Orbifolds and systolic inequalities

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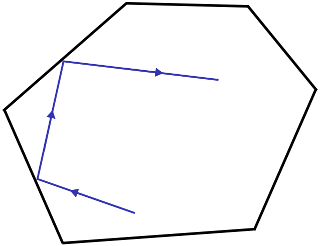
Symplectic Zoominar - January 13, 2023

Billiards



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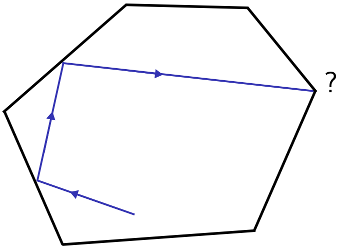

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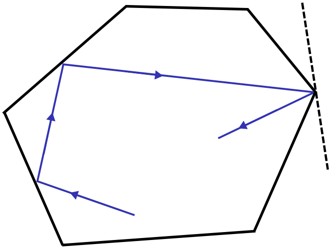

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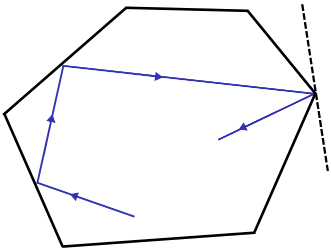
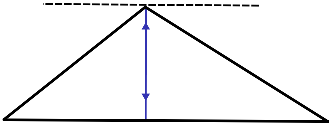
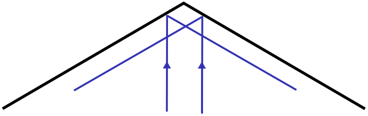




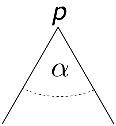


Question

Which polyhedral convex bodies admit a continuous billiard dynamic?

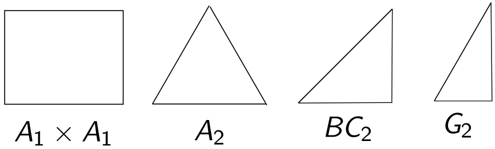
From billiards to orbifolds

Question: Which polyhedral convex bodies admit a continuous billiard dynamic?



Continuity at $p \Leftrightarrow \alpha = \frac{\pi}{n}$, $n \in \mathbb{N}$.

Dimension 2:



Theorem

A polyhedral convex body in $\mathbb{R}^n$ admits a continuous billiard dynamic if and only if it is a Riemannian orbifold.

cf. L., On continuous billiard and quasigeodesic flows characterizing alcoves and isosceles tetrahedra.

Riemannian orbifolds

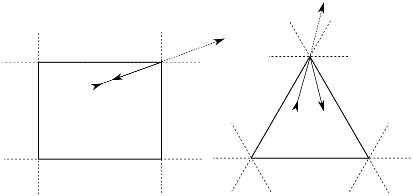
A polyhedral convex body in $\mathbb{R}^n$ admits a continuous billiard flow if and only if it is a Riemannian orbifold.

Definition

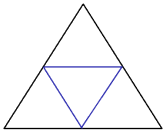
A *Riemannian orbifold* of dimension n is a metric length space $\mathcal{O}$ such that for each point $x \in \mathcal{O}$ there exists

- an open neighborhood U of x in $\mathcal{O}$
- a Riemannian n-manifold M
- a finite group G that acts by isometries on M

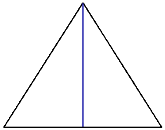
for which U and M/G are isometric.



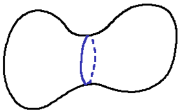
Systolic geometry



Systolic geometry



The so-called Calabi–Croke sphere $\Delta \cup_{\partial\Delta} \Delta$ is a Riemannian orbifold.



Theorem (Croke, 1988)

Let $L_{\min}$ be the length of the shortest nontrivial closed geodesic on a Riemannian 2-sphere S^2. Then $\rho(S^2) := \frac{L_{\min}^2}{\mathrm{area}} < 32$.

$$\rho(S^2_{\mathrm{round}}) = \pi < 2\sqrt{3} = \rho(S^2_{\mathrm{Calabi-Croke}}).$$

Observation

The Calabi–Croke sphere is the global maximizer for the systolic ratio among Riemannian orbifolds of type $S^2(3,3,3)$.

Loewner's theorem and Calabi–Croke's sphere

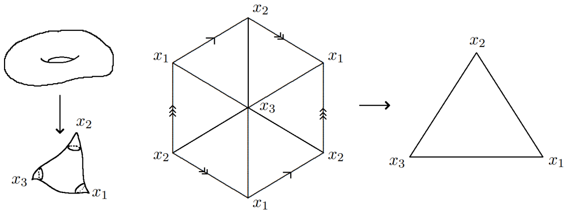
Theorem (Loewner, 1949)

The systole, i.e. the length of a shortest non-contractible closed geodesic, of a Riemannian 2-torus T^2 satisfies

$$\mathrm{sys}^2 \leq \frac{2}{\sqrt{3}} \mathrm{area}(\mathbb{T}^2)$$

with equality if and only if $\mathbb{T}^2$ is an equilateral torus.

Proof (that the Calabi–Croke metric maximizes ρ on $S^2(3,3,3)$)



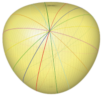
$$\mathrm{area}(\mathbb{T}^2) = 3\mathrm{area}(S^2(3,3,3)), \;\; L_{\min}(S^2(3,3,3)) \leq \mathrm{sys}(\mathbb{T}^2) \; \square.$$

Local maximizers and Zoll metrics

Theorem (Abbondandolo, Bramham, Hryniewicz, Salomão '18)

Zoll metrics on S^2 are local maximizers of the systolic ratio ρ with respect to the $\mathcal{C}^3$ topology.

Zoll means that all geodesics are closed and have the same length (picture by K. Polthier & M. Schmies).



Theorem (ABHS, 2021)

The systolic ratio of a sphere of revolution in $\mathbb{R}^3$ does not exceed π. It equals π if and only if S is Zoll.

Let L_{contr} be the length of shortest nontrivial closed geodesic whose lift to the unit sphere bundle is contractible.

Conjecture

$\rho_{\mathrm{contr}}(S^2) := \frac{L_{\mathrm{contr}}(S^2)^2}{\mathrm{area}(S^2)} \leq \rho_{\mathrm{contr}}(S^2_{\mathrm{round}}) = 4\pi$ for Riemannian S^2.

Local maximizers and Besse orbifolds

Theorem (L., Soethe, 2023)

The contractible systolic ratio of a rotationally symmetric Riemannian 2-sphere S does not exceed 4π. It equals 4π if and only if S is Zoll.

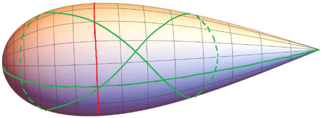
Local maximizers and Besse orbifolds

Theorem (L., Soethe, 2023)

Let $\mathcal{O} = S^2(m,n)$ be a rotationally symmetric spindle orbifold. Then

$$\rho_{\mathrm{contr}}(\mathcal{O}) \leq 2(m+n)\pi$$

with equality if and only if $\mathcal{O}$ is Besse.



Besse $\mathcal{O} = S^2(3,1)$ orbifold in $\mathbb{R}^3$ (picture K. P.

- *Besse* means that all geodesics are periodic. In this case there exists a common period due to a theorem by Wadsley.

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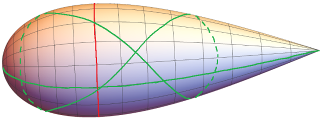


Besse $\mathcal{O} = S^2(3, 1)$ orbifold in $\mathbb{R}^3$ (picture K. P.

- *Besse*: the geodesic flow is periodic.
- For $k \in \mathbb{N}$ we set L_k to be the infimum of all lengths $l > 0$ such that there are at least k closed geodesics of length $\leq l$ and $\rho_k = \frac{L_k^2}{\mathrm{area}}$.

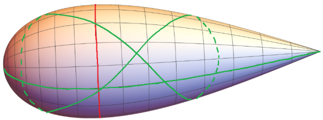
Local maximizers and Besse orbifolds

Theorem (L., Soethe, 2023)

Let $\mathcal{O} = S^2(m,n)$ be a rotationally symmetric spindle orbifold. There are $k = k(m+n) \in \mathbb{N}$ and $C = C(m+n) > 0$ such that

$$a)\ \rho_{\mathrm{contr}}(\mathcal{O}) \leq 2(m+n)\pi \quad \textit{and} \quad b)\ \rho_k(\mathcal{O}) \leq C,$$

each with equality if and only if $\mathcal{O}$ is Besse.



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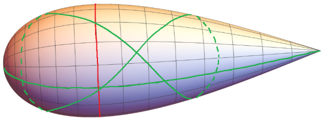


Besse $\mathcal{O} = S^2(3,1)$ orbifold in $\mathbb{R}^3$ (picture K. P.

Remark

certain Weyl Besse $S^2(m,n)$ orbifolds $\longleftrightarrow$ Finsler 2-spheres of constant flag curvature 1

see L.–Mettler, Deformations of the Veronese Embedding and Finsler 2-Spheres of Constant Curvature.

Besse $S^2(3,1)$ Tannery surface

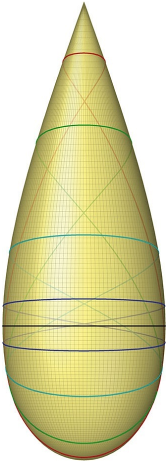


Figure: Picture by Konrad Polthier and Markus Schmies.

Systolic ratio for contact manifolds

- (M, λ) *closed contact* $(2n-1)$*-manifold*, i.e. $\lambda \wedge d\lambda^{n-1}$ is a volume form.
- *Volume of* (M, λ): $\mathrm{vol}(M, \alpha) = \int_M \lambda \wedge d\lambda^{n-1}$.
- *Reeb vector field* R_λ: $\iota_{R_\lambda} d\lambda = 0$, $\iota_{R_\lambda} \lambda = 1$.
- *Systolic ratio* of (M, λ):

$$\rho(M, \lambda) = \frac{T_{\min}(\lambda)^n}{\mathrm{vol}(M, \alpha)}.$$

$T_{\min} :=$ minimum of all periods of closed orbits of R_λ (if such exist like for $2n-1=3$ by Taubes' theorem).

Example

The unit tangent bundle $T^1\mathcal{O}$ of a Riemannian orbifold $\mathcal{O}$ with isolated singularities (like $S^2(m,n)$) equipped with the Liouville λ_L form. Here R_{λ_L} = geodesic vector field, $\mathrm{vol}(T^1\mathcal{O}, \lambda_L) = 2\pi\mathrm{vol}(\mathcal{O})$.

Systolic inequalities for Reeb flows

Alvarez Paiva, Balacheff, 2014: Local maximizers of ρ are Zoll.

Theorem (ABHS '18 for S^3, Benedetti–Kang for M^3 '21, Abbondandolo–Benedetti, 2019)

A Zoll contact form λ_0 on a connected closed $(2n-1)$-manifold M has a C^3 neighborhood $\mathcal{U}$ such that $\rho(\lambda) \leq \rho(\lambda_0)$ for all $\lambda \in \mathcal{U}$ with equality if and only if λ is Zoll.

Corollary (local sharp Viterbo conjecture)

There is a C^3-neigborhood $\mathcal{U}$ of the smooth ball in the space of smooth convex bounded open subsets of $\mathbb{R}^{2n}$ such that

$$T_{\min}(\lambda_0|_{\partial C})^n (= c_{\mathrm{EHZ}}^n) \leq \mathrm{vol}(C, (d\lambda_0)) \ \ \forall C \in \mathcal{U}$$

with equality if and only if C is symplectomorphic to a ball.

Higher systolic inequalities and Besse Reeb flows

For a $(2n-1)$-contact manifold (M, λ) we define higher systolic ratios for $k \in \mathbb{N}$:

$$\rho_k(\lambda) = \frac{\tau_k(\lambda)^n}{\mathrm{vol}(M, \lambda \wedge d\lambda)}$$

where

$$\tau_k(\lambda) = \inf\{t \mid \exists \geq k \text{ closed Reeb orbits of period } < t\}$$

if there exists a closed Reeb orbit.

Observation: Local maximizers of ρ_k are Besse, i.e. all Reeb orbits are closed and have a common period.

Besse Reeb flows

Examples

- The standard Liouville 1-form $\lambda_0 = \frac{1}{2}\sum_{j=1}^{n}(x_j dy_j - y_j dx_j)$ restricted to the boundary of the solid ellipsoid

$$E(p_1, ..., p_n) = \left\{ z \in \mathbb{C}^n \;\middle|\; \sum_{j=1}^{n} \frac{|z_j|^2}{p_j} \leq \frac{1}{\pi} \right\} \subset \mathbb{C}^n = \mathbb{R}^{2n}.$$

  for coprime integers $p_1, \ldots, p_n \in \mathbb{N}$.
- Unit cotangent bundles of Besse orbifolds with isolated singularities.
- Ustilovsky '99: For $n \geq 3$ there are infinitely many pairwise non-contactomorphic Besse contact spheres (S^{2n-1}, λ).

Spectral characterizations of contact Besse manifolds

Action spectrum: $\sigma(M, \lambda) = \{T(\gamma) \mid \gamma \text{ periodic Reeb orbit}\}$ with $T(\gamma)$ the period of a periodic Reeb orbit γ of (M, λ).

Theorem (Cristofaro-Gardiner, Mazzucchelli, 2019)

A closed contact manifold (M^3, λ) is Besse if and only if $\sigma(M, \lambda) \subset r\mathbb{N}$ for some $r > 0$.

$(Y^{2n+1}, \lambda) \subset \mathbb{C}^{n+1}$ convex contact sphere.

Ekeland-Hofer action selectors: $c_k = c_k(Y) \in \sigma(Y, \lambda)$.

$\min \sigma(Y, \lambda) = c_1 \leq c_2 \leq c_3 \leq \dots$

Theorem (Ginzburg, Gürel, Mazzucchelli, 2019)

(Y^{2n+1}, λ) is Besse if and only if $c_k = c_{k+n}$ for some k.

Boothby–Wang construction

(M, λ) Besse contact $(2n-1)$ manifold.

- Reeb flow induces an almost free (i.e. with finite stabilizers) S^1-action on M with an orbifold quotient $\pi : M \to M/S^1$.
- M/S^1 has an integral symplectic form represented by the image $i(\hat{e}) \in H^2_{S^1}(M;\mathbb{R})$ of the Euler class $\hat{e} \in H^2_{S^1}(M;\mathbb{Z})$ of π such that $d\lambda = T_{mc}(\lambda)\pi^* w$, where T_{mc} is the minimal common period.
- The Euler class e induces isomorphisms (Gysin sequence)

$$\hat{e} \cup \cdot : H^i_{S^1}(M;\mathbb{Z}) \to H^{i+2}_{S^1}(M;\mathbb{Z}) \tag{1}$$

for all $i \geq 2n-1$.

Remark: The construction can be seen as a special case of a symplectic reduction (applied to $(M \times \mathbb{R}_{>0}, d(r\lambda))$ with the trivially extended S^1-action and the Hamiltonian $H(x,t) = t$). In general a symplectic reduction gives rise to symplectic orbifolds.

Orbifold cohomology

- Every (Riemannian) orbifold $\mathcal{O}$ can be realized as the quotient of an almost free (isometric) action of a compact Lie group G on a (Riemannian) manifold M (take the orthonormal frame bundle $M = \mathrm{Fr}(\mathcal{O})$ and $G = O(n)$).
- Then $H^*_{\mathrm{orb}}(\mathcal{O}) := H^*_G(M)$ is independent of the representation M/G of $\mathcal{O}$.
- If M is contractible and G is finite (like for an orbifold chart) then $H^*_G(M) = H^*(G)$ is nontrivial in infinitely many degrees.

How to recognize manifolds among orbifolds?

Theorem (Quillen, 1971)

An n-dimensional orbifold $\mathcal{O}$ is a manifold if and only if $H^i_{\mathrm{orb}}(\mathcal{O}) = 0$ for all $i > n$.

Recognizing manifolds among orbifolds

Some consequences of Quillen's characterization:

Theorem (Amann, L., Radeschi, 2021)

Odd-dimensional simply connected Besse orbifolds are manifolds.

Here "simply connected" refers to the orbifold fundamental group.

Theorem (L., Radeschi, 2022)

An n-connected n-orbifold is a manifold for $n \geq 1$.
A compact $2n$-connected $(2n+1)$-orbifold is a manifold for $n \geq 1$.
A compact $(2n-2)$-connected $2n$-orbifold is a manifold for $n \geq 3$.

Remark: For $n \geq 4$ there exist compact $\lfloor n/2 - 1 \rfloor$-connected bad (i.e. not covered by a manifold) n-orbifolds [L., Radeschi, 2022].

Converse Boothby–Wang construction

Conversely, if $(\mathcal{O}, \omega)$ is a symplectic $(2n-2)$-orbifold with integral symplectic form $[\omega] \in \mathrm{Im}(H^2_{orb}(\mathcal{O};\mathbb{Z}) \to H^2_{orb}(\mathcal{O};\mathbb{R}))$, then for each integral lift $\hat{e} \in H^2_{orb}(\mathcal{O};\mathbb{Z})$ of w for which (cf. L.–Kegel)

$$\hat{e} \cup \cdot : H^i_{orb}(\mathcal{O};\mathbb{Z}) \to H^{i+2}_{orb}(\mathcal{O};\mathbb{Z}) \qquad (2)$$

is an isomorphism for all $i \geq 2n-1$, there exists a contact $(2n-1)$-manifold (M, λ) such that

- (M, λ) has Euler class $\hat{e}$,
- the reverse construction above gives rise to $(\mathcal{O}, \omega)$.

Dimension 3: An almost free S^1-action on an orientable 3-manifold M with Euler class $\hat{e}$ can be realized by a Reeb flow if and only if its Euler number $e = \langle \hat{e}, [M/S^1] \rangle$ is negative.

Besse 3-manifolds and higher systolic inequalities

(M, λ) be a closed Besse contact 3-manifold. Recall that for $k \in \mathbb{N}$ we defined $\rho_k(M, \lambda) = \frac{\tau_k(\lambda)^2}{\mathrm{vol}(M, \lambda \wedge d\lambda)}$ with

$$\tau_k(\lambda) = \inf\{t \mid \exists \geq k \text{ closed Reeb orbits of period } < t\}.$$

- there are finitely many singular Reeb orbits $\gamma_1, \ldots, \gamma_h$ with multiplicities $\alpha_1, \ldots, \alpha_h$. Set $k_0(\lambda) = \alpha_1 + \ldots + \alpha_h - h + 1$.
- $T_{\min} = \tau_1(\lambda) \leq \tau_2(\lambda) \leq \ldots \leq \tau_{k_0}(\lambda) = \tau_{k_0+1}(\lambda) = \ldots.$

Theorem (Abbondandolo, L., Mazzucchelli, 2022)

Let Y be a closed connected orientable 3-manifold and k a positive integer.

i) *If a contact form λ_0 on Y is a local maximizer of ρ_k, then λ_0 is Besse and $k_0(\lambda_0) = k$.*

Besse 3-manifolds and higher systolic inequalities

Theorem (Abbondandolo, L., Mazzucchelli, 2022)

Let Y be a closed, connected, orientable 3-manifold and k a positive integer.

i) *If a contact form λ_0 on Y is a local maximizer of ρ_k, then λ_0 is Besse and $k_0(\lambda_0) = k$.*

ii) *Every Besse contact form λ_0 on Y with $k_0(\lambda_0) = k$ has a C^3-neighborhood $\mathcal{U}$ in the space of contact forms on Y such that*

$$\rho_k(\lambda) \leq \rho_k(\lambda_0) = -\frac{1}{e(\lambda_0)}, \quad \forall \lambda \in \mathcal{U},$$

with equality if and only if λ is Besse.

Remarks:

- All $\rho_k \geq \rho_1$ are unbounded on the space of contact forms inducing a given contact structure ξ on Y [Sağlam, 2021].
- All $\rho_k \leq k^2 \rho_1$ are bounded on the space of smooth convex contact spheres.

Example: Ellipsoids

Let $p, q \in \mathbb{N}$ be coprime. Consider the boundary of an ellipsoid

$$E(p,q) = \left\{ z \in \mathbb{C}^n \,\middle|\, \frac{|z_1|^2}{p} + \frac{|z_2|^2}{q} \leq \frac{1}{\pi} \right\} \subset \mathbb{C}^{=}\mathbb{R}^4.$$

equipped with the restriction $\lambda_{p,q}$ of the standard Liouville 1-form $\lambda_0 = \frac{1}{2}\sum_{j=1}^{2}(x_j dy_j - y_j dx_j)$

One closed orbit of minimal period p, one closed orbit of minimal period q, all other orbits have minimal period pq. Hence,

$$k_0(\lambda_{p,q}) = p + q - 1.$$

Moreover, $\mathrm{vol}(E(p,q)) = pq$ and hence $\rho_{k_0}(\lambda_{p,q}) = pq$.

Comments on the proof of ii) - global surfaces of sections

Like in the Zoll case the proof uses global surfaces of sections. For us a global surface of section for a contact 3-manifold (Y, λ) is a smooth map $\iota : \Sigma \to Y$ from an oriented connected compact surface with non-empty boundary such that

- (Boundary) The restriction $\iota|_{\partial\Sigma}$ is an immersion positively tangent to the Reeb vector field R_λ
- (Transversality) The restriction $\iota|_{\mathrm{int}(\Sigma)}$ is an embedding into $Y\backslash(\partial\Sigma)$ transverse to the Reeb vector field R_λ.
- (Globality) Reeb orbits starting in any $y \in Y$ intersect Σ in both positive and negative time.

Comments on the proof of ii) - proof in the Zoll case

For S^3 in the Zoll case the proof works by contradiction:

1. Suppose contact forms λ arbitrarily close to a Zoll contact form λ_0 violate the systolic inequality.
2. Find a global surface of section $\iota : \Sigma = D^2 \to Y$ for λ bounding a Reeb orbit γ_m of minimal period $T_{\min}$ of λ such that the first return map $\phi : \Sigma \to \Sigma$ is close to the identity.
3. Find a fixed point of the first return map $\phi : \Sigma \to \Sigma$ with period $< T_{\min}$.

Step 2 is problematic in the Besse case if the minimal Reeb orbit of λ bifurcates from the regular orbits but approaches an iterate of a singular orbit of λ_0 when λ becomes closer to λ_0.

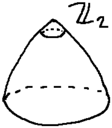
Comments on the proof of ii) - problems in the Besse case

Step 2 is problematic in the Besse case if the minimal Reeb orbit of λ bifurcates from the regular orbits but approaches an iterate of a singular orbit of λ_0 when λ becomes closer to λ_0.

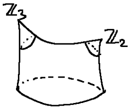
In this case the topology of Σ jumps in the limit and one cannot control the C^k norms of the first return map.

Example: Ellipsoid $E(2,3)$

- A global surface of section bounding an interate of a singular orbit covers $D^2(2)$ (or $D^2(3)$) and is hence a disk.



- A global surface of section bounding a regular orbit covers $D^2(2,3)$ and is hence hyperbolic.



Comments on the proof of ii) - strategy in the Besse case

Step 2 is problematic in the Besse case if the minimal Reeb orbit of λ bifurcates from the regular orbits but approaches an iterate of a singular orbit of λ_0 when λ becomes closer to λ_0.

Instead we prove

Theorem (Abbondandolo, L., Mazzucchelli, 2022)

If (Y, λ_0) is Besse and γ is any orbit of R_{λ_0}, then there exists a global surface of section with $\iota(\partial\Sigma) = \gamma$ (with explicit control on the topology).

as well as a stronger fixed point theorem for the Calabi homomorphism $\mathrm{Cal} : \widetilde{\mathrm{Ham}}_0(\Sigma, \iota^*\lambda) \to \mathbb{R}$ that can also be applied if the bounding orbit γ is not minimal. (Here, the condition that $\partial\Sigma$ covers a single Reeb orbit is needed to assure that a required vanishing flux condition is satisfied.)