{array}$$

we want to define

$$\mathrm{GW}^{(X,\omega)}_{\beta,g,n} : H^*(X^n,\mathbb{Q}) \to H^*(\overline{\mathcal{M}}_{g,n},\mathbb{Q})$$

by

$$\mathrm{GW}^{(X,\omega)}_{\beta,g,n}(\alpha) = \mathrm{PD}(\mathrm{st}_*(\mathrm{ev}^*\alpha \cap [\overline{\mathcal{M}}^J_{g,n}(X,\beta)])).$$

Problem: $\overline{\mathcal{M}}^J_{g,n}(X,\beta)$ often does not admit a fundamental class in the expected degree $\rightsquigarrow$ replace with a suitable *virtual fundamental class*

Global Kuranishi chart

A *global Kuranishi chart* $\mathcal{K}$ for a compact space $\mathcal{M}$ consists of

- a compact Lie group G,
- a G-manifold $\mathcal{T}$, the *thickening*, with finite isotropy
- a G-vector bundle $\mathcal{E} \to \mathcal{T}$, the *obstruction bundle*,
- an equivariant section $\mathfrak{s} \colon \mathcal{T} \to \mathcal{E}$ such that

$$\mathfrak{s}^{-1}(0)/G \cong \mathcal{M}.$$

The *virtual dimension* of $\mathcal{M}$ is

$$\mathrm{vdim}_{\mathcal{K}}(\mathcal{M}) := \dim(\mathcal{T}) - \dim(G) - \mathrm{rank}(\mathcal{E}).$$

Virtual fundamental class

An *orientation* of $\mathcal{K}$ is a G-orientation of $[T\mathcal{T} - \underline{\mathfrak{g}}]$ and of $\mathcal{E}$.

Given this, the *virtual fundamental class* $[\mathcal{M}]^{\mathrm{vir}} \in \check{H}^d(\mathcal{M}, \mathbb{Q})^*$ is the composition

$$\check{H}^{\mathrm{vdim}}(\mathcal{M}; \mathbb{Q}) \xrightarrow{\simeq} H^G_{\mathrm{rank}(\mathcal{E})}(\mathcal{T}, \mathcal{T} \backslash \mathfrak{s}^{-1}(0); \mathbb{Q}) \xrightarrow{\mathfrak{s}^* \tau \cdot} \mathbb{Q}$$

where τ is the equivariant Thom class of $\mathcal{E}$.

Remark

We use $\mathbb{Q}$-coefficients due to the G-action.

Main result

Theorem (H.-Swaminathan, 2022)

Let $g, n \geqslant 0$ be arbitrary.

a) $\overline{\mathcal{M}}^J_{g,n}(X,\beta)$ *admits an oriented global Kuranishi chart of the expected virtual dimension.*

b) *Its virtual fundamental class is independent of the auxiliary choices made during the construction.*

c) *Given another $J' \in \mathcal{J}_\tau(X,\omega)$, we can make the auxiliary choices, so that there exists a cobordism between the global Kuranishi charts associated to the respective moduli space.*

$$\rightsquigarrow \quad \mathrm{GW}^{(X,\omega)}_{\beta,g,n} := \mathrm{PD}(\mathrm{st}_*(\mathrm{ev}^*(-) \cap [\overline{\mathcal{M}}^J_{g,n}(X,\beta)]^{\mathrm{vir}}))$$

is well-defined

Remarks

1. *Local* Kuranishi chart has been used in several approaches before. A global Kuranishi chart removes the need for a complicated patching-together of the virtual fundamental class.

2. Abouzaid, McLean and Smith (AMS) were the first to construct a global Kuranishi chart in genus 0. (arXiv:2110.14320)
   They have an independent construction in higher genus.

3. AMS construct a Morava K-theory valued virtual fundamental class $\rightsquigarrow$ quantum K-theory

4. Bai-Xu: $\mathbb{Z}$-valued Gromov–Witten type invariants (arXiv:2201.02688)

Product formula

Theorem (H.-Swaminathan, 2022)

If $(X,\omega) = (X_0,\omega_0) \times (X_1,\omega_1)$*, then*

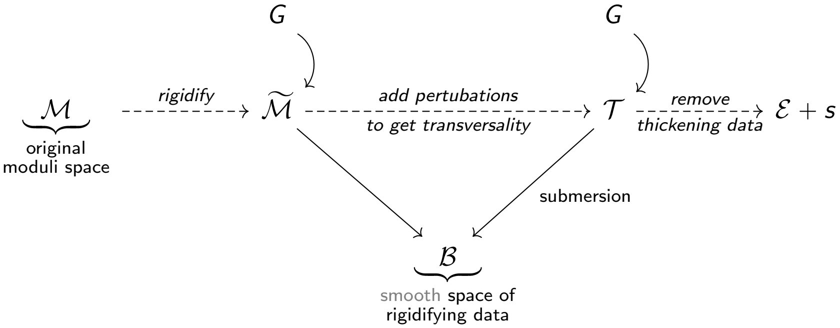
$$\sum_{p_{i*}\beta=\beta_i} \mathrm{GW}^{(X,\omega)}_{\beta,g,n}(\alpha_0 \times \alpha_1) = \mathrm{GW}^{(X_0,\omega_0)}_{\beta_0,g,n}(\alpha_0) \smile \mathrm{GW}^{(X_1,\omega_1)}_{\beta_1,g,n}(\alpha_1)$$

for any $(g,n) \notin \{(1,1),(2,0)\}$ *and* $\alpha_i \in H^*(X_i^n,\mathbb{Q})$*.*

In particular, the small quantum cohomology ring splits over the universal Novikov ring Λ_0*:*

$$\mathrm{QH}^*(X) \cong \mathrm{QH}^*(X_0) \otimes_{\Lambda_0} \mathrm{QH}^*(X_1).$$

Constructing a global Kuranishi chart from scratch



In our case (for $n = 0$):

- rigidifying data: highly positive embeddings $C \hookrightarrow \mathbb{P}^N$ (*framings*);

  $\rightsquigarrow \mathcal{B} = \overline{\mathcal{M}}_g^*(\mathbb{P}^N, m)$ consists of embedded regular curves in $\mathbb{P}^N$

- perturbations are elements of

$$H^0(C, \overline{\mathrm{Hom}}_{\mathbb{C}}(T_{\mathbb{P}^N}|_C, u^* T_X) \otimes \mathcal{O}_C(k)) \otimes \overline{H^0(\mathbb{P}^N, \mathcal{O}(k))}$$

  for some $k \gg 1$ (an auxiliary choice)

- $G = \mathrm{PU}(N+1)$ acts on the framing and the perturbation term

The case of $n > 0$ follows formally.

Equivalence of global Kuranishi charts

We call two global Kuraishi charts *equivalent* if they are related by a zigzag of the following moves

- (Germ equivalence) Given $U \subseteq \mathcal{T}$ open G-invariant neighbourhood of $\mathfrak{s}^{-1}(0)$: $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s}) \rightsquigarrow (G, U, \mathcal{E}|_U, \mathfrak{s}|_U)$
- (Group enlargement) Given a principal G'-bundle $q \colon \mathcal{P} \to \mathcal{T}$ with compatible G-action: $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s}) \rightsquigarrow (G \times G', \mathcal{P}, q^*\mathcal{E}, q^*\mathfrak{s})$
- (Stabilisation) Given a G-vector bundle $p \colon \mathcal{W} \to \mathcal{T}$: $(G, \mathcal{T}, \mathcal{E}, \mathfrak{s}) \rightsquigarrow (G, \mathcal{W}, p^*\mathcal{E} \oplus p^*\mathcal{W}, p^*\mathfrak{s} \oplus \delta_{\mathcal{W}})$

Constructing a global Kuranishi chart from a given one

Question: Given a map $\psi\colon \mathcal{M} \to \mathcal{N}$ between moduli spaces, can we lift it to a map $\mathcal{K}_{\mathcal{M}} \to \mathcal{K}_{\mathcal{N}}$ between global Kuranishi charts?

Answer: Usually not.

But: *Global Kuranishi charts can be pulled back along maps to their base space.*

$\rightsquigarrow$ use $\mathcal{K}_{\mathcal{N}}$ to construct a new global Kuranishi chart $\widetilde{\mathcal{K}}_{\mathcal{M}}$ which

i) comes with a natural map $\widetilde{\mathcal{K}}_{\mathcal{M}} \to \mathcal{K}_{\mathcal{N}}$
ii) is equivalent to $\mathcal{K}_{\mathcal{M}}$ (work required).

Remark: This is the key idea in the proof of the product formula, using the map

$$\overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^{N_0} \times \mathbb{P}^{N_1}, (m_0, m_1)) \to \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^{N_0}, m_0) \times \overline{\mathcal{M}}^*_{g,n}(\mathbb{P}^{N_1}, m_1).$$

Thank you for your attention!