Symplectic Barriers

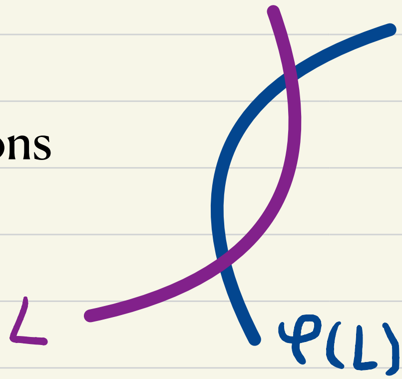
Symplectic Zoominar

March 24, 2023

Joint with Pazit Haim-Kislev & Richard Hind

Lagrangian Intersection Phenomenon

Arnold's Conjectures
Gromov's P.H.C. Theory
Theory of Generating Functions
Floer Homology
⋮



"Lagrangian sub-manifolds have more intersection points than is required by topology."

On the other hand: "non-Lagrangians" are flexible as long as there are no topological obstructions.*
(Polterovich, Laudenbach-Sikorav, Gurel,....)

* Infinitesimally displaceable provided that the normal bundle has a nowhere-vanishing section.

Biran's Lagrangian Barriers

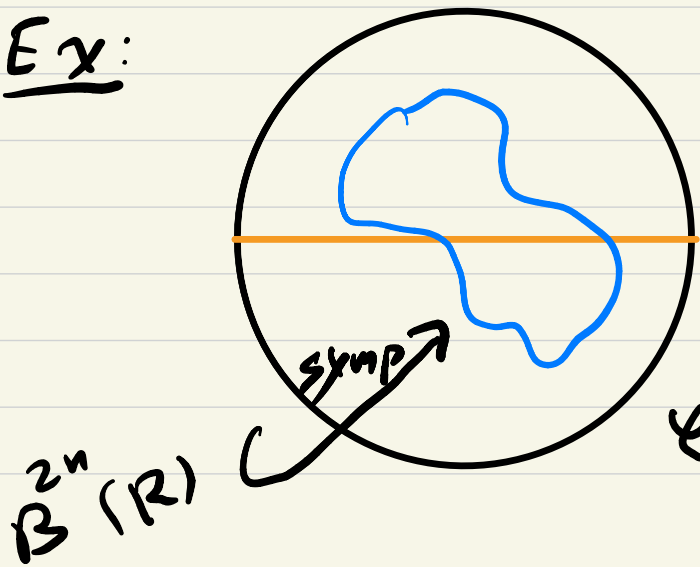
Existence of irremovable intersections between contractible domains & Lagrangian submanifolds.

Biran's decomposition theorem:

$$(M, \omega, J) = \underset{\text{skeleton } \Delta}{\text{CW-complex}} \sqcup \underset{\text{symp disc}}{\text{bundle}}$$

Theorem [Biran]: $P = (M, \omega, J, \Sigma)$ polarized Kähler of degree k.
For every $\phi : B(R) \xrightarrow{s} M$ with $R^2 \geq 1/\pi k$ one has $\phi(B(R)) \cap \Delta_P \neq \emptyset$.

Ex:



$\mathbb{C}P^n$

$\mathbb{R}P^n$

$R^2 \geq \frac{1}{2}$

$\Downarrow$

$\varphi(B^{2n}(R)) \cap \mathbb{R}P^n \neq \emptyset.$

Biran's Lagrangian Barriers

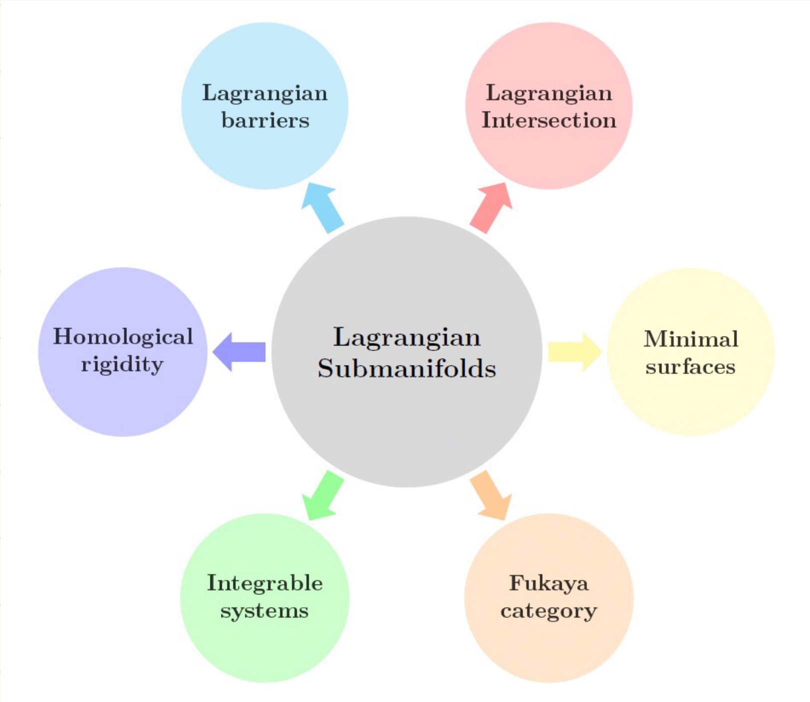
Lagrangian barriers are Lagrangian submanifolds such that:

$$c_G(M \setminus L) < c_G(M).$$

$$c_G(\mathbb{C}P^n \setminus \mathbb{R}P^n) = 1/2, \quad \text{(Biran)}$$

Other examples: Biran-Cornea (Clifford torus), Brendel-Schlenk (Markov pinwheels), Lee-Oh-Vianna (special Lagrangian tori), ...

Theorem [McDuff-Polterovich]: $c_G(\mathbb{C}P^n \setminus \Gamma) = 1$, where Γ is a complex submanifold.

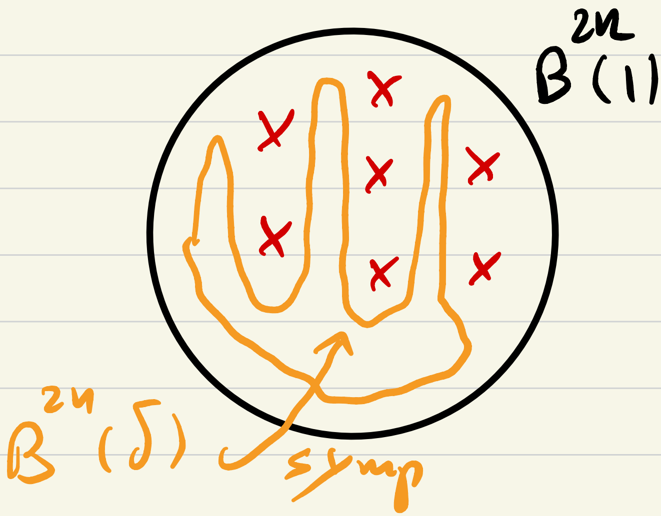


"Lagrangian submanifolds are the most interesting submanifolds for several reasons…….these are the submanifolds that exhibit "symplectic rigidity" (Abbondandolo & Schlenk).

"Everything is a Lagrangian submanifold" (Weinstein)

The Existence of Symplectic Barriers

Theorem [Haim-Kislev, Hind, O]: $\forall \delta > 0$ there is a finite union of codimension two disjoint symplectic hyperplanes Σ such that every symplectic embedding $B^{2n}(\delta) \overset{s}{\hookrightarrow} B^{2n}(1)$ must intersect Σ. In other words, $c_G(B^{2n}(1) \setminus \Sigma) < \pi\delta^2$.



Main Point: Lagrangians are not the only source of symplectic rigidity. Non-Lagrangian submanifolds are not always "flexible".

The Existence of Symplectic Barriers

Theorem [Haim-Kislev, Hind, O]: $\forall \delta > 0$ there is a finite union of codimension two disjoint symplectic hyperplanes Σ such that every symplectic embedding $B^{2n}(\delta) \overset{s}{\hookrightarrow} B^{2n}(1)$ must intersect Σ. In other words, $c_G(B^{2n}(1) \setminus \Sigma) < \pi\delta^2$.

$\boxed{n \geq 2}$

Remarks:

1) $\forall$ union $\tilde{\Sigma}$ of J-hol planes

$$c_G(B \setminus \tilde{\Sigma}) = c_G(B)$$

2) $\exists$ "explicit construction" for Σ

3) After: Sackel-Song-Varolgunes-Zhou

$$c_G(B^4 \setminus V) = ?, \quad V \in Gr(4,2)$$

Sketch of the proof

$\boxed{\Sigma_\varepsilon} = \{ (z_1, z_2) \in \mathbb{C}^2 \mid z_2 \in \varepsilon \mathbb{Z}^2 \}$

$A^L \quad z_1 \longmapsto z_1 \quad z_2 \longrightarrow L z_2$

__Step I:__ $\forall D \subseteq \mathbb{C}^2$ convex, $\forall L > 1$, $\forall \varepsilon > 0$

$\exists \; D \setminus N(\Sigma_\varepsilon) \hookrightarrow \beta A^L D$

"small nbg"

$\beta \xrightarrow{\varepsilon \to 0} 1$

(Eliashberg)

__Step II:__ $\forall L > 1 \quad \exists \, s \in Sp(2n)$

s.t $\quad c(A^L \underbrace{s B^{2n}}_{D}) << \text{"}\delta\text{"}$

The Thm follows for $s^{-1} \Sigma_\varepsilon$

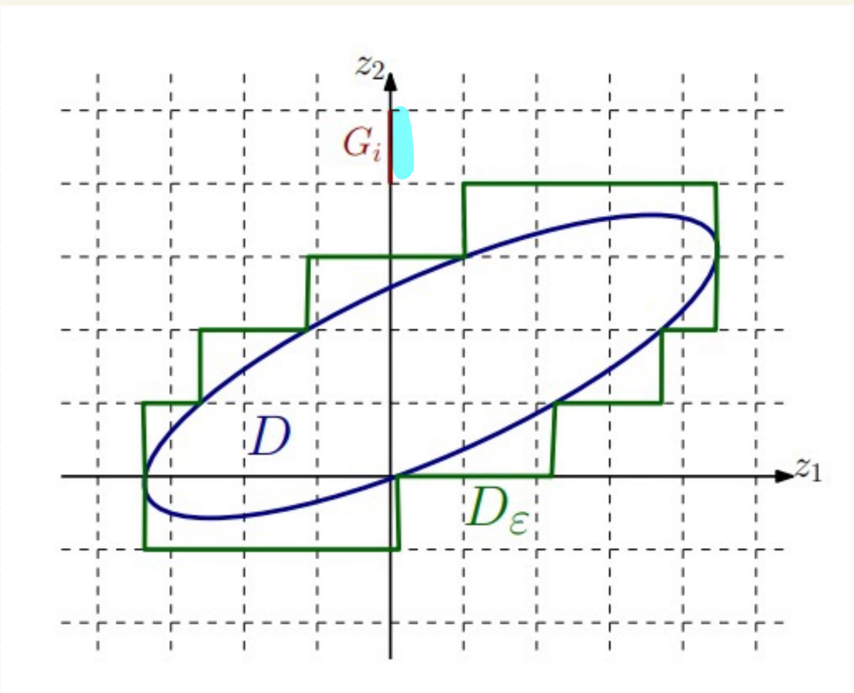
Approximation Argument

$$\Sigma_\varepsilon = \{(z_1, z_2) \in \mathbb{C}^2 \mid z_2 \in \varepsilon \mathbb{Z}^2\}$$

G_α 2 dim sq vertices in

$$\pi_2 \Sigma_\varepsilon \in \varepsilon \mathbb{Z}^2$$

$$D_\varepsilon = \bigsqcup_\alpha \left(\left\{ \pi_1 x \;\middle|\; \substack{x \in D \\ \pi_2 x \in G_\alpha} \right\} \times G_\alpha \right)$$



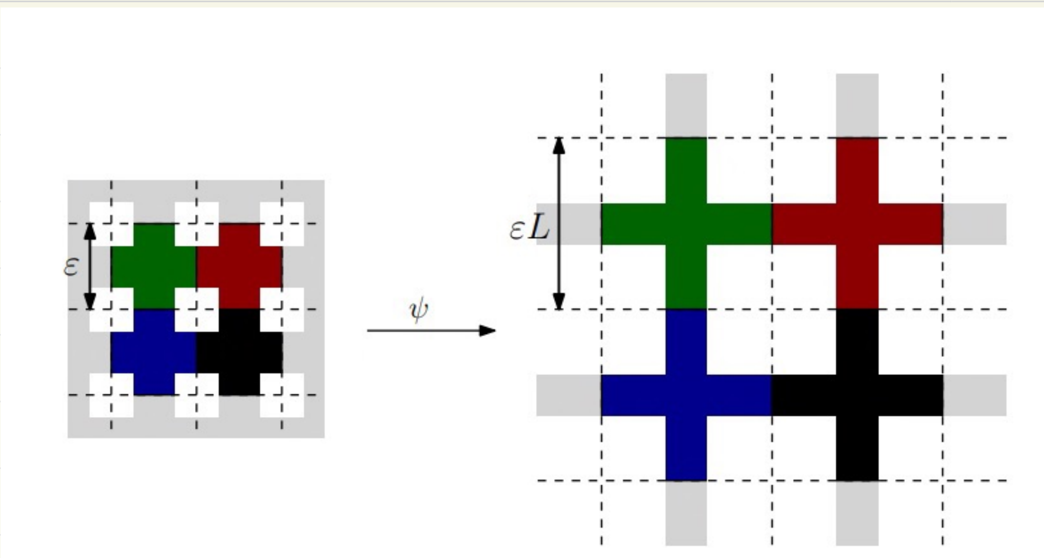
The Symplectic Embedding

$$D_\varepsilon \setminus N(\Sigma_\varepsilon) \xhookrightarrow{\text{symp}} A^L D_\varepsilon$$

$$\boxed{Id \times \Psi}$$

$$\psi : \mathbb{R}^2 \setminus N(\varepsilon \mathbb{Z}^2) \hookrightarrow \mathbb{R}^2 \setminus \tilde{N}(\varepsilon L \mathbb{Z}^2)$$

Area Preserving !!!



Open Questions

1) $c_G(B \setminus V) \stackrel{?}{=}$ $\quad V \in Gr(4,2)$

2) understand "symplectic barriers" in terms of dynamics

3) $\tilde{N}(\varepsilon) := \# \{ \Sigma \mid c(B \setminus \Sigma) < \varepsilon \}$

Estimate $\tilde{N}(\varepsilon)$

THANKS FOR YOUR ATTENTION!

Any Questions?